Discontinuous Galerkin finite element method for two-phase flows in heterogeneous porous media

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joint work with Alexandre Ern[1] and Luciane Schuh [3]

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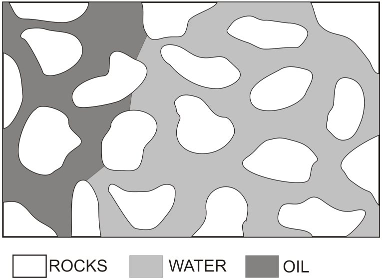
- Two-Phase Flow Models
- DG finite element methods
- Homogeneous media
  - Model $1D$ problem
  - Sequential DG scheme
  - Numerical results
- Heterogeneous media
  - General formulation
  - Model $1D$ problem
  - Sequential DG scheme
  - Numerical results
- Conclusions

Two-Phase Flow Model:

Let Ω be a domain in $\mathbb{R}^d$. The two-phase problem through porous media Ω is described by saturation s_α and pressure p_α of phase α, where $\alpha = w$ stays for wetting phase and $\alpha = n$ stays for non-wetting phase.



Two-phase flow system:

√ $s_w + s_n = 1, \quad p_n - p_w = \pi(s_w)$, where π is capillary pressure;

√ Darcy velocity equation for each phase;

√ mass conservation equation for each phase;

√ + boundary and initial conditions.

Global-pressure/fractional flow formulation for incompressible two-phase flow

Introducing global pressure (G.Chavent and J.Jaffré, 1978) obtain global-pressure/fractional flow formulation

$$
\begin{aligned}
&-\nabla\left(\lambda K \nabla p\right) = q_w + q_n \\
&u = -\lambda K \nabla p \\
&\phi \frac{\partial s_n}{\partial t} + \nabla \cdot \left(f(s_n) u\right) - \nabla \cdot \left(\epsilon(s_n) \nabla \pi(s_n)\right) = q_n
\end{aligned}
$$

where K is absolute permeability, ϕ is porosity, $\lambda = \lambda_w + \lambda_n$ is total mobility, $f = \frac{\lambda_n}{\lambda}$ denotes the fractional flux, $\epsilon = \lambda_w f K$ and q_α are sources or sinks of the medium.

- Nonlinear system is weakly coupled by the total velocity term.
- Permits efficient theoretical and numerical analysis R.Eymard, R.Herbin and A.Michel, MMNA, 2003.

- Requires accurate velocity reconstruction from pressure equation

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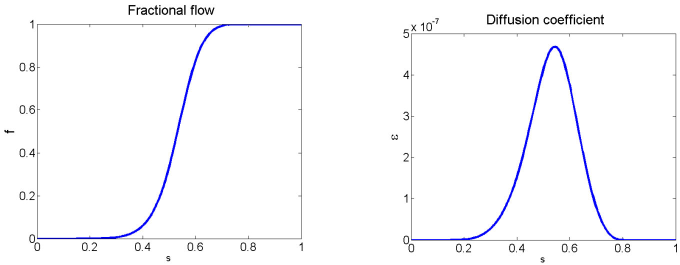
Brooks–Corey model



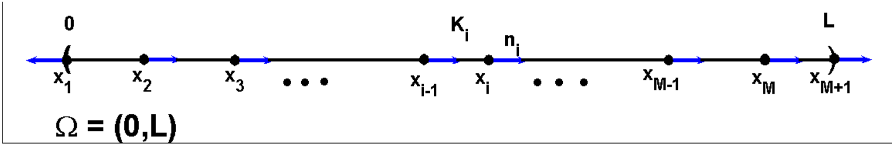
$$\lambda_w(s) = \frac{1}{\mu_w}(1-s_e)^{\frac{2+3\theta}{\theta}}, \quad \lambda_n(s) = \frac{1}{\mu_n}(s_e)^2(1-(1-s_e)^{\frac{2+\theta}{\theta}}),$$

$$\pi(s) = p_e(1-s_e)^{-\frac{1}{\theta}}, \quad s_e = \frac{s - s_{nr}}{1 - s_{wr} - s_{nr}}, \; s_{nr} \leq s \leq 1 - s_{wr}$$

Here p_e is a entry pressure, $s = s_n$, s_{wr}, s_{nr} are residual saturations and s_e is effective saturation of non-wetting phase.

