
Skeleton and Level Set for tubing.

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Plan

Geometry of tubing:

- Description with Level Set

- Description with Skeleton

- From 2D to 3D tubing

- Perspective with Skeleton: Blood vessel reconstruction

Level Set for flow simulation:

- Geometry with Level Set

- Multifluid with Level Set

Flow model

- Navier-Stokes bifluid with surface tension

Applications:

- Flows in tubing

- Flows water/air

Geometry of tubing

Tubing is easily defined by the axes of tubes and diameters.

The Skeleton is the simpler mathematical tool to define tubing.

Definition: For $\Omega \subset \mathbb{R}^d$ bounded with boundary $\partial\Omega$, the skeleton of Ω , S , is the smaller set of point of Ω such that

$$\Omega = \cup_{x \in S} B(x, r(x)),$$

for appropriate radius $r(x)$.

Remark: $\overline{B}(x, r(x)) \cap \partial\Omega$ contains at least 2 points.

Definition: Level Set of signed distance function to $\partial\Omega$:

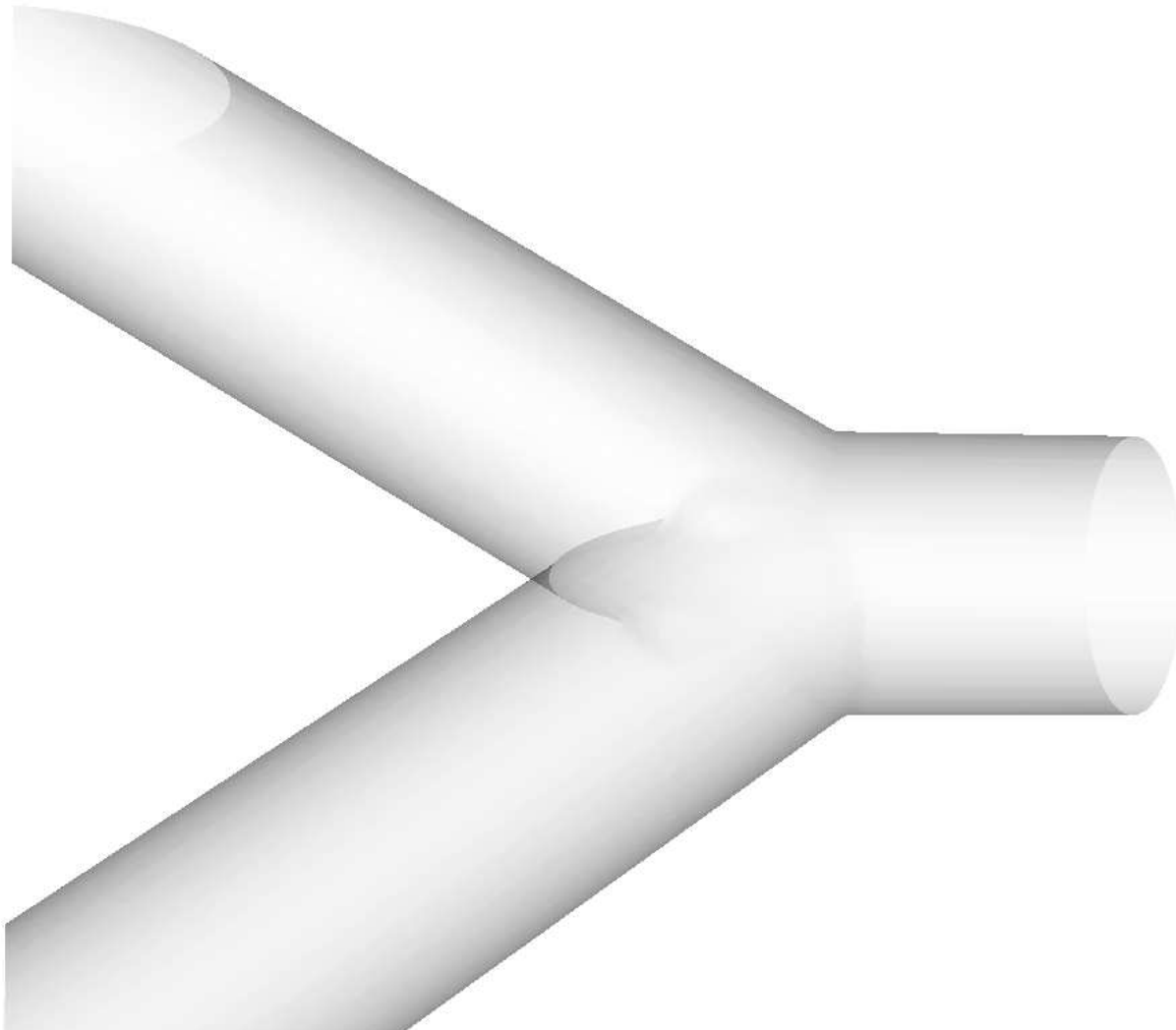
$$y \in \mathbb{R}^d, \psi(y) = \inf_{x \in S} \{\|x - y\| - r(x)\}.$$

ψ is negative inside Ω .

Remark: $\nabla\psi$ is singular on S .

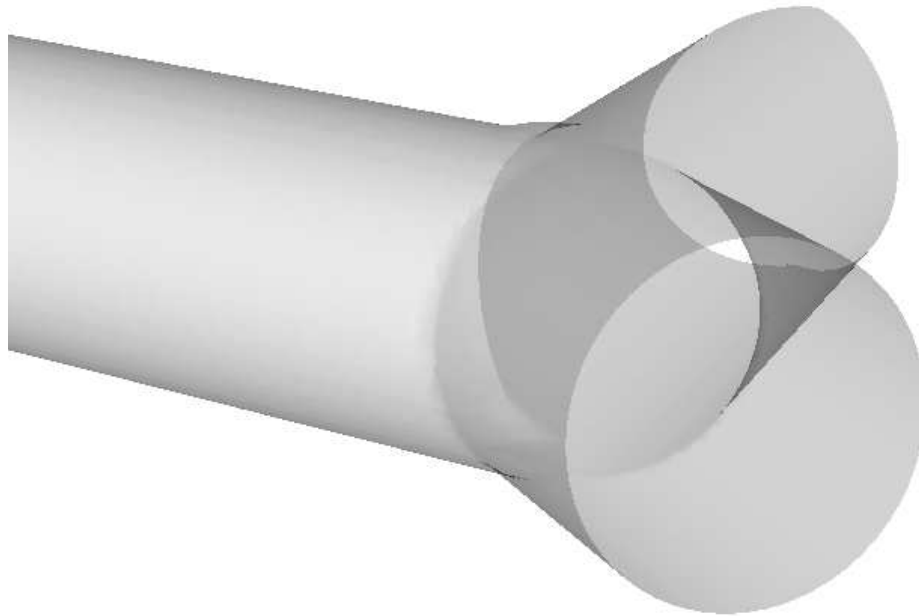
Skeleton for tubing

Define 3 segments forming a "Y" with given radius associated to each point : construct the Level Set function ψ :



