Discontinuous Galerkin Methods

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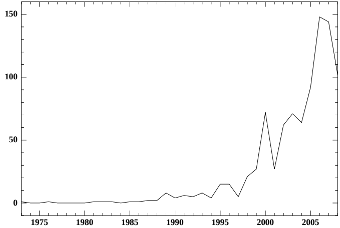
Introduction

- Discontinuous Galerkin methods (DG) can be viewed as
  - finite element methods allowing for discontinuous discrete functions
  - finite volume methods with more than one dof per mesh cell
- Advantages of such methods include
  - a high level of flexibility (choice of basis functions, nonmatching meshes, variable approximation order, local time stepping)
  - the possibility to enforce locally basic conservation principles
- The main drawback is higher computational costs w.r.t. FE's or FV's on the same mesh

A brief historical perspective I

- DG methods were introduced over three decades ago
  - moderate impact at that time
- Vigorous development over the last decade
  - numerical analysis
  - extensive range of applications
- DG-related publications/year (Source: Mathscinet)



A brief historical perspective II

First-order PDE's

- DG methods first coined for neutronics simulations [Reed & Hill '73]
- Convergence analysis
    - $O(h^k)$ L^2-error estimate if polynomials of degree k are used and exact solution is smooth enough [Lesaint & Raviart '74]
    - sharper $O(h^{k+1/2})$ L^2-estimate and $O(h^k)$ estimate for (broken) advective derivative [Johnson & Pitkäranta '86]
- Time-dependent conservation laws
    - low order DG method in space, forward Euler in time [Chavent & Cockburn '89]
    - Runge–Kutta DG & slope limiter [Cockburn & Shu '89-91] formal accuracy in smooth regions, sharp shock resolution, CV to entropy solution

A brief historical perspective III

Elliptic PDE's

- Boundary penalty methods [Nitsche '71]
- Interior penalty methods [Babuška '73, Douglas & Dupont '75, Baker '77, Wheeler '78, Arnold '82]
- Mixed approximation of elliptic PDE's
    - Navier–Stokes equations [Bassi & Rebay '97]
    - Local DG [Cockburn & Shu '98]
- Nonsymmetric interior penalty methods
    - introduced by [Oden, Babuška & Baumann '98]
    - further developed and analyzed in [Rivière, Wheeler & Girault '99]
- Unified analysis for Poisson problem [Arnold, Brezzi, Cockburn & Marini '01]

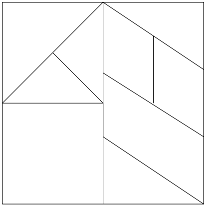
A brief historical perspective IV

Friedrichs' systems

- Introduced by Friedrichs in '58
- Linear systems of first-order PDE's with symmetry and positivity (L^2-coercivity) properties
- Encompass many important examples of elliptic and hyperbolic PDE's
  - advection–reaction, diffusion(–AR), elasticity, Stokes, Maxwell in diffusive regime, ...
- Unified analysis of DG methods based on Friedrichs' systems [AE & Guermond, '06, '07, '08]

Some basic notation I

- Mesh family $\{\mathcal{T}_h\}_h$ of computational domain $\Omega \subset \mathbb{R}^d$
    - shape-regularity in the usual sense
    - the meshes can be nonmatching (hanging nodes)
    - for simplicity, meshes cover Ω exactly
    - h: maximum mesh size
- Example of admissible mesh



Some basic notation II

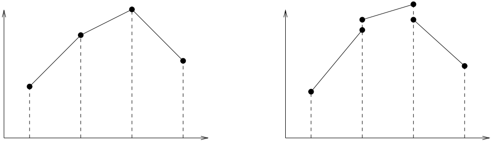
- Discontinuous finite element space ($k \geq 0$)

$$\mathbb{D}_h^k := \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_k\}$$

  - dof's can be taken elementwise (no matching condition at interfaces)

$$\dim(\mathbb{D}_h^k) = N_{\rm el} \dim(\mathbb{P}_k) = \binom{k+d}{k} N_{\rm el}$$

- Continuous finite element space $\mathbb{C}_h^k := \mathbb{D}_h^k \cap C^0(\Omega)$



Some basic notation III

- Comparison of the dimensions of $\mathbb{D}_h^k$ and $\mathbb{C}_h^k$
    - Euler's relations
    - 2D : $N_{\mathrm{el}} \simeq 2N_{\mathrm{ve}}$, $N_{\mathrm{ed}} \simeq 3N_{\mathrm{ve}}$
    - 3D : $N_{\mathrm{el}} \simeq 6N_{\mathrm{ve}}$, $N_{\mathrm{fa}} \simeq 12N_{\mathrm{ve}}$, $N_{\mathrm{ed}} \simeq 7N_{\mathrm{ve}}$

- Examples
    - $(d = 2,\ k = 1)$ $\dim(\mathbb{C}_h^1) = N_{\mathrm{ve}}$, $\dim(\mathbb{D}_h^1) = 3N_{\mathrm{el}} \simeq 6N_{\mathrm{ve}}$
    - $(d = 2,\ k = 2)$ $\dim(\mathbb{C}_h^2) \simeq 4N_{\mathrm{ve}}$, $\dim(\mathbb{D}_h^2) = 6N_{\mathrm{el}} \simeq 12N_{\mathrm{ve}}$

	$d = 2$		$d = 3$	
k	$\mathbb{D}_h^k$	$\mathbb{C}_h^k$	$\mathbb{D}_h^k$	$\mathbb{C}_h^k$
1	6	1	24	1
2	12	4	40	8
3	20	9	80	27

Some basic notation IV

- Broken Sobolev spaces $H^s(\mathcal{T}_h)$ $(s \geq 0)$
- Broken gradient operator (defined elementwise)

$$\nabla_h : H^1(\mathcal{T}_h) \to [L^2(\Omega)]^d$$

- Local approximation property: $\forall z \in H^{k+1}(\mathcal{T}_h)$, $\forall T \in \mathcal{T}_h$,

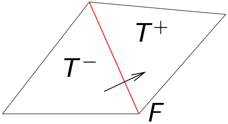
$$\|z - \pi_h^k z\|_T + h_T^{\frac{1}{2}} \|z - \pi_h^k z\|_{\partial T} + h_T \|\nabla(z - \pi_h^k z)\|_T \lesssim h_T^{k+1} \|z\|_{H^{k+1}(T)}$$

π_h^k: L^2-orthogonal projection onto $\mathbb{D}_h^k$

Some basic notation V

- Mesh faces collected into $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\partial}$ (split into interfaces and boundary faces)
- Discrete functions can be two-valued at interfaces
- Interface $\mathcal{F}_h^{\mathrm{i}} \ni F = T^- \cap T^+$, normal n_F from T^- to T^+
- Mean values and jumps at interfaces

$$\{\!\{\varphi\}\!\} := \tfrac{1}{2}(\varphi^- + \varphi^+) \qquad [\![\varphi]\!] := \varphi^- - \varphi^+$$



- On the boundary, $\{\!\{\varphi\}\!\} = [\![\varphi]\!] := \varphi$

Outline

- Nonconforming error analysis
- First-order PDE's
    - Advection–reaction
    - Friedrichs' systems
- Second-order scalar PDE's
    - Diffusion
    - Advection–diffusion–reaction
- Incompressible flows
    - Stokes
    - Navier–Stokes

Topics not covered in this lecture include

- Time-dependent problems
  - abundant numerical techniques with promising results
  - theoretical aspects are much less covered
  - see [Burman, AE & Fernandez, Mafelap '09]
- Implementation issues
  - see, e.g., [Karniadakis & Spencer '99, Hesthaven & Warburton '08]
- A posteriori error analysis
  - see lecture by M. Vohralík

Nonconforming error analysis

- Abstract problem: Seek $u \in W$ s.t.

$$a(u,w) = \int_\Omega fw, \qquad \forall w \in W$$

    - Banach space $W \subset [L^2(\Omega)]^m$ $(m \geq 1)$
    - bilinear form a bounded on $W \times W$
    - data f, say, in $[L^2(\Omega)]^m$

- Well-posedness of abstract problem BNB theorem

$$\forall v \in W, \qquad \|v\|_W \lesssim \sup_{w \in W \setminus \{0\}} \frac{a(v,w)}{\|w\|_W}$$

$$\forall w \in W, \qquad (\forall v \in W,\ a(v,w) = 0) \implies (w = 0)$$

- See [AE & Guermond '04]

The discrete setting

- Finite-dimensional space $W_h \not\subset W$ (nonconforming)
- Discrete problem: Seek $u_h \in W_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h, \qquad \forall w_h \in W_h$$

- Approximation error $(u - u_h)$ belongs to $Z := W + W_h$
- We work with two norms: $\|\cdot\|$ and $\|\cdot\|_*$ both defined on Z
    - the approximation error will be estimated in the $\|\cdot\|$-norm
    - the $\|\cdot\|_*$-norm controls the $\|\cdot\|$-norm

$$\forall z \in Z, \quad \|z\| \lesssim \|z\|_*$$

Three key properties

- Stability (implies well-posedness of discrete problem)

$$\forall v_h \in W_h, \qquad \|v_h\| \lesssim \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{\|w_h\|}$$