A difficulty in development of numerical methods for the saturation equation is the degeneracy of the diffusion term that implicates the lack of regularity of the solution

DG nomenclature



$\{\mathcal{K}_h\}$ - partition of Ω;

$\kappa_i = (x_{i-1}, x_i)$;

$\mathcal{E}_h^D$ the set of boundary nodes;

$\mathcal{E}_h^\circ = \{x_i : x_i \subset \Omega\}$ the set of all interior nodes.

$\mathcal{E}_h = \mathcal{E}_h^D \cup \mathcal{E}_h^\circ$

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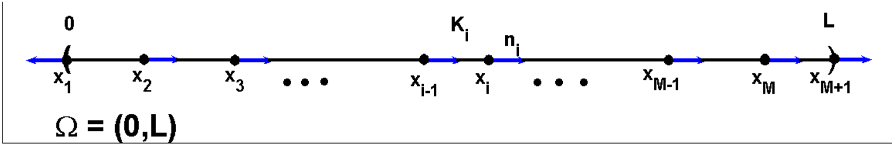

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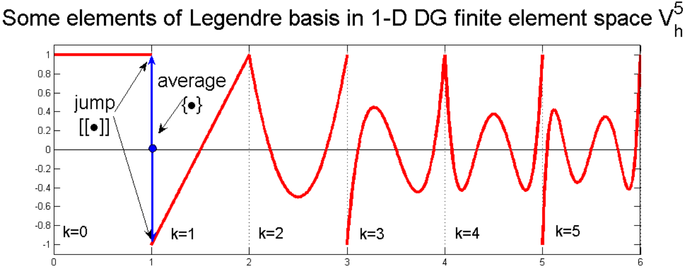
DG finite element space

Let $\mathbb{P}_k(\kappa_i)$ denotes the k-th order polynomial space on κ_i. Consider the discontinuous finite element space of uniform approximation order k $V_h^k = \{v \in L^2(\Omega) : v|_{\kappa_i} \in \mathbb{P}_k(\kappa_i) \;\; \forall i = 1, \dots, M\}$.



$\forall v \in H^1(\mathcal{K}_h)$, where $H^1(\mathcal{K}_h)$ denotes the broken Sobolev space, define the jump and the average operator:

$$[\![v]\!]_i = v|_{K_{i+1}}(x_i) - v|_{K_i}(x_i), \quad \{v'\}_i = \frac{1}{2}\left(v'|_{K_{i+1}}(x_i) + v'_{K_i}(x_i)\right), \quad x_i \in \mathcal{E}_h^\circ;$$
$$[\![v]\!]_i = v(x_i), \quad \{v'\}_i = \mathbf{n}_i v'(x_i) \quad x_i \in \mathcal{E}_h^\partial.$$

Why discontinuous finite elements?

- In DG finite element methods the smoothness of elements of the approximation space is imposed weakly. This helps avoiding non-physical oscillations in neighborhood of singularities of a solution
- DG methods are locally conservative. This is of crucial importance for two phase fluid flows in porous medium simulation
- DG methods are highly local finite element methods of high order (easy parallelization)
- Due to weak implementation of interelement connections the use of non-structured nonmatching meshes is facilitated
- DG FEMs are well suited for $h - p$ adaptative algorithms

- Relatively high computational cost

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Model 1D problem

Let us consider in $(0, L) \times (0, T)$ the system

$$-\partial_x \left(\lambda(s) K \partial_x p\right) = 0, \qquad u = -\lambda(s) K \partial_x p,$$
$$\phi \partial_t s + \partial_x \left(u f(s)\right) - \partial_x \left(\epsilon(s) \partial_x \pi(s)\right) = 0,$$

with the boundary and initial conditions:

$$p|_{x=0} = p_1,\ p|_{x=L} = p_2;$$
$$s|_{x=0} = s_1,\ -\epsilon(s)\partial_x \pi(s)|_{x=L} = 0;$$
$$s|_{t=0} = s_0.$$

Observe that owing to the first equation the total velocity u is constant in space, and only depends on time.