Skeleton for tubing

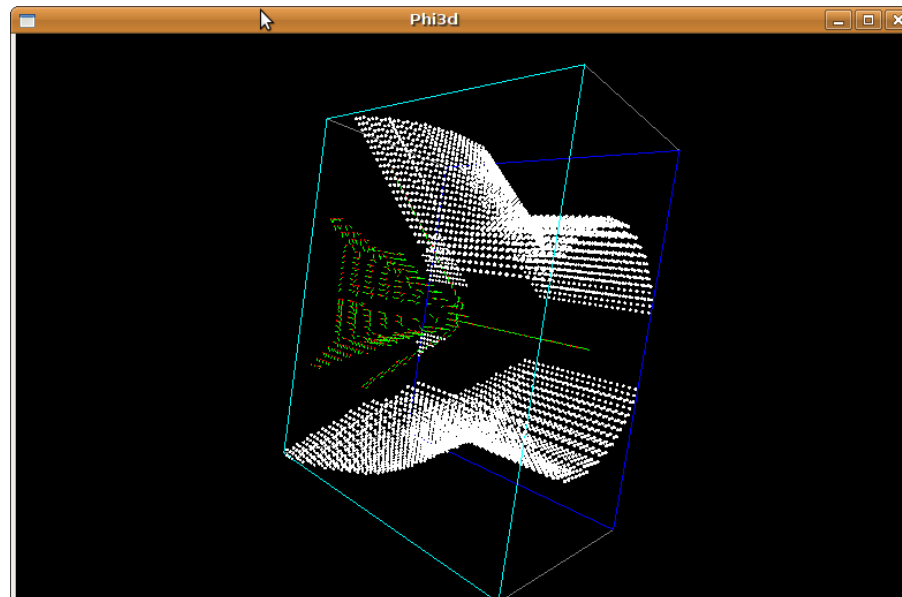
Define 3 segments forming a "Y" with given radius associated to each point : construct the Level Set function ψ :



Skeleton for 3D tubing from 2D view

- Give a planar view of a 2D/3D geometry (Image)
- Compute the Level Set signed distance function of the 2D geometry ψ
- Generate the 2D Skeleton as the set of non-smooth point of ψ .
- Construct the 3D Level Set function with the 2D Skeleton.

(being done by C. Nguyen (USTV)).



The Super-Skeleton

The skeleton: $\cup_{x \in S} B(x, r(x))$

The Super-Skeleton: an extension of skeleton

position: $x \in \mathbb{R}^3$

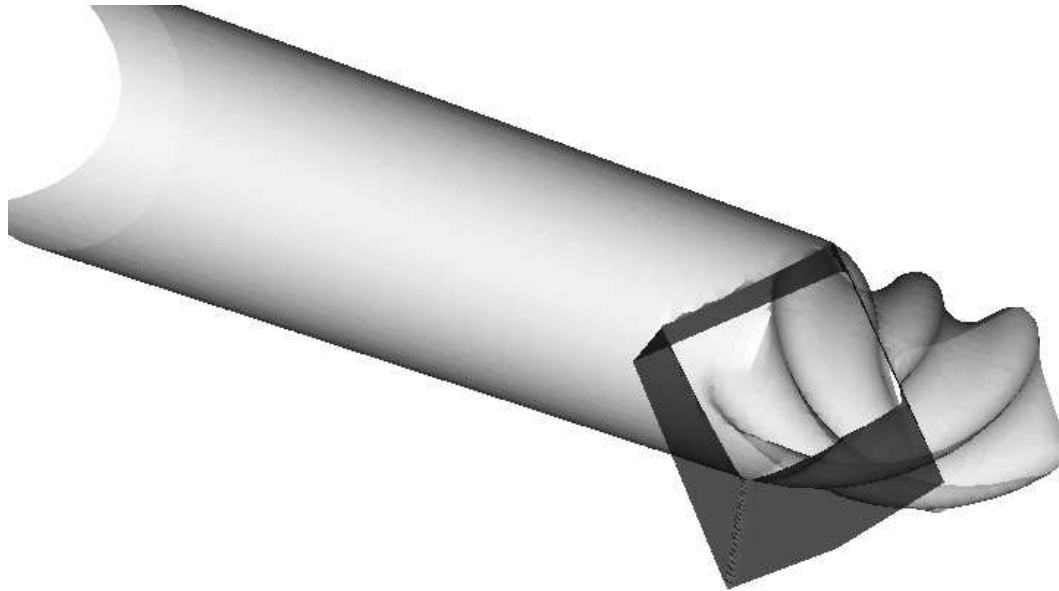
radius: $r(x) \in \mathbb{R}^+$

flow rate: $d(x) \in \mathbb{R}$ (useful for flow boundary conditions)