- Consistency (DG methods are consistent!)

$$\forall w_h \in W_h, \qquad a_h(u, w_h) = a(u, w_h) = \int_\Omega f w_h$$

- Continuity

$$\forall z \in Z, \; \forall w_h \in W_h, \qquad a_h(z, w_h) \lesssim \|z\|_* \|w_h\|$$

Error estimate I

- There holds

$$\|u - u_h\| \lesssim \inf_{y_h \in W_h} \|u - y_h\|_*$$

- Proof. Let $y_h \in W_h$.
    - stability, consistency, and continuity imply

$$\begin{aligned} \|u_h - y_h\| &\lesssim \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(u_h - y_h, w_h)}{\|w_h\|} \\ &= \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(u - y_h, w_h)}{\|w_h\|} \\ &\lesssim \|u - y_h\|_* \end{aligned}$$

    - conclude using the triangle inequality

Error estimate II

- Recall that
$$\|u - u_h\| \lesssim \inf_{y_h \in W_h} \|u - y_h\|_*$$
- The estimate is not optimal since different norms are used
- The estimate is quasi-optimal if the upper bound has the same CV order as the optimal bound $\inf_{y_h \in W_h} \|u - y_h\|$
- This is the case in the sequel for the analysis of DG methods (and many other stabilized FEM's)

First-order PDE's

- Advection–reaction
- Friedrichs' systems

Advection–reaction

- Let $\beta \in [L^\infty(\Omega)]^d$, $\nabla\cdot\beta \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$
- Inflow and outflow parts of boundary $\partial\Omega$

$$\partial\Omega^\pm := \{x \in \partial\Omega;\ \pm\beta(x)\cdot n(x) > 0\}$$

- The model problem is

$$\begin{cases} \mu u + \beta\cdot\nabla u = f & \text{in } L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega^- \end{cases}$$

A mathematical framework I

- Graph space $W := \{v \in L^2(\Omega);\ \beta\cdot\nabla v \in L^2(\Omega)\}$
- This is a Hilbert space with the norm $\|v\|_W^2 := \|v\|_\Omega^2 + \|\beta\cdot\nabla v\|_\Omega^2$
- Assume that $\partial\Omega^-$ and $\partial\Omega^+$ are well-separated
- There is a continuous trace operator from W onto

$$L^2(\partial\Omega; |\beta\cdot n|) := \{v \text{ is measurable on } \partial\Omega \quad \int_{\partial\Omega} |\beta\cdot n| v^2 < +\infty\}$$

A mathematical framework II

- Define on $W \times W$ the bilinear form

$$a(v,w) := \int_\Omega [\mu v + (\beta \cdot \nabla v)]w + \int_{\partial\Omega} (\beta \cdot n)^\ominus vw$$

where for $x \in \mathbb{R}$, $x^\oplus := \frac{1}{2}(|x| + x)$ and $x^\ominus := \frac{1}{2}(|x| - x)$

- Assume that

$$\exists \mu_0 > 0, \qquad \mu - \tfrac{1}{2}\nabla \cdot \beta \geq \mu_0 \quad \text{a.e. in } \Omega$$

- This implies the L^2-coercivity of a on W since

$$\begin{aligned} a(v,v) &= \int_\Omega (\mu - \tfrac{1}{2}\nabla \cdot \beta) v^2 + \tfrac{1}{2}\int_{\partial\Omega} (\beta \cdot n) v^2 + \int_{\partial\Omega} (\beta \cdot n)^\ominus v^2 \\ &\geq \mu_0 \|v\|_\Omega^2 + \tfrac{1}{2}\int_{\partial\Omega} |\beta \cdot n| v^2 \end{aligned}$$

A mathematical framework III

- Consider the following problem: Seek $u \in W$ s.t.

$$a(u, w) = \int_\Omega fw, \qquad \forall w \in W$$

- Theorem. This problem is well-posed
  - L^2-coercivity readily implies uniqueness
  - existence by BNB theorem (using L^2-coercivity of a)
  - BC's are weakly enforced

Centered fluxes I

- DG approximation in $W_h := \mathbb{D}_h^k$
- Discrete problem: Seek $u_h \in W_h$ s.t.

$$a_h(u_h, w_h) = \int_\Omega f w_h, \qquad \forall w_h \in W_h$$

- Guidelines for the design of discrete bilinear form a_h
  - discrete L^2-coercivity on W_h
  - consistency
- Two assumptions on the exact solution u
  - locally smooth enough, $u \in H^{k+1}(\mathcal{T}_h)$
  - mesh fitted to possible singularities, $\beta \cdot n_F [\![u]\!] = 0$ on all $F \in \mathcal{F}_h^{\mathrm{i}}$

Centered fluxes II

- Step 1: Localize gradient

$$a_h(v_h, w_h) := \int_\Omega [\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h] + \sum_{F\in\mathcal{F}_h^\partial} \int_F (\beta\cdot n)^\ominus v_h w_h$$

- a_h is not L^2-coercive on W_h

$$a(v_h, v_h) := \int_\Omega [\mu v_h^2 + (\beta\cdot\nabla_h v_h) v_h] + \sum_{F\in\mathcal{F}_h^\partial} \int_F (\beta\cdot n)^\ominus v_h^2$$

$$= \int_\Omega (\mu - \tfrac{1}{2}\nabla\cdot\beta) v_h^2 + \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^\partial} \int_F |\beta\cdot n| v_h^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F \{\!\{\beta\cdot n v_h^2\}\!\}$$

and the last term has no sign a priori

Centered fluxes III

- Step 2: Recover discrete L^2-coercivity in a consistent way by setting

$$a_h(v_h, w_h) := \int_\Omega [\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h] + \sum_{F\in\mathcal{F}_h^\partial} \int_F (\beta\cdot n)^\ominus v_h w_h$$

$$- \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F \beta\cdot n_F [\![v_h]\!] \{\!\{ w_h \}\!\}$$

so that $a_h(v_h, v_h) \geq |\!|\!| v_h |\!|\!|^2$ on W_h with

$$|\!|\!| v_h |\!|\!|^2 := \mu_0 \| v_h \|_\Omega^2 + \sum_{F\in\mathcal{F}_h^\partial} \int_F \tfrac{1}{2} |\beta\cdot n| v_h^2$$

Centered fluxes IV

- Convergence estimate $\|\!| u - u_h \|\!| \lesssim h^k$
    - discrete stability

$$\|\!| w_h \|\!|^2 \lesssim a_h(w_h, w_h) = \left(\frac{a_h(w_h, w_h)}{\|\!| w_h \|\!|} \right) \|\!| w_h \|\!| \leq \left(\sup_{y_h \in W_h} \frac{a_h(w_h, y_h)}{\|\!| y_h \|\!|} \right) \|\!| w_h \|\!|$$

-
    - consistency by construction
    - continuity: $a_h(z, w_h) \lesssim \|\!| z \|\!|_* \|\!| w_h \|\!|$ with

$$\|\!| z \|\!|_*^2 := \|\!| z \|\!|^2 + \|\beta{\cdot}\nabla_h z\|_\Omega^2 + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} h_F^{-1} \int_F |\beta{\cdot}n_F| [\![z]\!]^2$$

-
    - standard approximation properties on $\mathbb{D}_h^k$
- convergence for $k \geq 1$ with suboptimal rate

Centered fluxes V

- Local formulation
  - Recall that

$$a_h(v_h,w_h)=\int_\Omega[\mu v_h w_h+(\beta\cdot\nabla_h v_h)w_h]+\sum_{F\in\mathcal{F}_h^\partial}\int_F(\beta\cdot n)^\ominus v_h w_h$$
$$-\sum_{F\in\mathcal{F}_h^{\mathrm{i}}}\int_F\beta\cdot n_F[\![v_h]\!]\{\!\{w_h\}\!\}$$

  - Integration by parts yields

$$a_h(v_h,w_h)=\int_\Omega[(\mu-\nabla\cdot\beta)v_h w_h-v_h(\beta\cdot\nabla_h w_h)]+\sum_{F\in\mathcal{F}_h^\partial}\int_F(\beta\cdot n)^\oplus v_h w_h$$
$$+\sum_{F\in\mathcal{F}_h^{\mathrm{i}}}\int_F\beta\cdot n_F\{\!\{v_h\}\!\}[\![w_h]\!]$$

