# Some recent results on nonlinear frictional and memory stabilization of hyperbolic evolution equations for general feedbacks

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# Outline



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# Stabilization

We consider the following abstract equation

$$u'' + Au + \text{ feedback operator}[u] = 0$$
$$(u, u')(0) = (u^0, u^1)$$

Here $A$ stands for an unbounded linear operator in an Hilbert space $H$, which is closed, coercive self-adjoint and with dense domain in $H$.
The feedback operator can be bounded (in case of locally distributed feedbacks for instance) or unbounded operator in $H$, it can be linear or nonlinear, it can be frictional or of memory type (for instance).

We consider dissipative systems, that is: we can associate to the problem

a "natural" energy $E_u$ of a solution $u$ such that formally (indeed for strong solutions):

$$E_u'(t) = -\langle D_B[u], [u] \rangle \leq 0 \,.$$

i.e.: the feedback produces dissipation of energy

$D_B$ is an operator which depends on $B$ and $A$ in general.

The scalar product involved is, in general, different from the scalar product in $H$,

# Model examples

We consider the wave equation.

- Case of frictional feedbacks: the feedback depends locally on the velocity and on the space variable. Two possible situations
  - locally distributed feedbacks:

$$\begin{cases} u_{tt} - \Delta u + \rho(., u_t) = 0 \\ u = 0 \text{ in } [0, \infty) \times \Gamma \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega \, , \end{cases}$$

where $\Gamma = \partial\Omega$.

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# Model examples

- locally distributed $\Longleftrightarrow$ the feedback is "nonvanishing" in a subset $\omega$ of $\Omega$.
- dissipative $\Longleftrightarrow$ $\rho$ monotone nondecreasing with respect to the second variable.

In this case:
The natural energy is

$$E(t) = \frac{1}{2}\Big( \int_\Omega |u_t(t)|^2 + |\nabla u(t)|^2 \, dx \Big)$$

The dissipation relation (for strong solutions) is:

$$E'(t) = -\int_\Omega u_t \rho(., u_t) \, dx \leq 0 \,, \quad t \geq 0 \,.$$

# Model examples

- boundary case:

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u = 0 \text{ in } [0, \infty) \times \Gamma_1 \\ \frac{\partial u}{\partial \nu} + \eta(.)u + \rho(., u_t) = 0 \text{ in } [0, \infty) \times \Gamma_0 \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega\,, \end{cases}$$

where $\{\Gamma_0, \Gamma_1\}$ is a partition of $\Gamma$ and where $\eta$ is a nonnegative function.

# Model examples

The natural energy is

$$E(t) = \frac{1}{2}\Big( \int_{\Omega} |u_t(t)|^2 + |\nabla u(t)|^2 \, dx + \int_{\Gamma_0} \eta |u|^2 \, d\sigma \Big)$$

The dissipation relation (for strong solutions) is:

$$E'(t) = - \int_{\Gamma_0} u_t \rho(., u_t) \, dx \leq 0 \,, \quad t \geq 0 \,.$$

# Model examples

- Memory type feedbacks (several forms): the feedback depends on the memory of the "material", this is the case for viscoelastic materials and is nonlocal with respect to time.
  distributed memory feedbacks

$$\begin{cases} u_{tt} - \Delta u + k * \Delta u = 0 \\ u = 0 \text{ in } [0, \infty) \times \Gamma \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega \,, \end{cases}$$

$$(k * v)(t) = \int_0^t k(t-s) v(s)\, ds \,.$$

and the kernel $k$ is positive, and decaying at infinity.

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# Model examples

In this case, the natural energy is

$$E_u(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}\,dx + \frac{1}{2}\left(1 - \int_0^t k(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t k(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\,ds$$

and the dissipation relation is:

$$E_u'(t) = -\frac{1}{2}k(t)\|\nabla u(t)\|^2 + \frac{1}{2}\int_0^t k'(t)\|\nabla u(s) - \nabla u(t)\|^2\,ds \leq 0$$

# Model examples

- boundary memory feedbacks:

$$\begin{cases} u_{tt} - \Delta u = 0, \quad t > 0,\ x \in \Omega, \\ \partial_\nu u + \eta(x) u + k * u_t + \sigma(x) u_t = 0, \quad t > 0,\ x \in \Gamma_0, \\ u = 0, \quad t > 0,\ x \in \Gamma_1, \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega \end{cases}$$

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# Model examples

In this case, the natural energy is:

$$E(t) := \frac{1}{2}\Big( \int_\Omega (u_t^2 + |\nabla u|^2)\,dx + \int_{\Gamma_0} (a * v^2 + \eta u^2)\,ds \Big), \quad t \geq 0,$$

where

$$v = k * u_t$$

and $a$ is a kernel linked to $k$. We set $b_\infty := \gamma |a|_{L_1(\mathbb{R}_+)}$, where $\gamma$ is some positive parameter also linked to $k$.