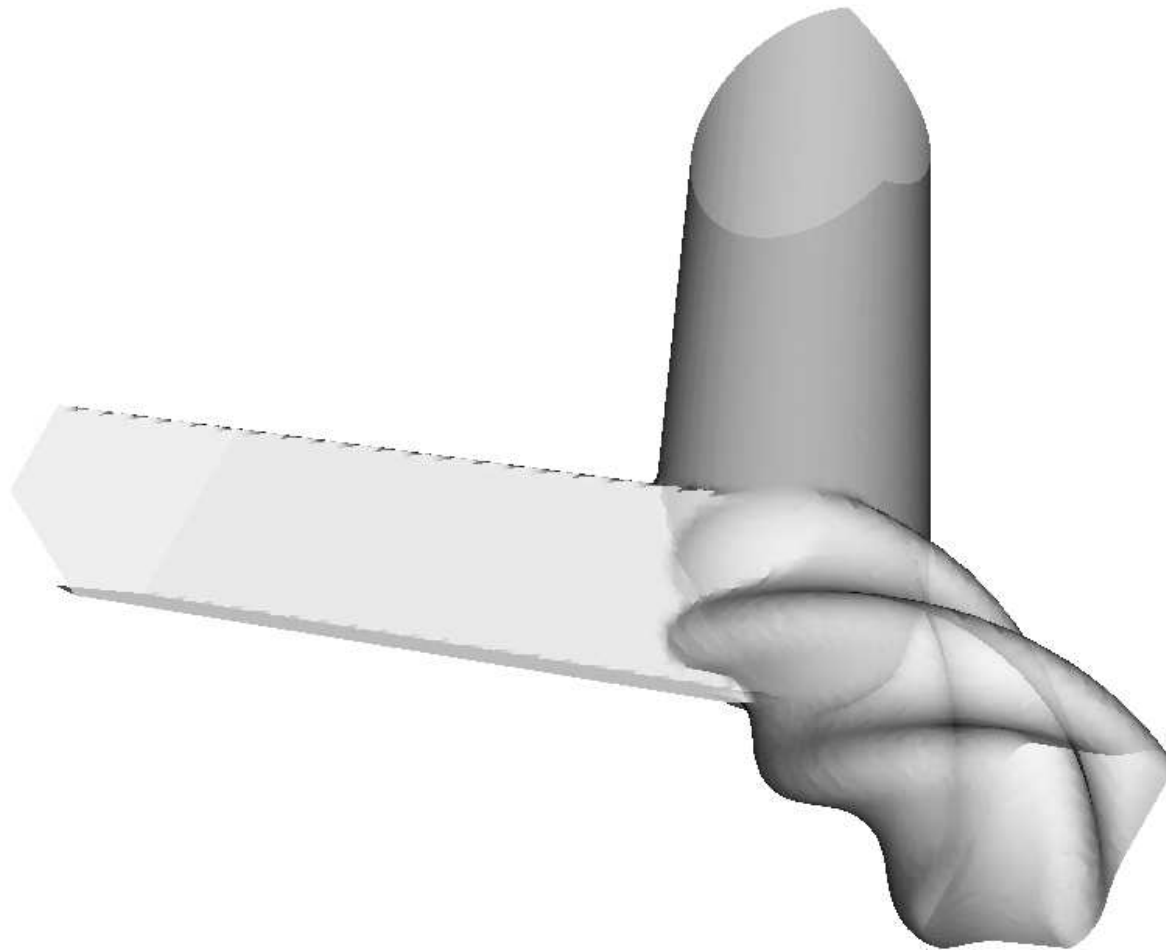
direction: $ds(x) \in \mathbb{R}^3$

transverse direction: $dst(x) \in \mathbb{R}^3$

choice of L^p norm: $p(x) \geq 1$



The Super-Skeleton



The Super-Skeleton

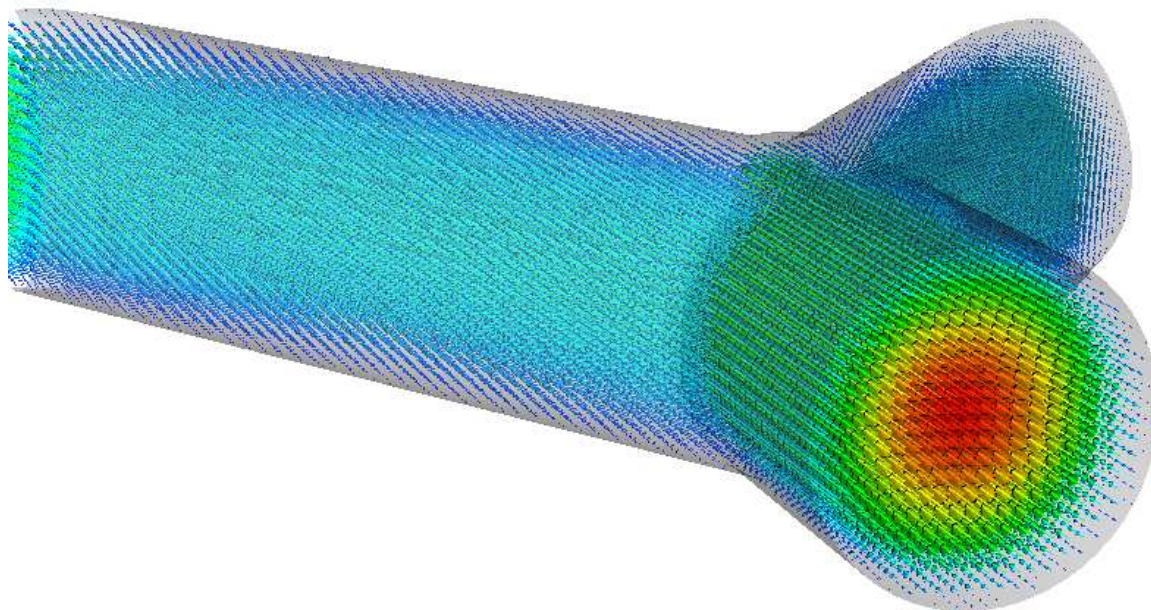
A natural flow associated to the skeleton S and the function ψ

The "Poiseuille like" velocities in tubing :

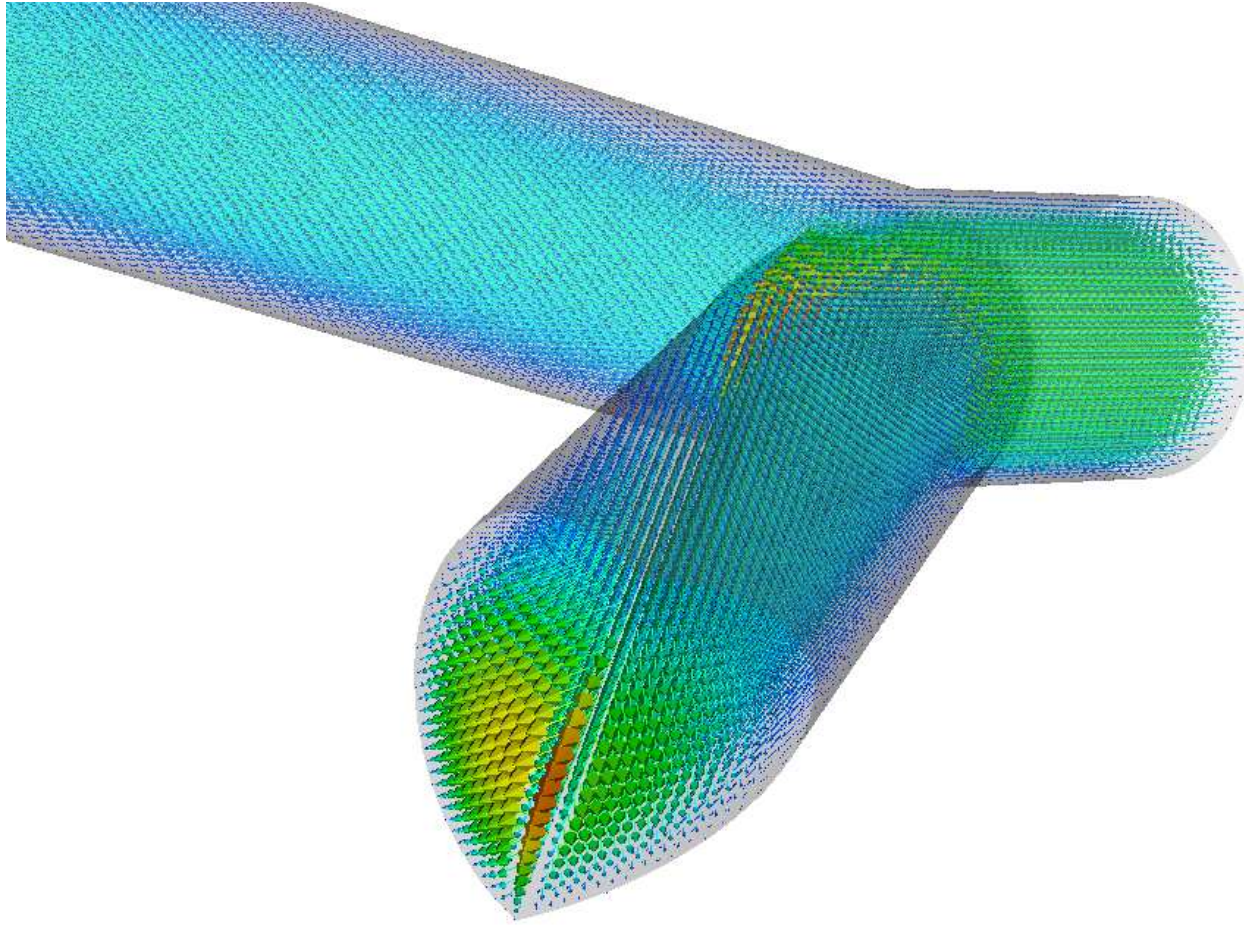
$$y \in \mathbb{R}^3, \text{ associated to } x \in S, V(y) = \alpha(\psi)\nabla\psi + \beta(\psi)ds(x),$$

$$\lim_{\psi \rightarrow 0} V(y) \cdot \nabla\psi = 0, \quad \lim_{\psi \rightarrow -r(x)} V(y) = \beta(-r(x))ds(x)$$