- Let $T\in\mathcal{T}_h$, $q\in\mathbb{P}_k(T)$ and take $w_h=q1_T$

Centered fluxes VI

- Since $[\![q 1_T]\!] = (n_T \cdot n_F) q|_F$ (n_T: outward normal to T), this yields

$$\int_T [(\mu - \nabla\cdot\beta) v_h q - v_h(\beta\cdot\nabla q)] + \sum_{F\subset\partial T} \int_F \phi_{T,F}(v_h) q = \int_T fq$$

with the numerical fluxes

$$\phi_{T,F}(v_h) := \begin{cases} \beta\cdot n_T \{\!\{v_h\}\!\} & (F \in \mathcal{F}_h^{\mathrm{i}}) \\ (\beta\cdot n)^{\oplus} v_h & (F \in \mathcal{F}_h^{\partial}) \end{cases}$$

- These fluxes are conservative ($F = T^- \cap T^+$)

$$\phi_{T^-,F}(v_h) + \phi_{T^+,F}(v_h) = 0$$

- They are also consistent with the exact fluxes: if u is the exact solution

$$\phi_{T,F}(u) = (\beta\cdot n_T) u$$

Upwind fluxes I

- Idea: sharpen discrete stability by penalizing interface jumps in a least-squares sense

$$a_h(v_h, w_h) := \int_\Omega [\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h] + \sum_{F\in\mathcal{F}_h^\partial} \int_F (\beta\cdot n)^\ominus v_h w_h$$
$$- \sum_{F\in\mathcal{F}_h^\mathrm{i}} \int_F \beta\cdot n_F [\![v_h]\!]\{\!\{w_h\}\!\} + \sum_{F\in\mathcal{F}_h^\mathrm{i}} \int_F \tfrac{1}{2}\alpha|\beta\cdot n_F|[\![v_h]\!][\![w_h]\!]$$

with user-defined parameter $\alpha > 0$

Upwind fluxes II

- Sharper discrete stability with stronger norm

$$|||w|||^2 := \mu_0\|w\|_\Omega^2 + \sum_{F\in\mathcal{F}_h^\partial}\int_F \tfrac{1}{2}|\beta\cdot n|w^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}\int_F \tfrac{\alpha}{2}|\beta\cdot n_F|[\![w]\!]^2 + \sum_{T\in\mathcal{T}_h} h_T\|\beta\cdot\nabla_h w\|_T^2$$

- Continuity holds with $|||w|||_*^2 := |||w|||^2 + \sum_{T\in\mathcal{T}_h}[h_T^{-1}\|w\|_T^2 + \|w\|_{\partial T}^2]$
- Convergence estimate $|||u-u_h||| \lesssim h^{k+1/2}$
  - quasi-optimal estimate
  - optimal estimate for advective derivative, jumps and boundary values
  - $h^{1/2}$-suboptimal in L^2
  - similar estimate for many other stabilized FEM's

Upwind fluxes III

- New numerical fluxes

$$\phi_{T,F}(v_h) := \begin{cases} \beta\cdot n_T \{\!\{v_h\}\!\} + \frac{1}{2}\alpha|\beta\cdot n_T|[\![v_h]\!]_T & (F \in \mathcal{F}_h^{\mathrm{i}}) \\ (\beta\cdot n)^{\oplus} v_h & (F \in \mathcal{F}_h^{\partial}) \end{cases}$$

where $[\![v_h]\!]_T = v_h^{\mathrm{int}} - v_h^{\mathrm{ext}}$

- Particular choice $\alpha = 1$ yields the upwind flux

$$\phi_{T,F}(v_h) = \begin{cases} (\beta\cdot n_T)^{\oplus} v_h^{\mathrm{int}} - (\beta\cdot n_T)^{\ominus} v_h^{\mathrm{ext}} & (F \in \mathcal{F}_h^{\mathrm{i}}) \\ (\beta\cdot n)^{\oplus} v_h & (F \in \mathcal{F}_h^{\partial}) \end{cases}$$

Friedrichs' systems

- Introduced by Friedrichs in 1958
- $(d+1)$ fields $\{\mathcal{A}^k\}_{0\le k\le d}$ in $[L^\infty(\Omega)]^{m,m}$ such that

$$\sum_{k=1}^{d} \partial_k \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m}$$

- Symmetry and positivity properties (a.e. in Ω)

$$\forall 1 \le k \le d,\ \mathcal{A}^k = (\mathcal{A}^k)^T$$

$$\exists \mu_0 > 0, \quad \mathcal{A}^0 + (\mathcal{A}^0)^T - \sum_{k=1}^{d} \partial_k \mathcal{A}^k \ge 2\mu_0 \mathcal{I}_m$$

- Data $f \in L := [L^2(\Omega)]^m$

A mathematical framework I

- The PDE is $A^0 z + Az = f$ in L
- Zero-order operator $A^0 : L \ni z \mapsto \mathcal{A}^0 z \in L$
- First-order differential operator

$$Az := \sum_{k=1}^{d} \mathcal{A}^k \partial_k z$$

- Graph space $W := \{z \in L;\ Az \in L\}$
- W is a Hilbert space with the norm $\|z\|_W^2 := \|z\|_L^2 + \|Az\|_L^2$
- $A \in \mathcal{L}(W; L)$ (bounded from W to L)

A mathematical framework II

- Admissible BC's for well-posedness
- Define $\tilde{A} \in \mathcal{L}(W; L)$ s.t. $\tilde{A}z := -\sum_{k=1}^{d} \partial_k(\mathcal{A}^k z)$
- Define $D \in \mathcal{L}(W; W')$ s.t.

$$\langle Dz, y \rangle_{W',W} := (Az, y)_L - (z, \tilde{A}y)_L$$

- The symmetry property implies that D is a boundary operator

$$[\mathfrak{C}_0^\infty(\Omega)]^m \subset \mathrm{Ker}(D)$$

A mathematical framework III

- Assume there is $M \in \mathcal{L}(W; W')$ s.t.

$$\begin{aligned} &\forall z \in W,\ \langle Mz, z\rangle_{W',W} \geq 0 \\ &W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M) \end{aligned}$$

- By construction, $\mathrm{Ker}(D) = \mathrm{Ker}(M)$ so that M is also a boundary operator
- An admissible BC is

$$z \in \mathrm{Ker}(M - D)$$

A mathematical framework IV

- In most cases, there is a boundary representation

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} y^T \mathcal{D} z \qquad \langle Mz, y\rangle_{W',W} = \int_{\partial\Omega} y^T \mathcal{M} z$$

with z, y smooth enough and $\mathcal{D}$, $\mathcal{M} \in [L^\infty(\partial\Omega)]^{m,m}$

$$\mathcal{D} = \sum_{k=1}^{d} n_k \mathcal{A}^k$$

  - D is by construction selfadjoint
  - $\mathcal{D}$ is by construction symmetric

A mathematical framework V

- Consider the model problem

$$\begin{cases} A^0 z + Az = f \quad \text{in } L \\ z \in \mathrm{Ker}(M - D) \end{cases}$$

- This amounts to seeking $z \in W$ s.t. $a(z, y) = (f, y)_L$, $\forall y \in W$, with

$$a(z, y) := (A^0 z, y)_L + (Az, y)_L + \tfrac{1}{2}\langle (M - D)z, y\rangle_{W',W}$$

  - a is L-coercive on W
  - BC's are weakly enforced

- The above problem is well-posed [AE & Guermond '06]

Examples I

- Advection–reaction Set $m = 1$,

$$\mathcal{A}^0 = \mu, \qquad \mathcal{A}^k = \beta^k$$

- The boundary operator D is such that

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} (\beta{\cdot}n) zy$$

- The BC $u|_{\partial\Omega^-} = 0$ can be enforced through the operator

$$\langle Mz, y\rangle_{W',W} = \int_{\partial\Omega} |\beta{\cdot}n| zy$$

- Boundary integral representation with $\mathcal{D} = \beta{\cdot}n$ and $\mathcal{M} = |\beta{\cdot}n|$

Examples II

- Maxwell's equations in diffusive regime (μ and ε uniformly bounded away from zero)

$$\begin{cases} \mu H + \nabla \times E = f \\ \varepsilon E - \nabla \times H = g \end{cases}$$

- 3D, $m = 6$, $\mathcal{A}^0(H,E) = (\mu H, \varepsilon E)$ and for $k \in \{1,2,3\}$,

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0_{3,3} & \mathcal{R}^k \\ \hdashline -\mathcal{R}^k & 0_{3,3} \end{array}\right] \qquad \mathcal{R}^k_{ij} = \epsilon_{ikj}$$

- Boundary operators

$$\mathcal{D} = \left[\begin{array}{c:c} 0_{3,3} & \mathcal{N} \\ \hdashline \mathcal{N}^T & 0_{3,3} \end{array}\right] \qquad \mathcal{M} = \left[\begin{array}{c:c} 0_{3,3} & -\mathcal{N} \\ \hdashline \mathcal{N}^T & 0_{3,3} \end{array}\right]$$

enforcing $n \times E|_{\partial\Omega} = 0$ ($\mathcal{N}\xi = n \times \xi$ for all $\xi \in \mathbb{R}^3$)

DG approximation I

- DG approximation in $W_h := [\mathbb{D}_h^k]^m$
- Discrete problem: Seek $z_h \in W_h$ s.t.