The dissipation relation is

$$E'(t) \leq -\sigma_0 \int_{\Gamma_0} u_t^2\,ds - \frac{b_\infty}{2} \int_{\Gamma_0} v^2\,ds - \frac{\gamma}{2} \int_{\Gamma_0} a * v^2\,ds, \quad \text{a.a. } t > 0.$$

# Model examples

If $\sigma$ vanishes, there is no frictional damping, this is the hard case.

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- **Weak, strong convergence of the energy of solutions?**
- **Obtention of energy decay rates?**
- **Identification of sharp conditions on**

  • the feedback operator: weak or strong dissipation, nonlinear, nondissipative . . .

  • the subsets $\omega$ or $\Gamma_0$ : for hyperbolic equations there are geometric conditions

  • for semilinear PDE's as well: critical exponents, explosion . . .

These questions have been the subject of a wide investigation in control theory.

# General goal

Dissipation implies that $E$ is a nonincreasing function of time

For the model examples given above, 0 is the unique equilibrium solution.

A natural question is: does the solution converges to 0 as time goes to $\infty$?

If so, at which rate?

# General goal

Can one assert that a rate is optimal or quasi-optimal in some way?

Is it possible to develop a somehow general "methodology" for general abstract equations which includes as peculiar cases the above examples of model problems?

Not for the pleasure of being abstract without object, but by "discovering" some common mathematical features which explain in some way the "physics" behind, that is how the dissipation mechanism acts on the energy.

Different mathematical tools have been developed or used in this context:

*microlocal analysis, multiplier methods, frequency domain approach or spectral analysis*, combined with *Liapunov type methods* for a modified energy or *integral inequalities* for the natural energy.

Many authors have developped these ideas and tools in the eighties and nineties until nowadays:

J.-L. Lions, Ho, Russell, Bardos-Lebeau-Rauch, Zuazua, Haraux, Nakao, Conrad, Burq-Gérard, Komornik, Lasiecka-Tataru, Triggiani, Puel-Tucsnak, Lasiecka-Benabdallah, Miller, Rao, Martinez, Vancostenoble, Ammari-Tucsnak-Tenenbaum, and many others
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We are interested in *optimal or quasi-optimal* energy decay rates of the energy at infinity for

- **General** abstract second order hyperbolic equations with applications to :

  wave, Petrowsky, systems elasticity, Timoshenko beams ...

- **locally distributed or boundary nonlinear damping**

- **Memory** feedbacks with:

  **general** decaying smooth kernels;

  **singular weak** memory dampings: the kernel exploses at initial time (more realistic)

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Example of singular kernels:

$$k(t) = g_{1-\alpha}(t)e^{-\gamma t}, \quad t > 0,\ \alpha \in (0,1),\ \gamma > 0,$$

where $g_\beta$ denotes the Riemann-Liouville kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0.$$

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# Frictional feedbacks: locally distributed case

For sake of clearness, we only consider localized damping case for the wave equation.
Thus we consider the case of the wave equation

$$\begin{cases} u_{tt} - \Delta u + \rho(., u_t) = 0 \\ u = 0 \text{ in } [0, \infty) \times \Omega \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega , \end{cases}$$

# Frictional feedbacks: locally distributed case

The *main point* here is to assume that $\rho(., v)$ can go to 0 as $v$ goes to 0 at "arbitrary" speed (strictly faster than the linear case).

Results for a linear growth or a polynomial growth go back the works of Zuazua (1990), Haraux, Nakao, Komornik (1994) ...

The first result for "arbitrary" growth is due to Lasiecka-Tataru 1993

Results in this direction have been obtained by Martinez 1999 and Liu-Zuazua 1999 and A.-B. (2004, 2005).

# Frictional feedbacks: locally distributed case

We assume that

$$a(x)g(|v|) \leq |\rho(x,v)| \leq Ca(x)g^{-1}(|v|) \quad \forall\, x \in \Omega \;, \forall\, |v| \leq 1 \,, \,,$$

where $g$ is continuously differentiable on $\mathbb{R}$ strictly increasing and such that $g(0) = 0$. Moreover, $\rho$ is assumed to have a linear growth with respect to the second variable at infinity.

$$\begin{cases} a \geq 0 \text{ in } \Omega \,, \\ a \geq a_- > 0 \;\text{ in } \omega \,, \end{cases}$$

Here $\omega$ is an open subset of $\Omega \subset \mathbb{R}^N$: the domain where the nonlinear feedback is active.