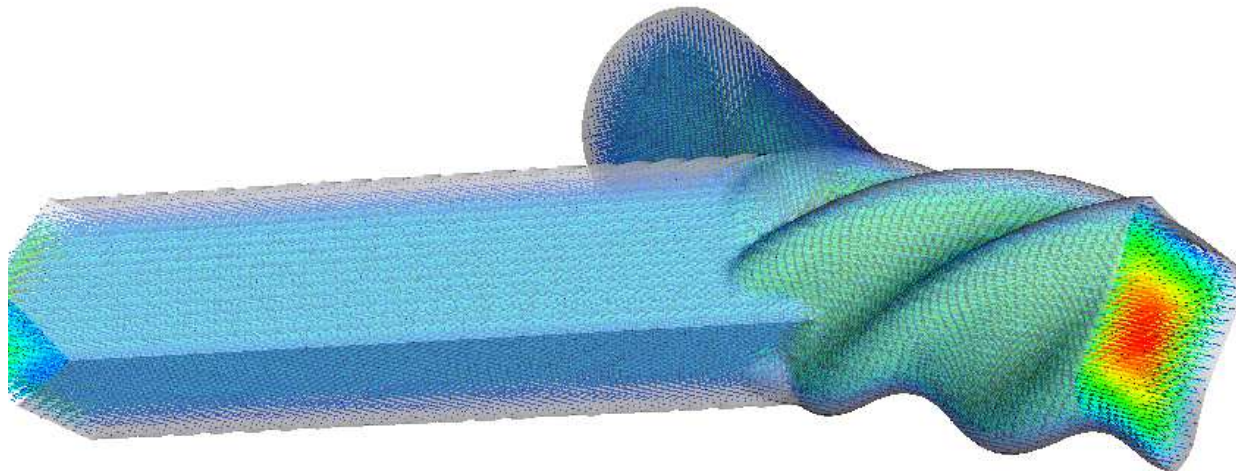
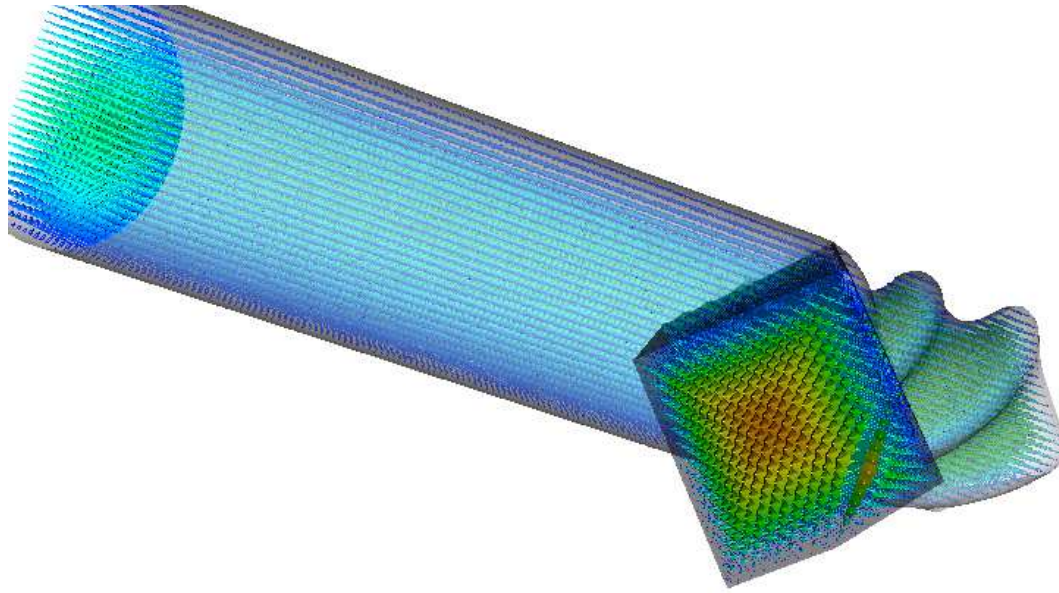
The flow is adapted to the flow rate $d(x)$ with parabolic profile depending on ψ .



The Super-Skeleton for velocities



The Super-Skeleton for velocities



Perspective for blood vessel reconstruction

A **Super-Skeleton** can describe blood vessels

A **PhD** supported by ANR Carpeinter (2009-2013) :

Image processing:

- Define region on cuts corresponding to vessels

- Associate flow rate to each cut

- Distinguish "in-flow" and "out-flow"

Define the mass transfer problem to connect "in-flow" and "out-flow"

First formulation:

- Direct formulation of a "Monge-Kantorovich" type problem

- Cuts of vessel are seen as points with mass corresponding to the flow rate.

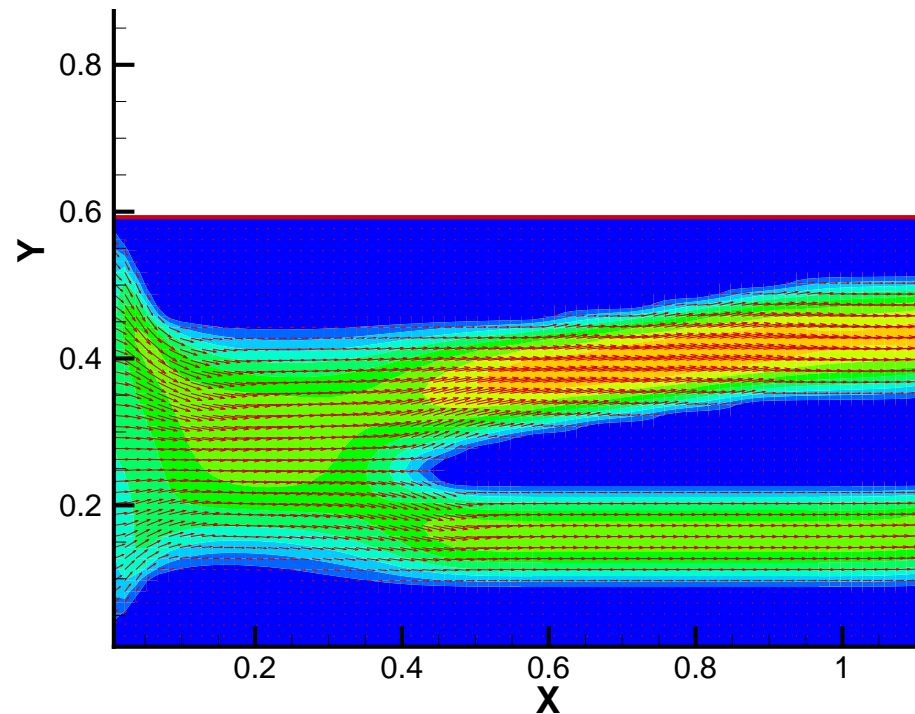
- The solution defines the Skeleton of the vessels.

- The radius of vessels depend on flow rates.

Perspective for blood vessel reconstruction

Second formulation:

An **incompressible fluid solver** in a porous media with permeability depending on the velocity (the pressure gradient is parallel to velocity with a modulus smaller than one)



Next steps:

Compute the Navier-Stokes flows with the level set geometry
Optimize the geometry of vessels to minimize drag.

Modelization

Notation: \vec{u}_i , p_i , ν_i , ρ_i : velocity, pressure and density of fluids i , $i = 1, 2$.

For each phase (i), **Navier-Stokes** incompressible equations:

$$\rho_i \frac{D}{Dt} \vec{u}_i - 2\nu_i \operatorname{div}(\mathbf{D}_i) + \nabla p_i = 0,$$

$$\operatorname{div} \vec{u}_i = 0,$$

$$\frac{D}{Dt} \vec{u} = \frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \nabla \vec{u},$$

$$\mathbf{D}_i = \frac{1}{2} (\nabla \vec{u}_i + (\nabla \vec{u}_i)^t).$$

Continuity on the fluid interface Γ :

$$\vec{u}_1 = \vec{u}_2$$

$$\nu_1 \mathbf{D}_1 n - \nu_2 \mathbf{D}_2 n = (p_1 - p_2 + \sigma \kappa) n.$$

κ : mean curvature, $\sigma \kappa n$: tension surface forces.