$$a_h(z_h, y_h) = (f, y_h)_L, \qquad \forall y_h \in W_h$$

- Two-valued extension of boundary field $\mathcal{D}$ to interfaces
  - for all $T \in \mathcal{T}_h$, $\mathcal{D}|_{\partial T} := \sum_{k=1}^{d} n_{T,k} \mathcal{A}^k$
  - $\mathcal{D}|_{\partial T}$ is symmetric
  - $\{\!\{\mathcal{D}\}\!\} = 0$
  - $\{\!\{\mathcal{D}z\}\!\} = 0$ for exact solution

DG approximation II

- Guidelines for the design of discrete bilinear form a_h
    - L^2-coercivity on W_h together with consistency
    - least-squares penalty of interface jumps

$$a_h(z,y) := \sum_{T\in\mathcal{T}_h} [(A^0 z, y)_{L,T} + (Az, y)_{L,T}] + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(\mathcal{M}_F z - \mathcal{D} z, y)_{L,F}$$

$$- \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\!\{\mathcal{D}z\}\!\}, \{\!\{y\}\!\})_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\mathcal{S}_F [\![z]\!], [\![y]\!])_{L,F}$$

- the fields $\{\mathcal{M}_F\}_{F\in\mathcal{F}_h^\partial}$ weakly enforce BC's
- the fields $\{\mathcal{S}_F\}_{F\in\mathcal{F}_h^{\mathrm{i}}}$ penalize interface jumps
- both fields are assumed to take nonnegative values

DG approximation III

- Consistency requires that for exact solution z

$$\forall F \in \mathcal{F}_h^{\partial},\ \mathcal{M}_F z = \mathcal{D}z|_F \qquad \text{and} \qquad \forall F \in \mathcal{F}_h^{\mathrm{i}},\ \mathcal{S}_F[\![z]\!] = 0$$

- Concerning L^2-coercivity, the positivity assumption and i.b.p. yield

$$a_h(y_h, y_h) \geq \mu_0 \|y_h\|_L^2 + \underbrace{\sum_{F \in \mathcal{F}_h^{\partial}} \tfrac{1}{2}(\mathcal{M}_F y_h, y_h)_{L,F}}_{|y_h|_M^2}$$
$$+ \underbrace{\sum_{F \in \mathcal{F}_h^{\mathrm{i}}} (\mathcal{S}_F[\![y_h]\!], [\![y_h]\!])_{L,F}}_{|y_h|_J^2}$$

DG approximation IV

- Additional assumptions on discrete fields

$$|(\mathcal{M}_F v - \mathcal{D}v, w)_{L,F}| \lesssim (\mathcal{M}_F v, v)_{L,F}^{1/2} \|w\|_{L,F}$$
$$|(\mathcal{M}_F v + \mathcal{D}v, w)_{L,F}| \lesssim \|v\|_{L,F} (\mathcal{M}_F w, w)_{L,F}^{1/2}$$
$$\mathcal{S}_F \simeq |\mathcal{D}|$$

- Assume that $T \in \mathcal{T}_h$, $\mathcal{A}^k|_T \in [\mathcal{C}^{0,\frac{1}{2}}(T)]^{m,m}$, $1 \leq k \leq d$
- Discrete stability result

$$\forall y_h \in W_h, \qquad |||y_h||| \lesssim \sup_{w_h \in W_h \setminus \{0\}} \frac{a_h(y_h, w_h)}{|||w_h|||}$$

with the norm

$$|||y|||^2 := \|y\|_L^2 + |y|_M^2 + |y|_J^2 + \sum_{T \in \mathcal{T}_h} h_T \|Ay\|_{L,T}^2$$

DG approximation V

- Continuity result: $a_h(z, y_h) \lesssim \|z\|_* \|y_h\|$ where

$$\|z\|_*^2 := \|z\|^2 + \sum_{T \in \mathcal{T}_h} [h_T^{-1} \|z\|_T^2 + \|z\|_{\partial T}^2]$$

- Convergence estimate $\|z - z_h\| \lesssim h^{k+1/2}$
  - quasi-optimal estimate
  - optimal estimate for broken A-derivative, jumps and boundary values
  - $h^{1/2}$-suboptimal in L^2

DG approximation VI

- Local formulation: For all $T \in \mathcal{T}_h$ and $q \in [\mathbb{P}_k(T)]^m$,

$$(\mathcal{A}^0 z_h, q)_{L,T} + (z_h, \tilde{A}q)_{L,T} + \sum_{F \subset \partial T} (\phi_{T,F}(z_h), q)_{L,F} = (f, q)_{L,T}$$

  with the numerical fluxes

$$\phi_{T,F}(z_h) := \begin{cases} \mathcal{D}|_{\partial T} \{\!\{z_h\}\!\} + \mathcal{S}_F [\![z_h]\!]_T & (F \in \mathcal{F}_h^{\mathrm{i}}) \\ \frac{1}{2}(\mathcal{M}_F + \mathcal{D}) z_h & (F \in \mathcal{F}_h^{\partial}) \end{cases}$$

- These fluxes are conservative and consistent with the exact fluxes

Examples

- Advection–reaction Set $(\alpha > 0)$

$$\mathcal{M}_F = |\beta{\cdot}n| \qquad \text{and} \qquad \mathcal{S}_F = \tfrac{1}{2}\alpha|\beta{\cdot}n_F|$$

- Maxwell's equations Set

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0_{3,3} & -\mathcal{N} \\ \hdashline \mathcal{N}^T & \varsigma\mathcal{N}^T\mathcal{N} \end{array}\right] \qquad \text{and} \qquad \mathcal{S}_F = \left[\begin{array}{c:c} \alpha_1\mathcal{N}_F^T\mathcal{N}_F & 0_{3,3} \\ \hdashline 0_{3,3} & \alpha_2\mathcal{N}_F^T\mathcal{N}_F \end{array}\right]$$

  - $\varsigma > 0$, $\alpha_1 > 0$, and $\alpha_2 > 0$
  - penalizes tangential jumps of both H and E at interfaces
  - penalizes tangentials comp. of E at boundary faces

Second-order scalar PDE's

- Diffusion
- Advection–diffusion–reaction

Diffusion

- Let $f \in L^2(\Omega)$
- Model problem $u \in H^1_0(\Omega)$ s.t.

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall v \in H^1_0(\Omega)$$

- Discrete problem $u_h \in V_h := \mathbb{D}^k_h$ s.t.

$$a_h(u_h, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h$$

Symmetric Interior Penalty (SIP) I

- [Arnold '82]
- Broken gradient in standard Galerkin formulation ⇒

$$a_h(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h := \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h$$

- a_h is not consistent for discontinuous v_h ⇒

$$a_h(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F n_F \cdot \{\nabla_h u_h\} [\![v_h]\!]$$

Symmetric Interior Penalty (SIP) II

- a_h is not symmetric $\Rightarrow$

$$a_h(u_h,v_h) := \int_\Omega \nabla_h u_h\cdot\nabla_h v_h - \sum_{F\in\mathcal{F}_h}\int_F (n_F\cdot\{\nabla_h u_h\}[\![v_h]\!] + n_F\cdot\{\nabla_h v_h\}[\![u_h]\!])$$

- a_h is not coercive $\Rightarrow$

$$\begin{aligned} a_h(u_h,v_h) := {}& \int_\Omega \nabla_h u_h\cdot\nabla_h v_h + \sum_{F\in\mathcal{F}_h} \varpi h_F^{-1}\int_F [\![u_h]\!][\![v_h]\!] \\ & - \sum_{F\in\mathcal{F}_h}\int_F (n_F\cdot\{\nabla_h u_h\}[\![v_h]\!] + n_F\cdot\{\nabla_h v_h\}[\![u_h]\!]) \end{aligned}$$

with user-defined parameter ϖ

Symmetric Interior Penalty (SIP) III

- Let $|\!|\!| v |\!|\!|^2 := \|\nabla_h v\|_\Omega^2 + \sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![v]\!]\|_F^2$
- If parameter ϖ is large enough (depends on mesh regularity)

$$\forall v_h \in V_h, \qquad a_h(v_h, v_h) \gtrsim |\!|\!| v_h |\!|\!|^2$$

- Convergence estimate: If $u \in H^{k+1}(\mathcal{T}_h)$, $|\!|\!| u - u_h |\!|\!| \lesssim h^k$
  - optimal for broken gradient, jumps and boundary values
  - (suboptimal) $h^{k+1/2}$ estimate in L^2-norm
  - optimal h^{k+1} estimate in L^2-norm using duality argument

Variants of SIP

$$a_h(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \sum_{F \in \mathcal{F}_h} \varpi h_F^{-1} \int_F [\![u_h]\!][\![v_h]\!] \\ - \sum_{F \in \mathcal{F}_h} \int_F (n_F \cdot \{\nabla_h u_h\}[\![v_h]\!] + \theta n_F \cdot \{\nabla_h v_h\}[\![u_h]\!])$$