# Geometrical conditions

Microlocal analysis: The subset $\omega$, or the part of the boundary on which the feedback is active has to satisfy geometrical conditions. References:

Bardos-Lebeau-Rauch 1992: gives a very general condition: the set $\omega$ must satisfy the condition of geometric optics, that is each ray must meet $\omega$.

it holds for smooth domains and smooth coefficients of the operator. It has been relaxed by Burq and Gérard 1997

Miller 2003

# Geometrical conditions

Multiplier type methods:

Ho, J.-L. Lions, Zuazua, Komornik, Lasiecka-Tataru, Triggiani, K. Liu, and many other authors ...

give explicit domains $\omega$ but of "large" size and restrictive conditions on the coefficients of the operator.

Frequency domain approach:

K. Liu, M. Tucsnak-Ramdani-Takahashi-Tenenbaum, ...

# Geometrical conditions

Here we consider either Zuazua's multiplier condition denoted later by $(MGC)$, which says that $\omega$ has to be a neighbourhood of the subset of the boundary where $(x - x_0) \cdot \nu > 0$ or

K. Liu's piecewise multiplier condition $(PWMGC)$ which is a generalization of Zuazua's condition, which allows several observation points.

So it is satisfied for instance for a set $\Omega$ which is a sphere and a coefficient $a$ in the feedback that vanishes in a neighbourhood of the two poles of the sphere, when one takes two suitable observation points.

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# Optimal and quasi-optimal energy decay rates

We no longer assume that $g$ is linear or polynomial close to 0.
We make the following assumptions on $g$ :

$$\begin{cases} g \in \mathcal{C}^2([0, r_0])\,, \ r_0 \text{ sufficiently small} \ , \\ H(.) = \sqrt{.}g(\sqrt{.}) \text{ is strictly convex on } [0, r_0^2]\,, \\ g \text{ is odd} \end{cases}$$

**Remark**

*The above assumptions can be relaxed, but they allow to obtain a semi-explicit decay rate for the energy. The last one is just made to make the presentation easier.*

# Optimal and quasi-optimal energy decay rates

We prove (AMO 2005)

## Theorem

*Assume (MGC) or (PWMGC) and that the feedback $\rho$ satisfies the above hypotheses. Then, there exists a $T_0 > 0$ (explicit) such that for all initial data, $E$ satisfies the following nonlinear integral inequality*

$$\int_S^T E(t) f(E(t))\, dt \leq T_0 E(S) \quad \forall\, 0 \leq S \leq T\,,$$

*where $f$ is determined in an "optimal" way using convexity arguments.*

# General decay

Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers and $F$ be a strictly increasing function from $[0, +\infty)$ on $[0, \eta)$, with $F(0) = 0$ and $\lim_{y \to +\infty} F(y) = \eta$.

For any $r \in (0, \eta)$, we define a function $K_r$ from $(0, r]$ on $[0, +\infty)$ by:

$$K_r(\tau) = \int_\tau^r \frac{dy}{yF^{-1}(y)}, \quad (1)$$

and a function $\psi_r$ which is a strictly increasing onto function defined from $[\frac{1}{F^{-1}(r)}, +\infty)$ on $[\frac{1}{F^{-1}(r)}, +\infty)$ by:

$$\psi_r(z) = z + K_r(F(\frac{1}{z})) \geq z, \quad \forall\, z \geq \frac{1}{F^{-1}(r)}, \quad (2)$$

# A general energy-weighted integral inequality

**Theorem**

*We assume that $E$ is a nonincreasing, absolutely continuous function from $[0, +\infty)$ on $[0, +\infty)$, satisfying $0 < E(0) < \eta$ and the inequality*

$$\int_S^T E(t)F^{-1}(E(t))\,dt \le T_0 E(S)\,, \quad \forall\, 0 \le S \le T\,. \tag{3}$$

*Then $E$ satisfies the following estimate:*

$$E(t) \le F\left(\frac{1}{\psi_r^{-1}\left(\frac{t}{T_0}\right)}\right)\,, \quad \forall\, t \ge \frac{T_0}{F^{-1}(r)}\,, \tag{4}$$

# Theorem continued

**Theorem**

*where r is any real such that*

$$\frac{1}{T_0}\int_0^{+\infty} E(\tau)F^{-1}(E(\tau))\,d\tau \leq r \leq \eta\,.$$