Sequential scheme

Denote by $\tau = \frac{T}{N}, N \in \mathbb{N}$ the time step and introduce

$$p_{\mathcal{T}_h,\tau}, s_{\mathcal{T}_h,\tau} : \Omega \times [0,T] \to \mathbb{R},$$
$$p_{\mathcal{T}_h,\tau} = p_h^n(x),\ s_{\mathcal{T}_h,\tau} = s_h^n(x),\ \ p_h^n(x), s_h^n(x) \in V_h,\ \ t \in [(n-1)\tau, n\tau].$$

Sequential scheme: time discretization

For given $s_h^0 \in V_h$ and for $n = 0, 1, \cdots, N$ find p_h^{n+1} and s_h^{n+1} such that

$$-\partial_x \left(\lambda(s_h^n) K \partial_x p_h^{n+1} \right) = 0, \qquad u_h^{n+1} = -\lambda(s_h^n) K \partial_x p_h^{n+1},$$

$$\phi \frac{s_h^{n+1} - s_h^n}{\tau} + \partial_x \left(u_h^{n+1} f(s_h^{n+1}) \right) - \partial_x \left(\epsilon(s_h^n) \pi'(s_h^n) \partial_x s_h^{n+1} \right) = 0,$$

Scheme:

DG space discretization

We consider

- symmetric interior penalty DG method for the global pressure equation
- symmetric interior penalty DG method for the diffusion term in the saturation equation
- DG method with Godunov fluxes for the nonlinear hyperbolic term in the saturation equation.

DG course at this Winter School

Discontinuous Galerkin Methods
Alexandre Ern (CERMICS, ENPC, Paris, France)

- Wednesday 9h – 10h30
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DG scheme design

Pressure equation

$$-\partial_x \left(\lambda(s_h^n) K \partial_x p_h^{n+1} \right) = 0$$

DG space disctretization

$$\sum_{T \in \mathcal{T}_h} \int_T \lambda(s_h^n) K (p_h^{n+1})' z'$$
$$- \sum_{x_i \in \mathcal{E}_h} \left(\{\lambda(s_h^n) K (p_h^{n+1})'\}_i [\![z]\!]_i + \{\lambda(s_h^n) K z'\}_i [\![p_h^{n+1}]\!]_i \right)$$
$$+ \sum_{x_i \in \mathcal{E}_h} \frac{\sigma_p k^2}{h} \gamma_i [\![p_h^{n+1}]\!]_i [\![z]\!]_i$$
$$= \lambda(s_n^i(0)) K p_1 z'(0) - \lambda(s_n^i(L)) K p_2 z'(L)$$
$$+ \frac{\sigma_p k^2}{h} \gamma_1 p_1 z(0) + \frac{\sigma_p k^2}{h} \gamma_{M+1} p_2 z(L) \quad \forall z \in V_h,$$

DG scheme design

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DG scheme design

Saturation equation

$$\phi \frac{s_h^{n+1} - s_h^n}{\tau} + \partial_x \left(u_h^{n+1} f(s_h^{n+1}) \right) - \partial_x \left(\epsilon(s_h^n) \pi'(s_h^n) \partial_x s_h^{n+1} \right) = 0$$