- $\theta = 1 \to$ SIP
- $\theta = -1 \to$ Nonsymmetric IP
  - $\varpi = 0$ [Oden, Babuška & Baumann '98] $\Rightarrow a_h(v_h, v_h) \geq \|\nabla_h v_h\|_\Omega^2$
  - $\varpi > 0$ [Rivière, Wheeler & Girault '99]: optimal error estimates for broken gradient, jumps and boundary values
  - optimal L^2-error estimates?
- $\theta = 0 \to$ Incomplete IP

Design from FS theory I

- [AE & Guermond '07]
- Mixed formulation

$$\begin{cases} \sigma + \nabla u = 0 & \text{in } \Omega \\ \nabla\cdot\sigma + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- $z = (\sigma, u)$, u : primal variable; σ : diffusive flux

Design from FS theory II

- Friedrichs' system with $m = d + 1$, $\mathcal{A}^0 = I_{d+1}$,

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0_{d,d} & e_k \\ \hdashline (e_k)^t & 0 \end{array}\right] \qquad k \in \{1, \dots, d\}$$

$(e_1, \dots, e_d)$ canonical basis of $\mathbb{R}^d$

- Boundary fields

$$\mathcal{D} = \left[\begin{array}{c:c} 0_{d,d} & n \\ \hdashline n^t & 0 \end{array}\right] \qquad \mathcal{M} = \left[\begin{array}{c:c} 0_{d,d} & -n \\ \hdashline n^t & 0 \end{array}\right]$$

Design from FS theory III

- $z_h = (\sigma_h, u_h)$, $u_h \in U_h := V_h = \mathbb{D}_h^k$, $\sigma_h \in \Sigma_h := [U_h]^d$
- DG approximation for FS

$$\mathcal{S}_F = \left[\begin{array}{c:c} \eta^\sigma n_F \otimes n_F & 0_{d,1} \\ \hdashline 0_{1,d} & \eta^u \end{array}\right] \qquad \mathcal{M}_F = \left[\begin{array}{c:c} 0_{d,d} & -n \\ \hdashline n^t & \eta^u \end{array}\right]$$

  with positive parameters η^σ and η^u
  - penalizes jumps of u_h and of normal comp. of σ_h at interfaces
  - penalizes values of u_h at boundary faces
- Convergence estimate: If $(\sigma, u) \in [H^{k+1}(\mathcal{T}_h)]^d \times H^{k+1}(\mathcal{T}_h)$,
  - optimal h^k estimate for broken gradient of primal variable, broken div. of flux, penalized jumps and boundary values
  - (suboptimal) $h^{k+1/2}$ estimate in L^2-norm for primal variable and flux

Design from FS theory IV

- Local formulation

$$\int_T [\sigma_h \cdot q - u_h \nabla\cdot q] + \sum_{F\subset\partial T} \int_F \phi^{\sigma}_{T,F}(z_h)\cdot q = 0 \qquad \forall q \in [\mathbb{P}_k(T)]^d$$

$$\int_T [-\sigma_h \cdot \nabla r] + \sum_{F\subset\partial T} \int_F \phi^{u}_{T,F}(z_h) r = \int_T fr \qquad \forall r \in \mathbb{P}_k(T)$$

with the numerical fluxes

$$\phi^{\sigma}_{T,F}(z_h) := \begin{cases} n_T(\{\!\{u_h\}\!\} + \eta^{\sigma} n_F \cdot [\![\sigma_h]\!]) & F \in \mathcal{F}_h^{\mathrm{i}} \\ 0 & F \in \mathcal{F}_h^{\partial} \end{cases}$$

and

$$\phi^{u}_{T,F}(z_h) := \begin{cases} n_T \cdot(\{\!\{\sigma_h\}\!\} + \eta^{u} n_F [\![u_h]\!]) & F \in \mathcal{F}_h^{\mathrm{i}} \\ n\cdot(\sigma_h + \eta^{u} n u_h) & F \in \mathcal{F}_h^{\partial} \end{cases}$$

Design from FS theory V

- Numerical fluxes are conservative and consistent
- The above so-called two-field approach is robust and accurate ...
- but quite expensive since it is not possible to eliminate locally the flux σ_h
- New penalty strategy ($\varpi > 0$)

$$\eta^\sigma = 0 \qquad \text{et} \qquad \eta^u = \varpi h_F^{-1}$$

Design from FS theory VI

- For $F \in \mathcal{F}_h$, $r_F : L^2(F) \ni v \longmapsto r_F(v) \in \Sigma_h$ is s.t.

$$\forall \tau_h \in \Sigma_h \qquad \int_\Omega r_F(v)\cdot\tau_h := \int_F v n_F \cdot \{\!\{\tau_h\}\!\}$$

  - r_F is vector-valued, colinear to n_F
  - support of r_F reduces to mesh element(s) of which F is a face

- Global lifting operator

$$R_h : H^1(\mathcal{T}_h) \ni u_h \longmapsto R_h(u_h) := \sum_{F \in \mathcal{F}_h} r_F([\![u_h]\!]) \in \Sigma_h$$

Design from FS theory VII

- Recall: $\forall \tau_h \in \Sigma_h$,

$$\int_\Omega (\sigma_h + \nabla_h u_h)\cdot\tau_h - \sum_{F\in\mathcal{F}_h} \int_F n_F\cdot\{\!\{\tau_h\}\!\}[\![u_h]\!] = 0$$

  which can be rewritten as

$$\int_\Omega (\sigma_h + G_h(u_h))\cdot\tau_h = 0$$

  with the discrete gradient

$$G_h(u_h) := \nabla_h u_h - R_h(u_h)$$

- $\sigma_h = -G_h(u_h)$

Design from FS theory VIII

- Recall: $\forall v_h \in U_h$,

$$\int_\Omega [-\sigma_h \cdot \nabla_h v_h] + \sum_{F \in \mathcal{F}_h} \int_F n_F \cdot \{\!\{\sigma_h\}\!\} [\![v_h]\!] + \sum_{F \in \mathcal{F}_h} \varpi h_F^{-1} \int_F [\![u_h]\!] [\![v_h]\!] = \int_\Omega f v_h$$

whence

$$\int_\Omega G_h(u_h) \cdot G_h(v_h) + \sum_{F \in \mathcal{F}_h} \varpi h_F^{-1} \int_F [\![u_h]\!] [\![v_h]\!] = \int_\Omega f v_h$$

- Local DG [Cockburn-Shu '98]

Local DG I

- LDG method ($\varpi > 0$)

$$\int_\Omega (\nabla_h u_h - R_h(u_h))\cdot(\nabla_h v_h - R_h(v_h)) + \sum_{F\in\mathcal{F}_h} \varpi h_F^{-1} \int_F [\![u_h]\!][\![v_h]\!] = \int_\Omega f v_h$$

- Convergence estimate: If $u \in H^{k+1}(\mathcal{T}_h)$,
  - $|\!|\!| u - u_h |\!|\!| \lesssim h^k$ (optimal)
  - (suboptimal) $h^{k+1/2}$ estimate in L^2-norm for primal variable
  - optimal h^{k+1} estimate in L^2-norm using duality argument

- Writing DG methods with lifting operators loses exact consistency; asymptotic consistency is preserved

Local DG II

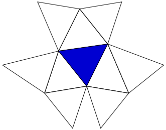
- Comparison with SIP method

$$\int_\Omega (\nabla_h u_h - R_h(u_h))\cdot(\nabla_h v_h - R_h(v_h)) + \left(\sum_{F\in\mathcal{F}_h} \varpi h_F^{-1} \int_F [\![u_h]\!][\![v_h]\!] - \int_\Omega R_h(u_h)\cdot R_h(v_h) \right) = \int_\Omega f v_h$$

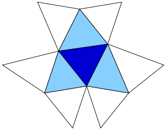
- Clear why ϖ must be large enough (depends on mesh regularity): the negative term $-\int_\Omega R_h(u_h)\cdot R_h(v_h)$ must be controlled
- The term $\int_\Omega R_h(u_h)\cdot R_h(v_h)$ widens the stencil of LDG w.r.t. SIP
  - there are other variants of LDG avoiding the wider stencil

Local DG III

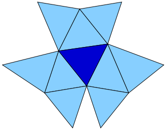
$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h$$



$$-\int_\Omega [R_h(u_h)\cdot\nabla_h v_h + \nabla_h u_h \cdot R_h(v_h)] + \sum_{F\in\mathcal{F}_h} \varpi h_F^{-1} \int_F \llbracket u_h \rrbracket \llbracket v_h \rrbracket$$



$$\int_\Omega R_h(u_h)\cdot R_h(v_h)$$



Local DG IV

- BRMPS variant [Bassi, Rebay et al. '97]

$$\int_\Omega (\nabla_h u_h - R_h(u_h))\cdot(\nabla_h v_h - R_h(v_h))$$
$$+ \left(\sum_{F\in\mathcal{F}_h} \varpi \int_\Omega r_F([\![u_h]\!])\cdot r_F([\![v_h]\!]) - \int_\Omega R_h(u_h)\cdot R_h(v_h) \right) = \int_\Omega f v_h$$