*Thus, we have* $\lim_{t\to+\infty} E(t) = 0$, *the decay rate being given by the estimate (4).*

# Optimal energy-weight

We now choose the weight function $f$ such that

$$\frac{s}{2\beta} = \frac{H^*(f(s))}{f(s)} \,,$$

for an appropriate $\beta$.
This optimal weight function $f$ is uniquely determined in the following way :

$$f(s) = F^{-1}(s/2\beta) \quad s \in [0, 2\beta r_0^2)\,,$$

where $\beta$ is of the form $\max(\eta_1, \eta_2 E(0))$. In the particular polynomial case, one gets back the usual polynomial weight $f(s) = s^{(p-1)/2}$.

# Optimal energy-weight

The function $F$ is defined as follows :

$$F(y) = \begin{cases} \dfrac{\hat{H}^*(y)}{y} \text{ if } y \in (0, +\infty), \\ 0 \text{ if } y = 0, \end{cases}$$

where $\hat{H}$ is defined as follows.

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# Optimal energy-weight

$$\hat{H}(x) = \begin{cases} H(x) \text{ if } x \in [0, r_0^2] \,, \\ +\infty \text{ otherwise} \,, \end{cases}$$

where $\hat{H}^*$ stands for the convex conjugate of $\hat{H}$, i.e.

$$\hat{H}^*(y) = \sup_{x \in \mathbb{R}} \{x\, y - \hat{H}(x)\} \,,$$

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# How convexity comes

Geometric hypotheses combined with multiplier method allow us to prove

$$\int_S^T E(t)f(E(t))\,dt \leq \int_S^T f(E(t))\Big[\int_\Omega |\rho(.,u_t)|^2\,dx + \int_\omega |u_t|^2\,dx\Big]$$

$$+cE(S)f(E(S)), \quad \forall\, 0 \leq S \leq T.$$

The terms in purple are the nonlinear and localized linear kinetic energies.

If we want to prove a nonlinear integral inequality, we have to estimate these terms in a nice way.

That is where convexity properties and weight $f(E)$ are involved.

# How convexity comes

For the sake of clearness, let us assume formally that $H$ is convex on all $[0, \infty)$.
Thanks to Jensen's inequality

$$\begin{cases} H\Big(\frac{1}{|\Omega|}\int_\Omega |\rho(.,u_t)|^2\,dt\Big) \leq \\ \frac{1}{|\Omega|}\int_\Omega |\rho(.,u_t)|\,g(|\rho(.,u_t)|) \end{cases}$$

Thanks to the assumption on $\rho$ close to 0.

$$g(|\rho(.,u_t)|) \leq c|u_t|\,,$$

# Optimal energy-weight

Hence if $f$ is nonnegative function (to be determined), we deduce from the above estimate that

$$\int_S^T f(E(t))\,dt \int_\Omega |\rho(.,u_t)|^2\,dx \leq \int_S^T |\Omega| f(E(t))\,dt \times H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u_t\rho(.,u_t)\,dx\Big)\,.$$

# Optimal energy-weight

Now, we use Young's inequality

$$f(E)H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u_t\rho(.,u_t)\,dx\Big) \leq$$
$$H^*(f(E)) + \frac{1}{|\Omega|}\int_\Omega u_t\rho(.,u_t)\,dx\,,$$

But thanks to the dissipation relation

$$E'(t) = -\int_\Omega u_t\rho(.,u_t)\,dx$$

We deduce that

# Optimal energy-weight

$$\int_S^T f(E(t))\,dt \int_\Omega |\rho(.,u_t)|^2\,dx \le |\Omega| \int_S^T H^*(f(E(t))) + cE(S)\,.$$

We can treat the linear kinetic energy term in a similar way.
Using multiplier method, we finally show that

$$\int_S^T E(t)f(E(t))\,dt \le \beta \int_S^T H^*(f(E(t))) + cE(S)$$

But $\beta \int_S^T H^*(f(E(t))) = \frac{1}{2}\int_S^T E(t)f(E(t))\,dt$ by construction of $f$.

# Optimal energy-weight

So $E$ satisfies an integral inequality and thanks to our nonlinear integral inequality, we deduce a decay rate of the energy $E$ at $\infty$.

We recover the optimal decay in polynomial case.

Optimality can be proved for boundary frictional feedbacks for the wave equation, applying optimality results by Vancostenoble, Vancostenoble-Martinez.