DG space disctretization

$$\sum_{T \in \mathcal{T}_h} \int_T \phi \tau^{-1} s_h^{n+1} v_h - \sum_{T \in \mathcal{T}_h} \int_T u_h^{n+1} f(s_h^{n+1}) v_h' + \sum_{e \in \mathcal{E}_h} \Phi_{hi}^{n+1} [\![v_h]\!]$$
$$+ \sum_{T \in \mathcal{T}_h} \int_T \epsilon(s_h^n) \pi'(s_h^n) (s_h^{n+1})' v_h' -$$
$$\sum_{x_i \in \mathcal{E}_h \setminus x_{M+1}} \left(\left\{ \epsilon(s_h^n) \pi'(s_h^n) (s_h^{n+1})' \right\} [\![v_h]\!] + [\![s_h^{n+1}]\!] \left\{ \epsilon(s_h^n) \pi'(s_h^n) v_h' \right\} \right)$$
$$+ \sum_{x_i \in \mathcal{E}_h \setminus x_{M+1}} \frac{\sigma_s k^2}{h} \delta_i [\![s_h^{n+1}]\!] [\![v_h]\!] = \sum_{T \in \mathcal{T}_h} \int_T \phi \tau^{-1} s_h^n v_h$$
$$+ \epsilon(s_1) \pi'(s_1) v'(0) + \frac{\sigma_s k^2}{h} \delta_1 s_1 v_h(0) \quad \forall v \in V_h.$$

DG scheme design

Saturation equation

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DG space disctretization

$$\sum_{T \in \mathcal{T}_h} \int_T \phi \tau^{-1} s_h^{n+1} v_h - \sum_{T \in \mathcal{T}_h} \int_T u_h^{n+1} f(s_h^{n+1}) v_h' + \sum_{e \in \mathcal{E}_h} \Phi_{hi}^{n+1} [\![v_h]\!]$$
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$$+ \sum_{x_i \in \mathcal{E}_h \setminus x_{M+1}} \frac{\sigma_s k^2}{h} \delta_i [\![s_h^{n+1}]\!] [\![v_h]\!] = \sum_{T \in \mathcal{T}_h} \int_T \phi \tau^{-1} s_h^n v_h$$
$$+ \epsilon(s_1) \pi'(s_1) v'(0) + \frac{\sigma_s k^2}{h} \delta_1 s_1 v_h(0) \quad \forall v \in V_h.$$

Scheme parameters:

Godunov numerical fluxes:

$$\Phi_{hi} = \begin{cases} \Phi_1 = f(s_{h,1}), \\ \min\limits_{s \in [s_{h,i-}, s_{h,i+})]} f(s) \ \text{ if, } s_{h,i-} < s_{h,i+}, \\ \max\limits_{s \in [s_{h,i+}, s_{h,i-}]} f(s) \ \text{ if, } s_{h,i-} > s_{h,i+}, \\ \Phi_{nEl+1} = f(s_{h,nEl+1}). \end{cases} , e_i \in \mathcal{E}_h^\circ.$$

$$\sigma_p \in [5, 10],$$
$$\gamma_i = \min(\lambda(s_{h,i-}^n)K, \lambda(s_{h,i+}^n)K),$$
$$\delta_i = \min(\epsilon(s_{h,i-}^n)\pi'(s_{h,i-}^n), \epsilon(s_{h,i+}^n)\pi'(s_{h,i+}^n), e_i \in \mathcal{E}_h^\circ.$$

Total velocity reconstruction

For the velocity reconstruction we use the ideas from Ern, Nicaise and Vohralik,07.
In $1D$ discrete $\mathbf{H}(\mathrm{div})$ space is spanned by *continuous*, piecewise polynomials of degree $\leq k+1$: $W_h^k = \{v \in C(\Omega) : v|_{\kappa_i} \in \mathbb{P}_{k+1}(\kappa_i) \ \ \forall i = 1, \dots, M\}$.