- Nice feature: simpler coercivity condition

$$\varpi > N_\partial$$

N_∂: maximum the number of faces per mesh cell

Advection–diffusion–reaction

- Model problem

$$-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f \quad \text{in } \Omega$$

  - Dirichlet BC's for simplicity
  - standard assumptions for well-posedness
  - **K** symmetric, uniformly PD, $\beta \in [L^\infty(\Omega)]^d$, $\nabla\cdot\beta \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$, and $\mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0$ a.e. in Ω

- Specific difficulties
  - heterogeneities (**K** is highly contrasted)
  - anisotropies (**K** is tensor-valued)
  - advection and/or reaction dominate

Weighted DG I

- SIP for heter. anisotropic diffusion [Houston, Schwab & Süli '02]

$$a_h(v,w) := \int_\Omega \mathbf{K}\nabla_h v\cdot\nabla_h w + \sum_{F\in\mathcal{F}_h}\int_F \gamma_{\mathbf{K},F}[\![v]\!][\![w]\!] - \sum_{F\in\mathcal{F}_h}\int_F \left(n_F^t\{\mathbf{K}\nabla_h v\}[\![w]\!] + n_F^t\{\mathbf{K}\nabla_h w\}[\![v]\!]\right)$$

- Penalty parameter ($\varpi > 0$ large enough)

$$\gamma_{\mathbf{K},F} := \varpi h_F^{-1}\{n\mathbf{K}n\}$$

Weighted DG II

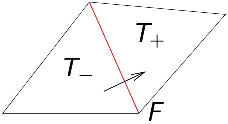
- In the presence of advection, diffusion heterogeneities can trigger internal layers
  - 1D isotropic diffusion [Gastaldi & Quarteroni '89]
  - semidefinite anisotropic diffusion [Di Pietro, AE & Guermond '08]
- As the diffusion coefficient in one of the subdomains goes to zero, penalizing the jump at this interface does not make good sense
- DG mortaring for domain decomposition [Burman & Zunino '06]
  - robustness w.r. to scalar diffusion heterogeneities needs the use of weighted averages

Weighted DG III

- Meshes fitted with the discontinuities of K
- Diffusivity dependent weights $(\omega^- + \omega^+ = 1)$

$$\omega^- := \frac{\delta_{\mathbf{K}+}}{\delta_{\mathbf{K}+} + \delta_{\mathbf{K}-}} \qquad \omega^+ := \frac{\delta_{\mathbf{K}-}}{\delta_{\mathbf{K}+} + \delta_{\mathbf{K}-}} \qquad \delta_{\mathbf{K}\mp} := n_F(\mathbf{K}|_{T\mp})n_F$$

- Weighted averages $\{\varphi\}_\omega := \omega^-\varphi^- + \omega^+\varphi^+$
- Usual average: $\omega^- = \omega^+ = \frac{1}{2}$



Weighted DG IV

- Pure diffusion
- (Symmetric) Weighted IP bilinear form (SWIP)

$$a_h(v,w) := \int_\Omega \mathbf{K}\nabla_h v \cdot \nabla_h w + \sum \int_F \gamma_{\mathbf{K},F} [\![v]\!][\![w]\!] - \sum_{F\in\mathcal{F}_h} \int_F (n_F^t \{\mathbf{K}\nabla_h v\}_\omega [\![w]\!] + n_F^t \{\mathbf{K}\nabla_h w\}_\omega [\![v]\!])$$

- Penalty parameter

$$\gamma_{\mathbf{K},F} := \varpi h_F^{-1} \left(\frac{\delta_{\mathbf{K}+}\delta_{\mathbf{K}-}}{\delta_{\mathbf{K}+}+\delta_{\mathbf{K}-}} \right)$$

$\varpi > 0$ large enough

Weighted DG V

- SWIP method with upwinding for ADR

$$\begin{aligned} a_h(v,w) := & \int_\Omega (\mathbf{K}\nabla_h v\cdot\nabla_h w + (\mu - \nabla\cdot\beta)vw - v(\beta\cdot\nabla_h w)) \\ & - \sum_{F\in\mathcal{F}_h}\int_F n_F^t\{\mathbf{K}\nabla_h v\}_\omega [\![w]\!] + n_F^t\{\mathbf{K}\nabla_h w\}_\omega [\![v]\!]) \\ & + \sum_{F\in\mathcal{F}_h}\int_F \gamma_F [\![v]\!][\![w]\!] + \sum_{F\in\mathcal{F}_h}\int_F \beta\cdot n_F\{v\}[\![w]\!] \end{aligned}$$

- Penalty parameter $\gamma_F = \gamma_{\mathbf{K},F} + \gamma_{\beta,F}$ ($\varpi > 0$ large enough)

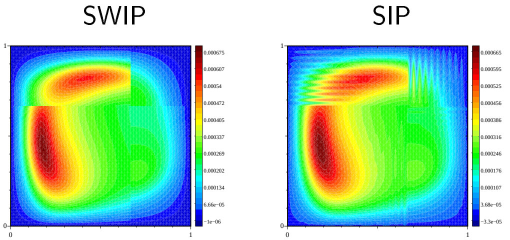
$$\gamma_{\mathbf{K},F} := \varpi h_F^{-1}\left(\frac{\delta_{K+}\delta_{K-}}{\delta_{K+}+\delta_{K-}}\right) \qquad \gamma_{\beta,F} := \frac{1}{2}|\beta\cdot n_F|$$

Weighted DG VI

- (Quasi-)optimal (in h) and robust (in $\mathbf{K}$) CV estimates
  - small, positive definite diffusion [AE, Stephansen & Zunino '08]
  - semidefinite diffusion [Di Pietro, AE & Guermond '08]

Comparison of SWIP with SIP

- Unit square divided into 4 subdomains
- Strong x-diffusion in 2 quadrants and strong y-diffusion in the others, anisotropy ratio 10^6
- Rotating advective field



- Robust DG methods for ADR with heterogeneous diffusion should use weighted averages

Incompressible flows

- For (steady) linear PDE's, the mathematical analysis of DG methods is well-understood
- For nonlinear PDE's, the situation is substantially different
    - FE-based techniques require strong regularity assumptions on the exact solution
    - the analysis of FV schemes proceeds along a different path, avoiding such assumptions [Eymard, Gallouët, Herbin et al., '00–08]
- Goal 1: extend discrete FV analysis tools to DG methods
- Goal 2: as an application, design a convergent DG method for incompressible NS equations
- [Di Pietro & AE '08, hal-00278925]

Outline

- Discrete functional analysis tools in DG spaces
- Convergence for Poisson problem
- Stokes
- Navier–Stokes

Discrete functional analysis

- Discrete Sobolev embeddings
- Compactness result for discrete gradients

DG spaces

- $V_h^k := \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_k(T)\}$ with norm

$$\|v_h\|_{\mathrm{DG}}^2 := \|\nabla_h v_h\|_{L^2(\Omega)^d}^2 + |v_h|_{\mathrm{J},\mathcal{F}_h,-1}^2$$

with broken gradient ∇_h and jump seminorm ($\mathcal{F} = \mathcal{F}_h$ or $\mathcal{F}_h^{\mathrm{i}}$)

$$|v_h|_{\mathrm{J},\mathcal{F},\pm1}^2 := \sum_{F \in \mathcal{F}} h_F^{\pm 1} \int_F |[\![v_h]\!]|^2$$

- For all $\varphi \in \mathfrak{C}_0^\infty(\Omega)$ and all $k \geq 1$,

$$\|\varphi - \pi_h^k \varphi\|_{\mathrm{DG}} \to 0 \quad \text{as } h \to 0$$

Discrete Sobolev embeddings I

- non-Hilbertian setting ($1 \leq p < +\infty$)

$$\|v_h\|_{\mathrm{DG},p}^p := \sum_{T\in\mathcal{T}_h} \int_T |\nabla v_h|_{\ell^p}^p + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![v_h]\!]|^p$$

- Main result: For all q such that
  (i) $1 \leq q \leq p^* := \frac{pd}{d-p}$ if $1 \leq p < d$;
  (ii) $1 \leq q < +\infty$ if $d \leq p < +\infty$;

  there is $\sigma_{q,p}$ such that

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^q(\Omega)} \leq \sigma_{p,q}\|v_h\|_{\mathrm{DG},p}$$

Discrete Sobolev embeddings II

- Discrete Poincaré–Friedrichs inequality ($q = 2$, $p = 2$) [Brenner '03]
- $q = 4$, $p = 2$ [Karakashian & Jureidini '98]
- Discrete Sobolev embeddings with $p = 2$ [Lasis & Süli '03]
- Two key differences
  - our technique of proof is much simpler: no elliptic regularity or nonconforming FE interpolation ⇒ general meshes can be used
  - embeddings are proven in discrete spaces, not in broken Sobolev spaces

