A class of Finite Element-Finite Volume schemes for the compressible barotropic Navier-Stokes equation

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We present a class of schemes for the compressible barotropic Navier-Stokes equations:

$$\partial_t \rho + \nabla \cdot (\rho\, u) = 0, \tag{1a}$$
$$\partial_t(\rho\, u) + \nabla \cdot (\rho\, u \otimes u) + \nabla p - \nabla \cdot \tau(u) = f_v, \tag{1b}$$
$$\rho = \varrho(p), \tag{1c}$$

where ρ, u and p are the density, velocity and pressure in the flow, f_v is a forcing term and $\tau(u)$ stands for the shear stress tensor. The function $\varrho(\cdot)$ is the equation of state used for the modelling of the particular flow at hand.

The discretization of the momentum balance equation (1b) is based on low-order nonconforming finite element spaces (Crouzeix-Raviart or Rannacher-Turek elements); standard finite element techniques are used for the discretization of the pressure gradient and viscous terms, while the first two terms are approximated by a finite volume scheme based on a dual mesh. The mass balance (1a) is approximated by an upwind finite volume scheme. The time stepping is performed by an incremental pressure correction scheme.

This scheme is proven to enjoy an unconditional stability property [2]: irrespectively of the time step, the discrete solution obeys the *a priori* estimates associated to the continuous problem, *i.e.* non-negativity (in fact, for the discrete case, positivity) of the density, bounds in L^∞-in-time norm of the entropy and in L^2-in-time norm of the viscous dissipation. Numerical convergence tests performed with a smooth solution show a first order convergence in time for the velocity and the pressure, and a space convergence order (in L^2 norm) between 1 and 2. In addition, when addressing the hyperbolic case (*i.e.* System (1) with $\tau(u) = 0$) with discontinuous solutions, the scheme is observed to converge when an upwind version of the discretization of the convection term in (1b) is used, or, with a centered approximation, if a slight viscosity is added in the discrete case.

To theoretically analyse the convergence of this scheme, we first address, as a model problem, the stationary Stokes problem (*i.e.* the stationary version of (1) where the $\nabla \cdot (\rho\, u \otimes u)$ term is removed) with equations of state of the form $p = \rho^\gamma$, with $\gamma \geq 1$. Restricting ourselves to Crouzeix-Raviart elements and adding a stabilization term to the discretization of (1a) to control the oscillations of the density, we are able to prove the convergence of the scheme to a solution of the continuous problem [3, 1]. The passage to the limit in the equation of state requires the a.e. convergence of the density. It is obtained by adapting at the discrete level the "effective viscous pressure lemma" of the theory of compressible Navier-Stokes equations.

The entropy preserving pressure correction scheme has been extended to diphasic flows [4]. An adaptation of the convergence studies to the the MAC scheme is underway.

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