Mountain-Pass algorithm for some indefinite problems

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(joint work with C. Troestler)

We are interested in the problem, defined on an open bounded domain $\Omega \subset \mathbb{R}^N$ or in all $\mathbb{R}^N$,

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-2}u, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega \end{cases},$$

with $2 < p < \frac{2N}{N-2}$, for $N \geq 2$.

For $V = 0$, Y. S. Choi and P. J. McKenna introduced in 1993 an algorithm to compute a non-zero solution of this problem. It is called the Mountain-Pass algorithm, a steepest descent method to approach a critical point of the energy

$$J_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + V(x)u\right) dx - \frac{1}{p}\int_\Omega |u|^p$$

restricted to the Nehari manifold $\mathscr{N}_p := \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u)(u) = 0\}$. In 2001, J. Zhou and Y. Li introduced a variant of this algorithm and proved its convergence. More generally, this algorithm can be used for any equation such that the associated energy respects a Mountain-Pass structure:

(i)0 is a local minimum of J_p, (ii) $\forall u \neq 0, \exists! t > 0 : J'(tu)(tu) = 0$
and (iii) $\forall u \neq 0, \lim_{t\to+\infty} J(tu) = -\infty$.

For V such that 0 is not eigenvalue of $-\Delta + V$ but some eigenvalues are negative, this algorithm cannot be used (0 is not a local minimum of J_p). We will present an alternative algorithm and prove its convergence. The main idea is to change the Nehari manifold. Now, we consider

$$\mathscr{N}_p^* := \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u)(u) = 0 \text{ and } J'(u)(v) = 0, \text{ for any } v \in H^-\},$$

where H^- is the space defined by the negative spectrum. This idea was introduced by A. Pankov and used by A. Szulkin and T. Weth to prove the existence of non-trivial solutions. Time permitting, we will illustrate it numerically.