Discrete Sobolev embeddings III

Principle of proof

- Inspired from [Eymard, Gallouët & Herbin '08]
- BV estimate $(\sum_{i=1}^{d} \sup\{\int_{\mathbb{R}^d} u\partial_i\varphi,\ \varphi \in \mathfrak{C}_0^\infty(\mathbb{R}^d),\ \|\varphi\|_{L^\infty(\mathbb{R}^d)} \le 1\})$

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{\mathrm{BV}} \lesssim \|v_h\|_{\mathrm{DG},1} \lesssim \|v_h\|_{\mathrm{DG},p} \quad (p \ge 1)$$

(v_h extended by zero outside Ω)

- Classical result ($1^* := \frac{d}{d-1}$): $\|v\|_{L^{1^*}(\mathbb{R}^d)} \le \frac{1}{2d}\|v\|_{\mathrm{BV}}$
- For $1 < p < d$, use $\|\cdot\|_{L^{1^*}(\mathbb{R}^d)}$-estimate for $|v_h|^\alpha$, Hölder's inequality and a trace inequality
- For $p \ge d$, simply use Hölder's inequality

Discrete Sobolev embeddings IV

- Main result for $p = 2$ and $d \in \{2, 3\}$: For all q such that
  - (i) $1 \leq q \leq 6$ if $d = 3$;
  - (ii) $1 \leq q < +\infty$ if $d = 2$;

  there is σ_q such that

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^q(\Omega)} \leq \sigma_q \|v_h\|_{\mathrm{DG}}$$

Compactness for discrete gradients I

- Let $l \geq 0$. For all $F \in \mathcal{F}_h$, let $r_F^l : L^2(F) \to [V_h^l]^d$ s.t.

$$\forall \tau_h \in [V_h^l]^d, \qquad \int_\Omega r_F^l(\phi){\cdot}\tau_h := \int_F \{\!\{\tau_h\}\!\}{\cdot}\nu_F \phi$$

- Let $k \geq 1$, define discrete gradient $G_h^l : V_h^k \to [V_h^{\max(k-1,l)}]^d$ as

$$\forall v_h \in V_h^k, \qquad G_h^l(v_h) := \nabla_h v_h - \sum_{F \in \mathcal{F}_h} r_F^l([\![v_h]\!])$$

- Usual values: $l = k$ or $l = k-1$
- Stability: $\forall v_h \in V_h^k$, $\|G_h^l(v_h)\|_{L^2(\Omega)^d} \lesssim \|v_h\|_{\mathrm{DG}}$

Compactness for discrete gradients II

- Sequence of meshes: $h \in \mathcal{H}$ with $h \to 0$
- Key compactness result
    - let $\{v_h\}_{h \in \mathcal{H}}$ be a sequence in V_h^k
    - bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm

  Then, there exists a subsequence of $\{v_h\}_{h \in \mathcal{H}}$ and a function $v \in H_0^1(\Omega)$ s.t. as $h \to 0$,

$$v_h \to v \quad \text{strongly in } L^2(\Omega)$$

and for all $l \geq 0$,

$$G_h^l(v_h) \rightharpoonup \nabla v \quad \text{weakly in } L^2(\Omega)^d$$

Compactness for discrete gradients III

- Proof inspired from FV analysis [Eymard, Gallouët & Herbin '08]
- Functions extended by zero outside Ω
- Uniform BV estimate on space translates

$$\|v_h(\cdot+\xi)-v_h\|_{L^1(\mathbb{R}^d)} \le |\xi|_{\ell^1}\|v_h\|_{\mathrm{BV}} \le C|\xi|_{\ell^1}$$

- Kolmogorov's Compactness Criterion in $L^1(\mathbb{R}^d)$
- Sobolev embedding: compactness in $L^2(\mathbb{R}^d)$

Compactness for discrete gradients IV

- bound on discrete gradient: $G_h^l(v_h) \rightharpoonup w$ in $L^2(\Omega)^d$
- For $\varphi \in \mathfrak{C}_0^\infty(\mathbb{R}^d)^d$,

$$\int_{\mathbb{R}^d} G_h^l(v_h)\cdot\varphi = -\int_{\mathbb{R}^d} v_h(\nabla\cdot\varphi) - \int_{\mathbb{R}^d} R_h^l([\![v_h]\!])\cdot(\varphi - \pi_h^0\varphi) + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\varphi - \pi_h^0\varphi\}\!\}\cdot\nu_F[\![v_h]\!]$$

  converges to $-\int_{\mathbb{R}^d} v(\nabla\cdot\varphi)$

- $\nabla v = w$, $v \in H^1(\mathbb{R}^d)$, and $v \equiv 0$ outside $\Omega \Rightarrow v \in H_0^1(\Omega)$.

Convergence for Poisson problem I

- Let $f \in L^r(\Omega)$ with $r \geq \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$
- $u \in H^1_0(\Omega)$ s.t. for all $v \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$$

- DG bilinear form (disc. grad. with $l = k$ or $k - 1$)

$$a_h(v_h, w_h) := \int_\Omega G_h(v_h) \cdot G_h(w_h) + j_h(v_h, w_h)$$

- BRMPS stabilization ($\varpi > N_\partial$)

$$j_h(v_h, w_h) := \sum_{F \in \mathcal{F}_h} \varpi \int_\Omega r_F([\![v_h]\!]) \cdot r_F([\![w_h]\!]) - \int_\Omega R_h([\![v_h]\!]) \cdot R_h([\![w_h]\!])$$

Convergence for Poisson problem II

- Stability result: For all $v_h \in V_h^k$,

$$\|G_h(v_h)\|^2_{L^2(\Omega)^d} + (\varpi - N_\partial) \sum_{F\in\mathcal{F}_h} \|r_F([\![v_h]\!])\|^2_{L^2(\Omega)^d} \leq a_h(v_h, v_h)$$

- Coercivity: $\exists \alpha > 0$ s.t. for all $v_h \in V_h^k$,

$$\alpha\|v_h\|^2_{\mathrm{DG}} \leq a_h(v_h, v_h)$$

Convergence for Poisson problem III

Let $\{u_h\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete Poisson problem on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $h \to 0$,

$$\begin{aligned}
u_h &\to u && \text{in } L^2(\Omega)\\
G_h(u_h) &\to \nabla u && \text{in } L^2(\Omega)^d\\
\nabla_h u_h &\to \nabla u && \text{in } L^2(\Omega)^d\\
|u_h|_{\mathrm{J},\mathcal{F}_h,-1} &\to 0
\end{aligned}$$

where $u \in H^1_0(\Omega)$ is the exact solution

Sketch of proof I

- A priori estimate:

$$\alpha \|u_h\|_{\mathrm{DG}}^2 \leq a(u_h, u_h) = \int_\Omega f u_h \leq \|f\|_{L^r(\Omega)} \|u_h\|_{L^{r'}(\Omega)}$$

  and Sobolev embedding yields

$$\|u_h\|_{\mathrm{DG}} \leq C$$

- Compactness: there exists a subsequence of $\{u_h\}_{h\in\mathcal{H}}$ and $u \in H_0^1(\Omega)$ s.t. as $h \to 0$,

$$\begin{aligned} u_h &\to u && \text{strongly in } L^2(\Omega) \\ G_h(u_h) &\rightharpoonup \nabla u && \text{weakly in } L^2(\Omega)^d \end{aligned}$$

Sketch of proof II

- Identification of the limit: For all $\varphi \in \mathfrak{C}_0^\infty(\Omega)$,

$$a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u \cdot \nabla \varphi$$

so that

$$\int_\Omega f\varphi \leftarrow \int_\Omega f\pi_h\varphi = a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u \cdot \nabla \varphi$$

- By density of $\mathfrak{C}_0^\infty(\Omega)$ in $H_0^1(\Omega)$, u solves the Poisson problem
- By uniqueness of the solution, the whole sequence converges

Sketch of proof III

- Owing to weak convergence

$$\liminf \|G_h(u_h)\|^2_{L^2(\Omega)^d} \geq \|\nabla u\|^2_{L^2(\Omega)^d}$$

- Owing to stability

$$\|G_h(u_h)\|^2_{L^2(\Omega)^d} \leq a_h(u_h, u_h) = \int_\Omega f u_h$$

so that

$$\limsup \|G_h(u_h)\|^2_{L^2(\Omega)^d} \leq \limsup \int_\Omega f u_h = \int_\Omega f u = \|\nabla u\|^2_{L^2(\Omega)^d}$$

- Hence, $\|G_h(u_h)\|_{L^2(\Omega)^d} \rightarrow \|\nabla u\|_{L^2(\Omega)^d}$ so that $G_h(u_h)$ strongly converges to ∇u in $L^2(\Omega)^d$