Velocity reconstruction

Calculate $u_h^{n+1} \in W_h^k$ such that

$$u_h^{n+1}(x_i) = -\{\lambda(s_h^n) K d_x p_h^{n+1}\}_i + \mathbf{n}_i \gamma_i([\![p_h^{n+1}]\!]_i - \mathcal{B}(x_i)), \quad \forall x_i \in \mathcal{E}_h,$$

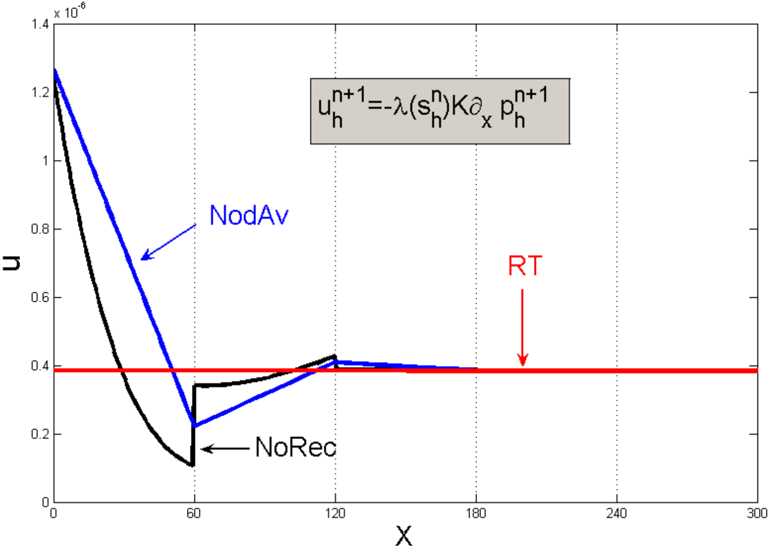
$$\int_K u_h^{n+1} w = -\int_K \lambda(s_h^n) K d_x p_h^{n+1} w + \sum_{x_i \in \partial K} \mathbf{n}_i \{\lambda(s_h^n) K w\}_i ([\![p_h^{n+1}]\!]_i - \mathcal{B}(x_i))$$

$$\forall w \in \mathbb{P}_{k-1}(K), \quad \forall K \in \mathcal{K}_h.$$

where $\mathcal{B}$ is boundary conditions operator defined on $\mathcal{E}_h$ as follows:

$$\mathcal{B}(x_1) = P1, \mathcal{B}(x_{M+1}) = P2 \text{ and } \mathcal{B}(x_i) = 0 \text{ otherwise.}$$

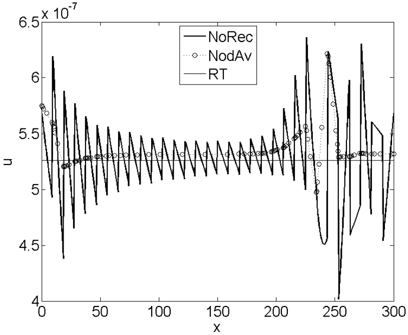
Total velocity reconstruction: how it works



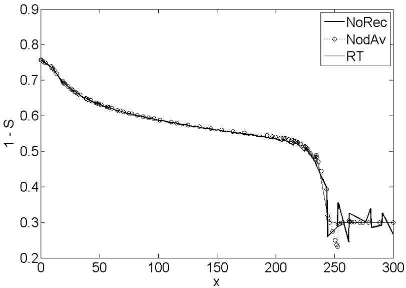
Some numerical results

To illustrate the impact of the above reconstruction, we present numerical results with the following data:

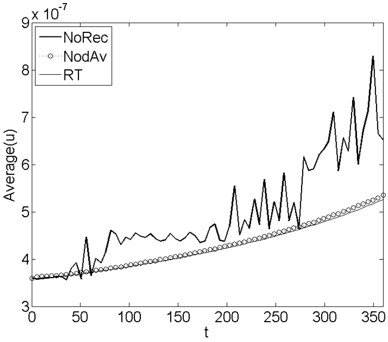
$$\Omega = (0, 300), \quad \mu_w = 0.001\text{kg/ms}, \quad \mu_n = 0.01\text{kg/ms}$$
$$\phi = 0.2, \quad K = 10^{-11}\text{m}^2, \quad s_{wr} = 0.2, \quad s_{nr} = 0.15, \quad \theta = 2,$$
$$p_e = 10^3\text{Pa}$$
$$p_{w1} = 0.3\text{MPa} \quad p_{w2} = 0.15\text{MPa} \quad s_1 = 0.25 \quad s_0 = 0.7$$
$$k = 1 \quad nEl = 32 \quad \gamma_* = \delta_* = 10 \quad \tau = 5 \text{ days} \quad T = 360 \text{ days}$$



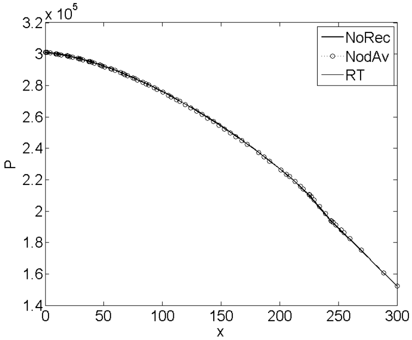
Total velocity at final time



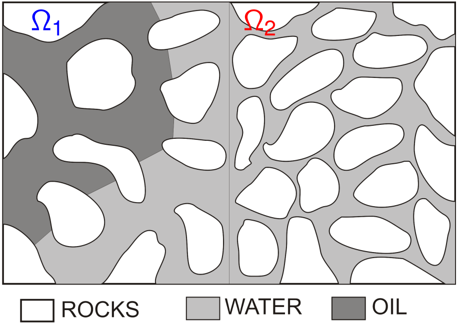
Wetting phase saturation at final time



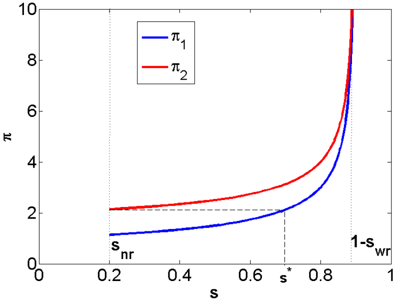
Mean value of total velocity as a function of time



Pressure at final time



Heterogeneous media



Capillary pressures

Problem formulation in heterogeneous media

Let us suppose that $\pi_1, \pi_2 : [s_{nr}, 1 - s_{wr}) \to \mathbb{R}^+$ are Lipschitzian strongly increasing functions such that

$$\pi_1(s) < \pi_2(s), \forall s \in [s_{nr}, 1 - s_{wr})$$

and denote by s^* such real number in $(s_{nr}, 1 - s_{wr})$ that $\pi_1(s^*) = \pi_2(s_{nr})$. Suppose that the mobilities and s_{nr}, s_{wr} are the same in both subdomains. Global-pressure/fractional flow formulation for incompressible two-phase flow in heterogeneous media reads as follows:

find $(p_\beta, s_\beta), \beta = 1, 2$, that satisfy

$$-\nabla\left(\lambda_\beta K_\beta \nabla p_\beta\right) = q_{w\beta} + q_{n\beta}, \quad \text{in } \Omega_\beta$$

$$u_\beta = -\lambda K_\beta \nabla p_\beta, \quad \text{in } \Omega_\beta$$

$$\phi \frac{\partial s_\beta}{\partial t} + \nabla \cdot (f(s_\beta) u_\beta) - \nabla \cdot (\epsilon_\beta(s_\beta) \nabla \pi_\beta(s_\beta)) = q_{n\beta}, \quad \text{in } \Omega_\beta;$$

$+$interface conditions;