Modelization

Monofluid formulation (Sussman *et al.* (1994)):

$$\vec{u} = \vec{u}_1 \text{ or } \vec{u}_2, p = p_1 \text{ or } p_2, \nu = \nu_1 \text{ or } \nu_2, \rho = \rho_1 \text{ or } \rho_2,$$

$$\rho \frac{D}{Dt} \vec{u} - 2 \operatorname{div}(\nu \mathbf{D}) + \nabla p = -\sigma \kappa \delta_{\Gamma} n,$$

$$\operatorname{div} \vec{u} = 0.$$

Interface tracking: Level set.

Let $\varphi > 0$ where the fluid 1 is.

Interface Γ : Level set 0 of φ .

viscosity: $\nu = \nu(\varphi)$, density: $\rho = \rho(\varphi)$

Choose φ such that $|\nabla \varphi| \sim 1$.

Interface displacement: transport equation

$$\frac{\partial}{\partial t} \varphi + \vec{u} \cdot \nabla \varphi = 0.$$

Redistanciation: sometimes solve

$$|\nabla \varphi| = 1.$$

with fixed level set $\varphi = 0$.

Discretization

Time Discretization

$$\rho(\varphi^n) \frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} - \operatorname{div}(\nu(\varphi^n)(\nabla \vec{u}^{n+1} + (\nabla \vec{u}^{n+1})^t)) + \rho(\varphi^n) \vec{u}^n \cdot \nabla \vec{u}^n + \nabla p^{n+1} = -\sigma \kappa^n \nabla(H(\varphi^n)),$$

$$\operatorname{div} \vec{u}^{n+1} = 0,$$

$$\kappa^n = \operatorname{div} \frac{\nabla \varphi^n}{|\nabla \varphi^n|},$$

$$\frac{\varphi^{n+1} - \varphi^n}{\Delta t} + \vec{u}^{n+1} \cdot \nabla \varphi^n = 0.$$

Augmented Lagrangian for incompressibility.

Space Discretization

Cartesian uniform grid on a box containing the domain.

Domain defined by the Level Set function $\psi < 0$.

Conclusion:

Geometry with Level Set.

Multifluid with Level Set.

Boundary condition

The geometry is approached by a cartesian grid.

No slip condition can be forced on the grid where $\psi < 0$.

How to cancel the grid effect?

No slip condition on the boundary can be replaced by a slip condition (Robin condition) so that

$$\vec{u} = 0 \text{ where } \psi = 0.$$

(See F. Chantalat et al., under review).

For $\psi \geq 0$, find \vec{u} so that

$$\vec{u} = \vec{u}_{ref} \text{ or } \vec{u} = \psi \nabla \psi \cdot \nabla \vec{u}$$

(\vec{u} is a linear extension outside the domain following the normal derivative).

$$\begin{aligned} \rho(\varphi^n) \frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} - \text{div}(\nu(\varphi^n)(2D\vec{u}^{n+1})) + \rho(\varphi^n) \vec{u}^n \cdot \nabla \vec{u}^n \\ + \frac{1}{\varepsilon} H(-\psi)(\vec{u}^{n+1} - \vec{u}_{ref}) + \nabla p^{n+1} = -\sigma \kappa^n \nabla(H(\varphi^n)), \end{aligned}$$

$$\text{div } \vec{u}^{n+1} = H(-\psi) \text{div } \vec{u}_{ref},$$

Boundary condition

Validation on a Laplace problem:

$$\Delta u = f \text{ on a disk}$$

$$u = 0 \text{ on the boundary}$$

The Robin condition $\vec{u} = \psi \nabla \psi \cdot \nabla \vec{u}$ is obtained by an iterative process.

Iteration 0: homogenous Dirichlet boundary condition.

The normal derivative $\vec{\alpha}_0 = \nabla \psi \cdot \nabla u_0$ is computed inside the domain and extended outside.

Iteration 1: inhomogenous Dirichlet boundary condition:

$$\vec{u}_1 = \psi \vec{\alpha}_0 \text{ where } \psi < 0, \vec{\alpha}_1 = \nabla \psi \cdot \nabla u_1.$$

Iteration 2: inhomogenous Dirichlet boundary condition:

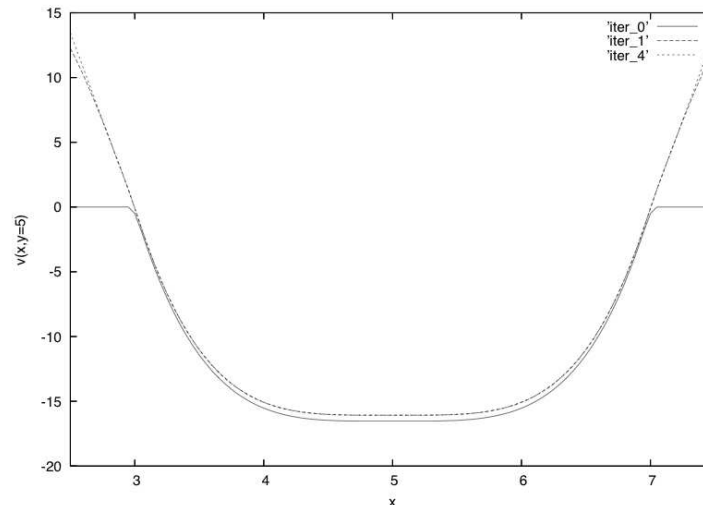
$$\vec{u}_2 = \psi \vec{\alpha}_1 \text{ where } \psi < 0.$$

Boundary condition improvement

Error on the normal derivative:

Mesh Resolution	iteration 0	iteration 1	iteration 2	iteration 3
101 x 101	0.2456	0.1259	0.1254	0.1254
201 x 201	0.2109	0.0679	0.0656	0.0655
401 x 401	0.2118	0.0369	0.0341	0.0339
801 x 801	0.2004	0.0207	0.0183	0.0181

The scheme is close to be of order 2 near the boundary (order 1 on derivative).



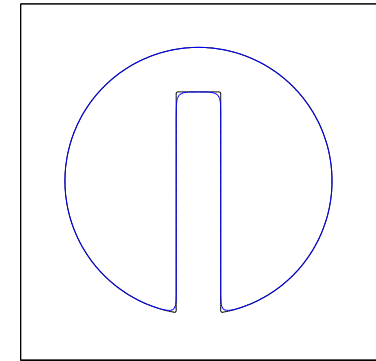
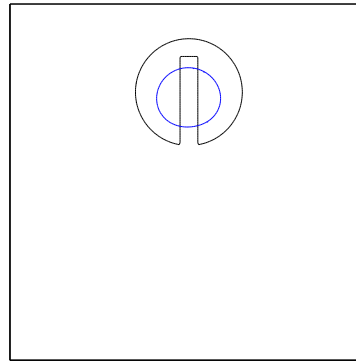
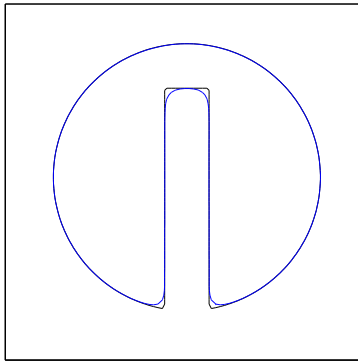
Conclusion: Drawbacks of cartesian mesh can be overcome!