Sketch of proof IV

- Owing to stability

$$(\varpi - N_\partial) \sum_{F \in \mathcal{F}_h} \|r_F([\![u_h]\!])\|^2_{L^2(\Omega)^d} \leq a_h(u_h, u_h) - \|G_h(u_h)\|^2_{L^2(\Omega)^d}$$

- Hence, $|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0$

Remark. If the exact solution is smooth, the usual optimal a priori error estimate is recovered

$$\|u - u_h\|_{\mathrm{DG}} \lesssim h^k$$

Nonsymmetric variants I

- Nonsymmetric DG bilinear form

$$a_h(v_h, w_h) := \int_\Omega \widehat{G}_h(v_h)\cdot G_h(w_h) + j_h'(v_h, w_h)$$

- Design conditions
  - $\widehat{G}_h$ strongly consistent for smooth functions
  - G_h weakly consistent for discrete functions
  - both gradients controlled by $\|\cdot\|_{\mathrm{DG}}$-norm
  - j_h' symmetric, nonnegative, controlled by jump seminorm and ensuring coercivity of a_h

Nonsymmetric variants II

- General convergence result can be proven as before
- Examples of nonsymmetric methods

$$G_h(v_h) := \nabla_h v_h + R_h([\![v_h]\!]) \qquad \text{(NIP)}$$
$$G_h(v_h) := \nabla_h v_h \qquad \text{(IIP)}$$

Stokes

- Let $f \in L^r(\Omega)^d$ with $r \geq \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$
- Let $\nu > 0$
- $(u, p) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ s.t. for all $(v, q) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$,

$$\nu \int_\Omega \nabla u \cdot \nabla v - \int_\Omega p \nabla\cdot v + \int_\Omega q \nabla\cdot u = \int_\Omega f \cdot v$$

- Equal-order polynomial spaces for velocity and pressure

$$U_h := [V_h^k]^d \qquad P_h := V_h^k \qquad X_h := U_h \times P_h$$

- Pressure stabilization $s_h(q_h, r_h) := \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} h_F \int_F [\![q_h]\!][\![r_h]\!]$

Pressure–velocity coupling

- Discrete divergence operator

$$\forall v_h \in U_h, \qquad D_h^l(v_h) := G_h^l(v_{h,j})\cdot e_j$$

- Pressure–velocity bilinear form

$$b_h(v_h, q_h) := -\int_\Omega q_h D_h^k(v_h)$$

- $(u_h, p_h) \in X_h$ s.t. $l_h((u_h, p_h), (v_h, q_h)) = \int_\Omega f\cdot v_h$, $\forall (v_h, q_h) \in X_h$ where

$$l_h((u_h, p_h), (v_h, q_h)) := \nu a_h(u_{h,i}, u_{h,i}) + b_h(v_h, p_h) - b_h(u_h, q_h) + s_h(p_h, q_h)$$

Convergence result

Let $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete Stokes problems on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $h \to 0$,

$$\begin{aligned} u_h &\to u && \text{in } L^2(\Omega)^d \\ \nabla_h u_h &\to \nabla u && \text{in } L^2(\Omega)^{d,d} \\ |u_h|_{\mathrm{J},\mathcal{F}_h,-1} &\to 0 && \\ p_h &\to p && \text{in } L^2(\Omega) \\ |p_h|_{\mathrm{J},\mathcal{F}_h^{\mathrm{i}},1} &\to 0 && \end{aligned}$$

where $(u,p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is the exact Stokes solution

Sketch of proof

- Coercivity on velocity and discrete inf-sup condition on pressure
- A priori estimate + compactness: $u_h \to u$ strongly in $L^2(\Omega)^d$, $G_h(u_{h,i}) \rightharpoonup \nabla u_i$ weakly in $L^2(\Omega)^d$ and $p_h \rightharpoonup p$ weakly in $L^2(\Omega)$
- Identification of the limit and convergence of the whole sequence
- Strong convergence of velocity gradient and jumps (as before)
- Strong convergence of the pressure using Nečas velocity

Remark. If the exact solution is smooth, the usual optimal a priori error estimates are recovered [Cockburn, Kanschat, Schötzau & Schwab '02, AE & Guermond '08]

$$\|u - u_h\|_{\mathrm{DG}} + \|p - p_h\|_{L^2(\Omega)} + |p_h|_{\mathrm{J},\mathcal{F}_h^{\mathrm{i}},1} \lesssim h^k$$

Navier–Stokes I

- Let $f \in L^r(\Omega)^d$ with $r \geq \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$
- Let $\nu > 0$
- $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ s.t. for all $(v, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\nu \int_\Omega \nabla u \cdot \nabla v + \int_\Omega v \cdot (\nabla \cdot F(u, p)) + \int_\Omega q \nabla \cdot u = \int_\Omega f \cdot v$$

  with incompressible Euler flux $F(u, p) := u \otimes u + pI$
- Existence of such a weak solution holds for $d \in \{2, 3\}$
- Uniqueness under small data assumption

Navier–Stokes II

- For all $u \in H^1_0(\Omega)^d$,

$$\int_\Omega u\cdot\nabla\cdot(u \otimes u) = \int_\Omega u\cdot(\tfrac{1}{2}(\nabla\cdot u)u) = -\int_\Omega u\cdot\nabla(\tfrac{1}{2}|u|^2)$$

- Temam's device for stability: add source term $-\int_\Omega \frac{1}{2}(\nabla\cdot u)u$
    - non-conservative form
    - source term vanishes at the limit for solenoidal velocity
- Modified Euler flux $\Phi(u,\overline{p}) := u \otimes u + \frac{1}{2}|u|^2 I + \overline{p} I$ with $\overline{p} := p - \frac{1}{2}|u|^2$
    - conservative form
    - hinted to in [Cockburn, Kanschat & Schötzau '05]

Discrete NS system I

- DG methods for incompressible NS
  - piecewise solenoidal velocity fields [Karakashian & Jureidini '98]
  - nonconservative method based on Temam's device [Girault, Rivière & Wheeler '04]
  - conservative LDG method [Cockburn, Kanschat & Schötzau '05] using BDM projection
- FV methods for incompressible NS
  - nonconservative form [Eymard, Herbin & Latché '07]
  - conservative form [Chénier, Eymard & Herbin '08]

Discrete NS system II

- $(u_h, p_h) \in X_h$ s.t. $\forall (v_h, q_h) \in X_h$,

$$l_h((u_h, p_h), (v_h, q_h)) + t_h(u_h, u_h, v_h) = \int_\Omega f \cdot v_h$$

  with Stokes bilinear form l_h and discrete trilinear form t_h

- Design conditions on t_h
    - Stability: $t_h(v_h, v_h, v_h) = 0$, $\forall v_h \in U_h$
    - Continuity on discrete space
    - Weak continuity: $t_h(u_h, u_h, \pi_h \varphi) \to t(u, u, \varphi)$

- Existence of discrete solution using topological degree argument (no small data assumption)

Convergence result for NS

Let $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ be a sequence of approximate solutions generated by solving the discrete NS problems on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $h \to 0$, up to a subsequence

$$\begin{aligned} u_h &\to u && \text{in } L^2(\Omega)^d \\ \nabla_h u_h &\to \nabla u && \text{in } L^2(\Omega)^{d,d} \\ |u_h|_{\mathrm{J},\mathcal{F}_h,-1} &\to 0 && \\ p_h &\to p && \text{in } L^2(\Omega) \\ |p_h|_{\mathrm{J},\mathcal{F}_h^{\mathrm{i}},1} &\to 0 && \end{aligned}$$

where $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is an exact solution

Examples of DG trilinear forms

- Non-conservative, based on Temam's device

$$t_h(w,u,v) := \int_\Omega (w\cdot\nabla_h u)\cdot v - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F \{\!\{w\}\!\}\cdot n_F [\![u]\!]\cdot\{\!\{v\}\!\}$$
$$+ \int_\Omega \tfrac{1}{2}\nabla_h\cdot w(u\cdot v) - \sum_{F\in\mathcal{F}_h} \int_F [\![w]\!]\cdot n_F \tfrac{1}{2}\{\!\{u\cdot v\}\!\}$$

- Conservative, based on Euler flux modification

$$t_h(w,u,v) := -\int_\Omega (w\otimes u):\nabla_h v + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F n_F\cdot\{\!\{u\}\!\}\{\!\{w\}\!\}\cdot[\![v]\!]$$
$$+ \int_\Omega \tfrac{1}{2} v\cdot\nabla_h(u\cdot w) - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F n_F\cdot\{\!\{v\}\!\}\tfrac{1}{2}[\![u\cdot w]\!]$$

Concluding remarks

- Uniqueness of discrete solution under small data assumption
- Upwinding of convective term
- Optimal a priori error analysis under strong regularity assumptions
- Confirmed by numerical tests on standard benchmark problems with moderate Reynolds (≤ 100)
- For higher Reynolds numbers, the artifical compressibility method of [Bassi, Di Pietro & Rebay '07], yet to be analyzed mathematically, yields better CV of nonlinear solver