Stabilized DDFV schemes for general Stokes problem

Stella KRELL, Université de Provence, LATP

In this communication, we propose a DDFV (Discrete Duality Finite Volume) scheme for the two-dimensional general linear Stokes problem. This class of schemes has been first introduced and studied in [H 00, DO 05] to approximate the Laplace equation on a large class of 2D meshes including non-conformal and distorted meshes. Unknowns are required on both vertices and "centers" of primal control volumes thus defining two meshes, the primal mesh $\mathfrak{M}$ and the dual mesh $\mathfrak{M}^*$. The dual set of unknowns $\mathbf{u}^{\mathcal{T}}$, $\mathcal{T} = \mathfrak{M} \cup \mathfrak{M}^*$, allows us to reconstitute two-dimensional discrete gradient $\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ on a partition $\mathfrak{D}$ of Ω, called the diamond mesh. A discrete divergence operator $\mathbf{div}^{\mathcal{T}} : (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \longrightarrow \left(\mathbb{R}^2\right)^{\mathcal{T}}$ is then defined by duality leading to the following discrete Stokes formula:

$$-< \mathbf{div}^{\mathcal{T}}\left(\xi^{\mathfrak{D}}\right), \mathbf{v}^{\mathcal{T}} > = < \xi^{\mathfrak{D}}, \nabla^{\mathfrak{D}}\mathbf{v}^{\mathcal{T}} >, \quad \forall\ \mathbf{v}^{\mathcal{T}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}},\ \forall\ \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

in case of homogeneous Dirichlet boundary conditions. We are interested here in general multifluid incompressible flows governed by the general linear Stokes equations: Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $p : \Omega \to \mathbb{R}$ such that:

$$\begin{aligned} \mathrm{div}\left(-\eta(.)(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}) + p\mathrm{Id}\right) &= \mathbf{f}, \quad \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0, \quad \text{on } \partial\Omega, \qquad \int_\Omega p(x)\mathrm{d}x = 0, \end{aligned}$$

where $\mathbf{f} \in (L^2(\Omega))^2$ and the viscosity $\eta \in L^\infty(\Omega)$, possibly discontinuous, satisfies $\inf \eta > 0$. At the continuous level, we refer to [OR 06] for the theoretical analysis of this Stokes problem with discontinuous viscosity. At the discrete level, it seems to be natural to approximate the velocity on both vertices and centers of primal control volumes and the pressure on the diamond cells. In order to have a wellposed scheme, we stabilize the mass conservation equation with the term $-\lambda(h^{\mathfrak{D}})^2\Delta^{\mathfrak{D}}p^{\mathfrak{D}}$ as inspired by the well known Brezzi-Pitkäranta scheme [BP 84] in the finite element framework. The scheme can be written as follows:

$$\begin{aligned} &\mathbf{div}^{\mathcal{T}}(-\eta^{\mathfrak{D}}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + {}^t\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathcal{T}} \quad + \text{ Dirichlet boundary conditions}, \\ &\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathcal{T}}) - \lambda(h^{\mathfrak{D}})^2\Delta^{\mathfrak{D}}p^{\mathfrak{D}} = 0, \qquad \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}p^{\mathcal{D}} = 0, \end{aligned}$$

where $\lambda > 0$ and $\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathcal{T}}) = \mathrm{Tr}\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}$. The discrete operator corresponding to $\nabla\mathbf{u} + {}^t\nabla\mathbf{u}$ is proved to satisfy a discrete Korn inequality (see [K]). In case of discontinuities of η, we use a suitable definition of the discrete gradient as in [BH 08]. Finally, the corresponding stabilized DDFV scheme is proved to be wellposed and first order convergent on general meshes, even for discontinuous viscosity.

References

[BH 08] F. Boyer and F. Hubert. Finite volume method for 2d linear and nonlinear elliptic problems with discontinuities. *SIAM J. Numer. Anal.*, 46(6):3032–3070, 2008.

[BP 84] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In *Efficient solutions of elliptic systems (Kiel, 1984)*, volume 10 of *Notes Numer. Fluid Mech.*, pages 11–19. Vieweg, Braunschweig, 1984.

[DO 05] K. Domelevo and P. Omnes. A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids. *M2AN Math. Model. Numer. Anal.*, 39(6):1203–1249, 2005.

[H 00] F. Hermeline. A finite volume method for the approximation of diffusion operators on distorted meshes. *J. Comput. Phys.*, 160(2):481–499, 2000.

[K] S. Krell. Stabilized DDFV schemes for general Stokes problem. *in preparation*, 2008.

[OR 06] M. A. Olshanskii and A. Reusken. Analysis of a Stokes interface problem. *Numer. Math.*, 103(1):129–149, 2006.

Stella KRELL, Université de Provence, Laboratoire d'Analyse, Topologie et Probabilités,
39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France
krell@cmi.univ-mrs.fr