Numerical cost

Fluid solver is costly

Choose a Level set transport on grid 3 times thinner than for the fluid solver:

- Interpolate the velocity on the fine grid
- less expensive than fluid solver.
- WENO5 scheme for transport of smooth function is efficient on fine grid.
- Alternative to Particle Level set method.



Zalesak 200*200 WENO5/ordre1, Zalesak 400*400 WENO5

Conclusion: no precision loss on interface tracking (with [mesh refinement](#)) compare to loss on fluid solver.

Numerical stability

Proposition 1. (C.G., P. Vigneaux 08) For low Reynolds, the above numerical scheme is stable under the condition:

$$\Delta t \leq \min(\Delta t_c, \Delta t_\sigma), \text{ avec } \Delta t_c = c_0 \|\vec{u}\|_{L^\infty(\Omega)}^{-1} \Delta x \text{ et}$$
$$\Delta t_\sigma = \frac{1}{2} \left(c_2 \frac{\eta}{\sigma} \Delta x + \sqrt{\left(c_2 \frac{\eta}{\sigma} \Delta x\right)^2 + 4c_1 \frac{\rho}{\sigma} \Delta x^3} \right) \quad (1)$$

where Δt is the time step, Δx the space time and c_0, c_1, c_2 do not depend on physical and numerical parameter.

Known time step:

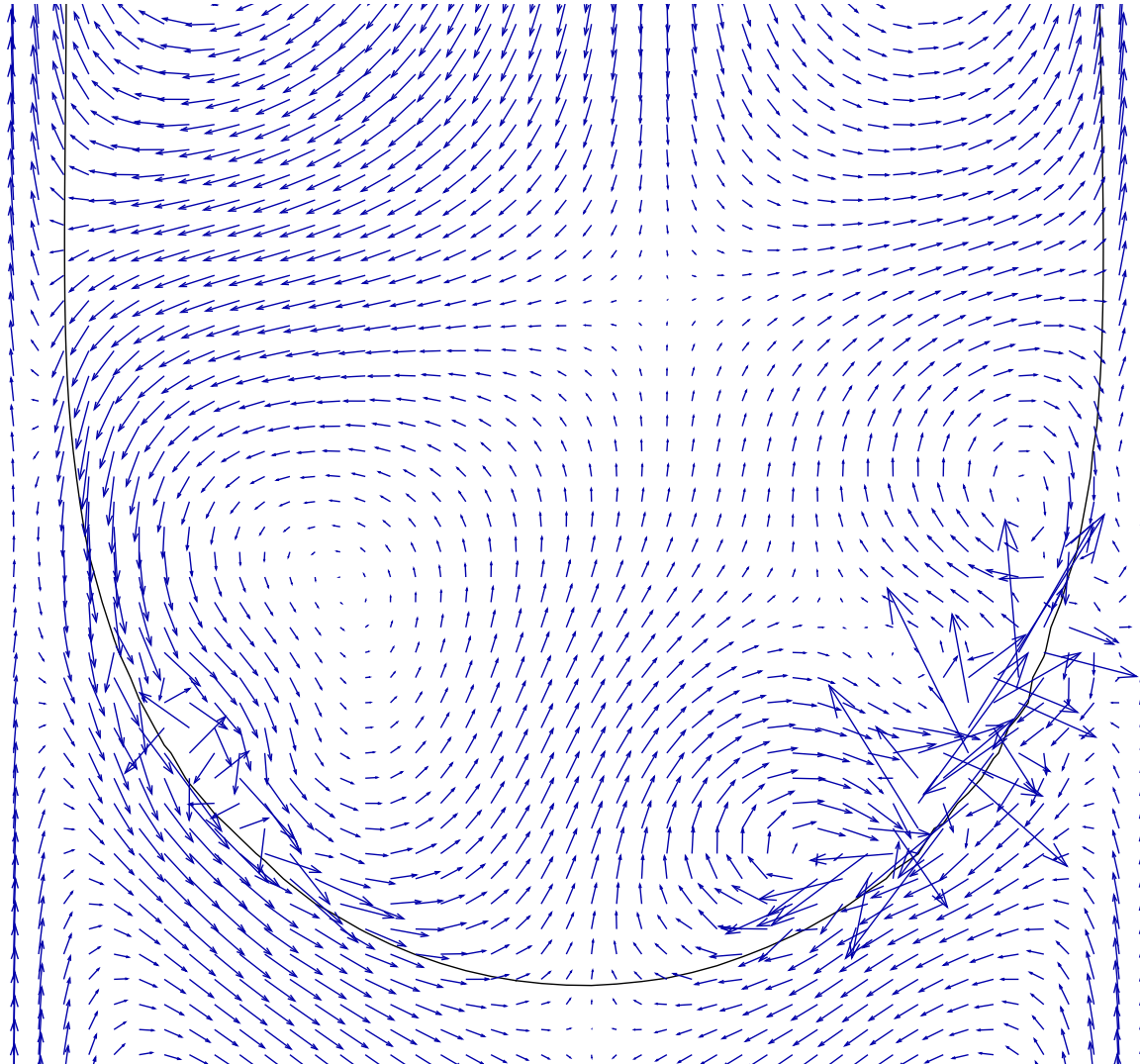
Brackbill (BKZ) capillary time step and Capillary time step for Stokes

$$\Delta t_{BKZ} = c_1 \sqrt{\frac{\rho}{\sigma} \Delta x^3} = \Delta t_\sigma(\rho, 0), \quad \Delta t_{STK} = c_2 \frac{\eta}{\sigma} \Delta x = \Delta t_\sigma(0, \eta).$$

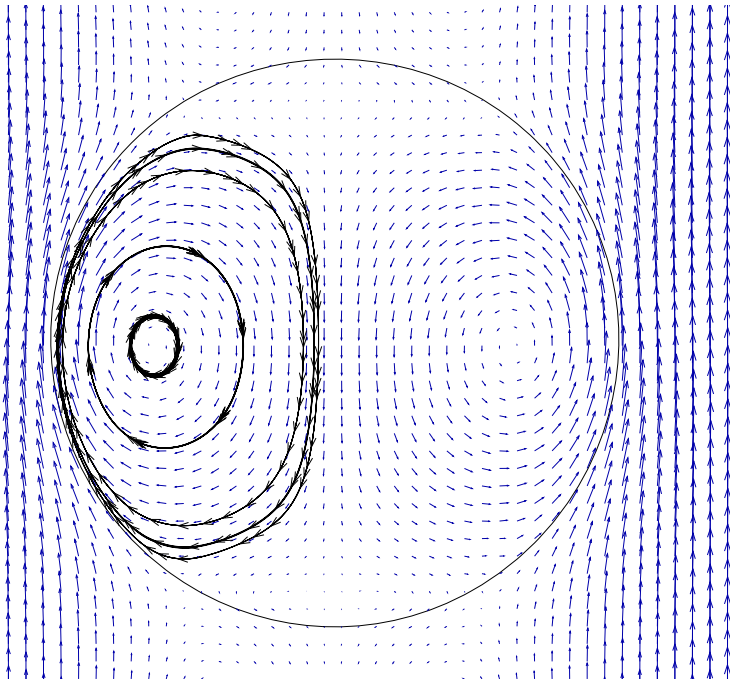
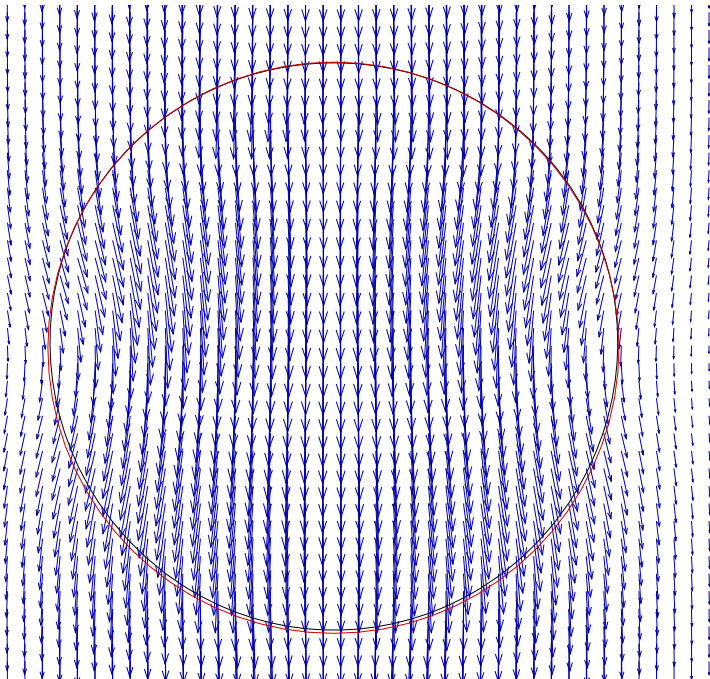
Remark:

$$\Delta t_\sigma \leq \frac{1 + \sqrt{5}}{2} \max(\Delta t_{STK}, \Delta t_{BKZ}). \quad (2)$$

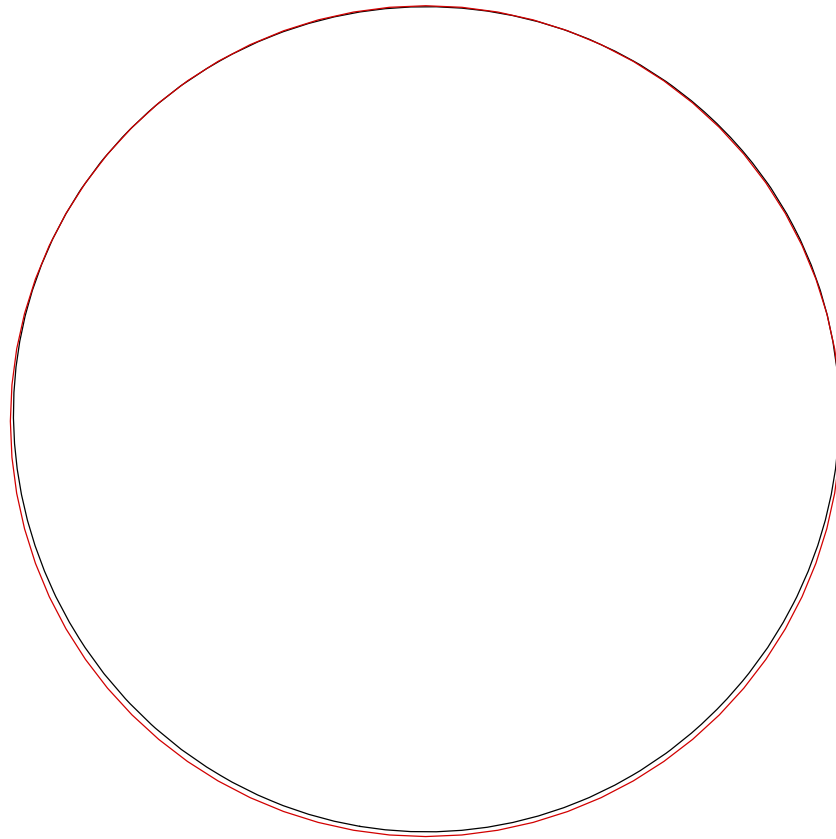
Numerical instability example



Stable flow

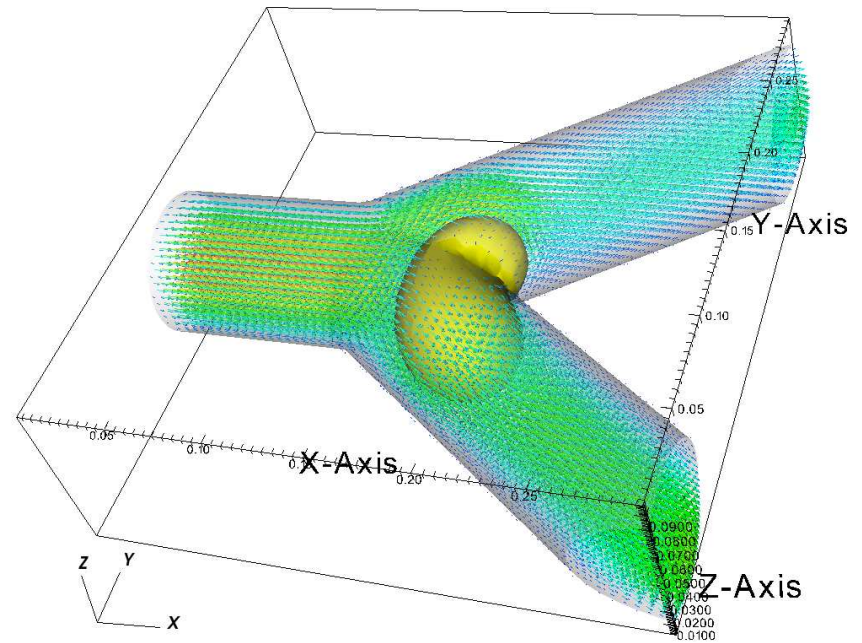
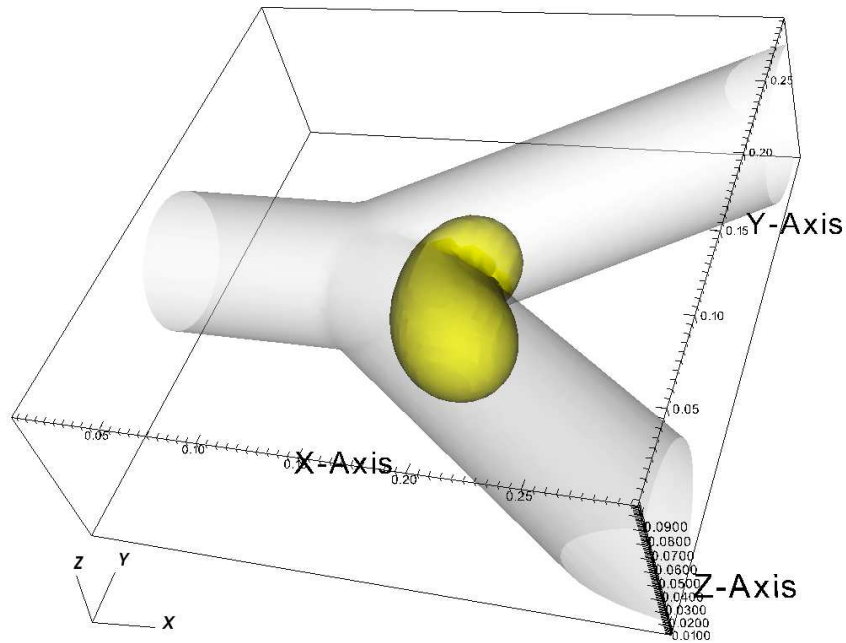