$+$boundary conditions

Interface conditions

Interface conditions for pressure:

$$-\lambda_1 K_1 \nabla p_1 \cdot \nu = -\lambda_1 K_2 \nabla p_2 \cdot \nu \quad \text{on } \Gamma,$$

$$p_1 - p_2 = \int_{s_{nr}}^{s_1} f(\xi)\pi_1'(\xi)d\xi + \pi_1(s_{nr}) - \pi_2(s_{nr}) \qquad \text{on } \Gamma, \text{ if } s_{nr} \leq s_1 \leq s^*,$$

$$p_1 - p_2 = \int_{s_{nr}}^{s_1} (f(\xi) - 1)\,\pi_1'(\xi)d\xi - \int_{s_{nr}}^{s_2} (f(\xi) - 1)\,\pi_2'(\xi)d\xi \text{ on } \Gamma, \text{if } s^* \leq s_1 \leq 1 - s_{wr}.$$

Interface conditions for saturation:

$$\begin{aligned}(u_1 f(s_1) + \epsilon_1(s_1)\nabla\pi_1(s_1)) \cdot \nu &= (u_2 f(s_2) + \epsilon_1(s_2)\nabla\pi_2(s_2)) \cdot \nu \quad \text{on } \Gamma,\\ s_2 &= 0 \qquad \text{on } \Gamma, \qquad \text{if } s_{nr} \leq s_1 \leq s^*,\\ \pi_1(s_1) &= \pi_2(s_2) \quad \text{on } \Gamma, \quad \text{if } s^* \leq s_1 \leq 1 - s_{wr}.\end{aligned}$$

Model heterogeneous 1D problem

Let $\Omega = (0, L), \Omega_1 = (0, x_{intf}), \Omega_2 = (x_{intf}, L), 0 < x_{intf} < L$. Suppose that $s_{nr} = s_{wr} = 0$.
Consider in $\Omega \times [0, T]$ the following boundary interface problem:

$$\phi \partial_t s_\beta - \partial_x \left(\epsilon(s_\beta) \partial_x \pi_\beta(s_\beta) \right) = 0 \quad \text{in } \Omega_\beta,$$

with interface conditions

$$\begin{cases} \epsilon(s_1)\partial_x \pi_1(s_1) = \epsilon(s_2)\partial_x \pi_2(s_2) \quad \text{on } \Gamma, & \\ s_2 = 0 \quad \text{on } \Gamma, & \text{if } 0 \leq s_1 \leq s^*, \\ \pi_1(s_1) = \pi_2(s_2) \quad \text{on } \Gamma, & \text{if } s^* \leq s_1 \leq 1, \end{cases} \tag{1}$$

boundary conditions

$$\partial_x s_1(0, t) = \partial_x s_2(L, t) = 0,$$

and initial condition

$$s(0, x) = s_0.$$

In more general setting the convergence of a finite volume scheme to a weak solution of this problem was proved in G. Enchéry, R.Eymard and A.Michel, SINUM, 2006, C. Cancés, NoDEA, 2008, C. Cancés, T.Gallouët and A.Porreta, 2008

DG scheme design

For interface problem the key advantage of discontinuous Galerkin is the possibility of accurate velocity reconstruction from the discontinuous pressure satisfying nonhomogeneous interface condition presented above.

We use implicit backward Euler time approximation of the saturation equation and the symmetric version of interior penalty discontinuous Galerkin method for the discretization of diffusion term.

For weak implementation of interface conditions for the saturation equation within the discontinuous Galerkin framework we introduce function $J : [0, 1) \to \mathbb{R}$ that describes a jump of the saturation at interface:

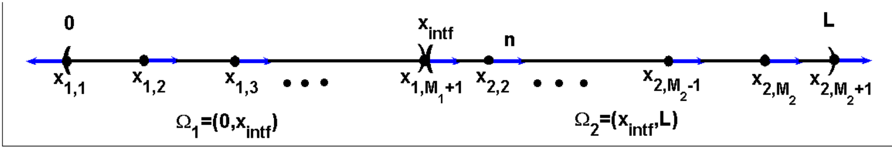
$$J(s) = \begin{cases} s, & \text{if } 0 \le s \le s^*, \\ s - \pi_2^{-1}(\pi_1(s)), & \text{if } s^* \le s \le 1. \end{cases}$$

Now the interface equation can be rewritten as

$$s_1 - s_2 = J(s_1).$$

DG space

Mesh is fitted with interface point:



We use the discontinuous finite element spaces of different mesh size and order of approximation in each subdomain: $\mathbf{k} = (k_1, k_2)$

$$V_{\mathbf{h}}^{\mathbf{k}}(\Omega) = V_{h_1}^{k_1}(\Omega_1) \times V_{h_2}^{k_2}(\Omega_2).$$

Problem Data

- $\Omega = (0, 2)$, $\quad x_{intf} = 1$; porosity $\phi = 0.2$
- Mobilities:

$$\lambda_w(s) = \begin{cases} 1 - s, & \text{if } 0 \le s \le 1; \\ 1, & \text{if } s < 0; \\ 0, & \text{otherwise.} \end{cases} \qquad \lambda_n(s) = \begin{cases} s, & \text{if } 0 \le s \le 1; \\ 0, & \text{if } s < 0; \\ 1, & \text{otherwise.} \end{cases}$$

- Capillary pressure:

$$\pi_1(s) = \begin{cases} 5s^2, & \text{if } 0 \le s \le 1; \\ 0, & \text{if } s < 0; \\ 5, & \text{otherwise.} \end{cases} \quad \pi_2(s) = \begin{cases} 4s^2 + 1, & \text{if } 0 \le s \le 1; \\ 1, & \text{if } s < 0; \\ 5, & \text{otherwise.} \end{cases}$$

- Initial conditions:

$$s_0^{\text{no-trapping}} = \begin{cases} 0.9, & \text{if } x < 0.7; \\ 0, & \text{otherwise.} \end{cases} \qquad s_0^{\text{trapping}} = \begin{cases} 0.4, & \text{if } x < 0.7; \\ 0, & \text{otherwise.} \end{cases}$$

- $nEl_1 = nEl_2 = 80, \quad k_1 = k_2 = 1, \quad \tau = 1e-4, \quad T = 0.5, 0.2$

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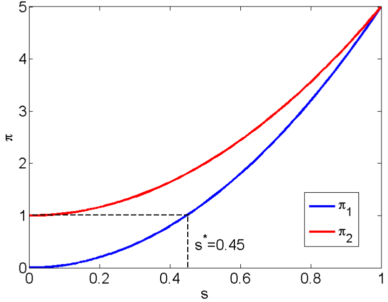
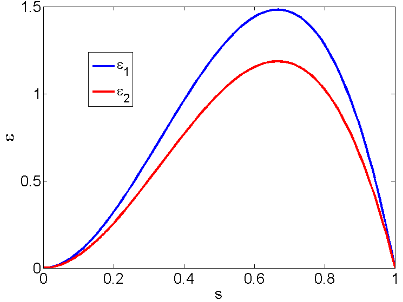
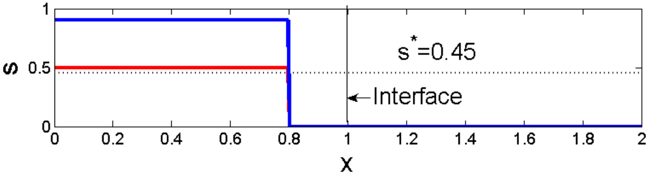


Capillary pressures



Diffusion coefficients

Initial date



Conclusions

- Discontinuous Galerkin method for two phase incompressible immiscible flows in porous media problem in global pressure / fractional flow formulation was introduced.
- We use accurate velocity reconstruction algorithm, which can be easily extended to multiple dimensions.
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