Low flow with surface tension



comparison with a circle

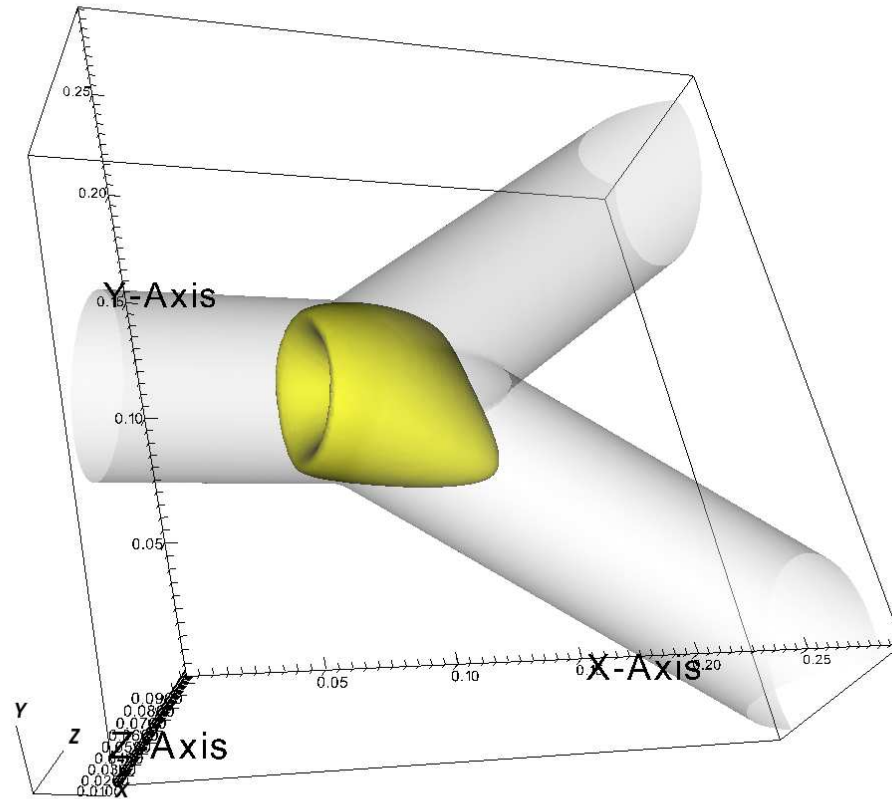
Application: bifluid flows in tubing



Bifluid flows with low velocities in tubing.

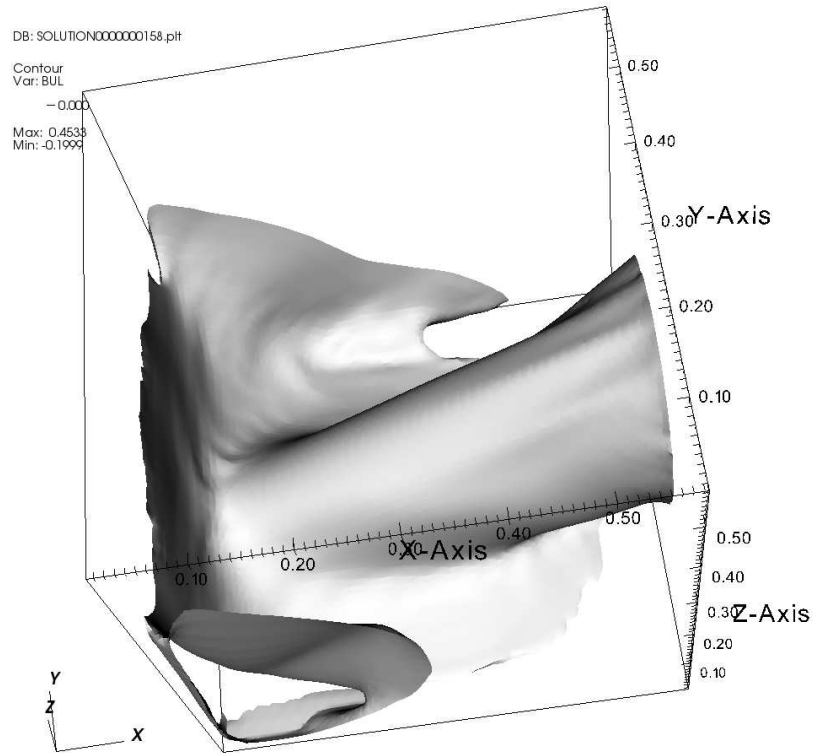
Stokes model corresponding to low dimension (microfluidic).

Application: bifluid flows in tubing

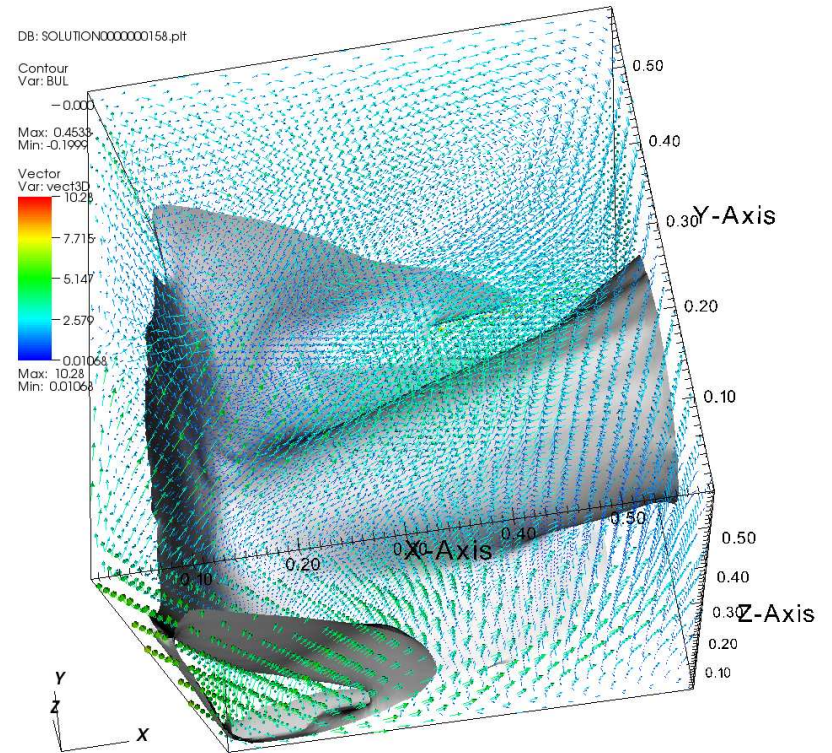


Bifluid flows without surface tension in tubing.

Application Air-Water



user: cedricgalusinski
Fri Jan 16 15:47:55 2009



user: cedricgalusinski
Fri Jan 16 15:46:33 2009

Water jet in an air box.

Conclusion Perspectives

Skeleton and Level set for geometry:

- A simple tool for complex geometry
- Uniform cartesian grid are used!

Level set for flow interface:

- Efficient and simple
- Use finer grids for transport!

Perspectives:

- Fluid solver on composite grid (3D DDFV solver, F. Hubert et al.)
- Compare to zoom techniques on cartesian grid
- Extension multifluid flows (more than three phases) (see Fedkiw).
- Penalize "solid fluid"
- Extend outlet boundary condition in tubing.
- EDP on fluid interface for surfactant.