# Optimal and quasi-optimal energy decay rates

- One can prove that the above formulas make sense, and in particular that $F$ is invertible and smooth.
- More precisely, $F$ is continuously differentiable strictly increasing, one-to-one function from $[0, +\infty)$ onto $[0, r_0^2)$.

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# Nonlinear wave equation with memory damping

What can be said in case of memory damping?
We consider the model problem

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + \int_0^t k(t-s)\Delta u(s,x)\,ds = |u(t,x)|^{\gamma} u(t,x) \\ u(t,\cdot)_{|\partial\Omega} = 0 \\ (u(0,\cdot), u_t(0,\cdot)) = (u_0, u_1) \end{cases}$$

second member is a source term $0 < \gamma \leq \frac{2}{N-2}$
The system is no longer autonomous, or local.

# Nonlinear wave equation with memory damping

The natural energy is:

$$E_u(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}\,dx + \frac{1}{2}\Big(1 - \int_0^t k(s)\,ds\Big)\|\nabla u(t)\|^2_{L^2(\Omega)}$$
$$-\frac{1}{\gamma+2}\|u(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} + \frac{1}{2}\int_0^t k(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\,ds$$

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# References

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- Ammar Khodja, Benabdallah, Munoz Rivera and Racke (2003); M. Cavalcanti and Oquendo (2003), M. Cavalcanti, V. Cavalcanti and Ma (2004);

# References continued

- Berrimi and Messaoudi (2006); M. Fabrizio, B. Lazzari and Munoz Rivera (2007);
- M. Cavalcanti and P. Martinez (2008): wave equations with both memory and frictional boundary dampings and other than exponential or polynomial kernels;
- and many other results...

# goals

- treat the case of general decaying kernels: i.e. meaning with other types than exponential or polynomial decays.
- in doing so : be sure to obtain "optimal" expected decay of energy
- extend analysis to a large class of evolution equations

# goals

- Results presented here come from a joint work with P. Cannarsa, Univ. Roma II) for general decaying kernels
- The cases of exponential and polynomial type kernels come from a joint work with P. Cannarsa and D. Sforza (Univ. Roma I) (JFA, 2008)
- The results for singular kernels and weak memory boundary damping are a joint work with J. Prüss and R. Zacher (Univ. of Halle, Germany)

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# Abstract set-up

$X$ real Hilbert space $\|\cdot\|$ norm $\langle\cdot,\cdot\rangle$ scalar product

$$\begin{cases} u''(t) + Au(t) - (k * Au)(t)\, ds = \nabla F(u(t)) \\ (u(0), u'(0)) = (u_0, u_1) \end{cases} \tag{5}$$

where $(k * Au)(t) := \int_0^t k(t-s)Au(s)\, ds$

- $A : D(A) \subset X \to X$ densely defined selfadjoint accretive

$$\exists M > 0 \ : \ \langle Ax\,, x\rangle \geq M\|x\|^2 \qquad \forall x \in D(A)$$

- $k : [0,\infty) \mapsto [0,\infty)$ locally absolutely continuous

$$k(0) > 0\,; \qquad k' \leq 0\,; \qquad \int_0^\infty k(t)\, dt < 1$$

# Abstract set-up continued

- $F : D(A^{1/2}) \to \mathbb{R}$ Gâteaux-differentiable with

$$\|\nabla F(x) - \nabla F(y)\| \leq c_R \|A^{1/2}x - A^{1/2}y\| \quad \forall \|A^{1/2}x\|, \|A^{1/2}y\| \leq R$$

One can show that local existence holds by a classical fixed point argument. Moreover mild solutions are strong if the initial data are smoother.

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# Global existence

Assume $\exists \psi : [0, \infty) \to [0, \infty)$ strictly increasing and such that $\psi(0) = 0$ and

$$F(x) \leq \psi(\|A^{1/2}\|)\|A^{1/2}x\|^2 \qquad \forall\ x \in D(A^{1/2})$$

Then $\exists \rho > 0$ such that

$$\|A^{1/2}u_0\| + \|u_1\| < \rho \quad \Longrightarrow \quad u \quad \text{defined} \quad [0, \infty)$$

Moreover $\quad \forall\ t \geq 0$

$$\begin{cases} (i) \quad E_u(t) \geq \frac{1}{2}\|u'(t)\|^2 + \frac{1}{4}(1 - \int_0^\infty k(s)\,ds)\|A^{1/2}u(t)\|^2 \\ (ii) \quad \psi(\|A^{1/2}u(t)\|) \leq \frac{1}{4}(1 - \int_0^\infty k(s)\,ds) \end{cases}$$

# Dissipation

Energy of a mild solution

$$E_u(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}(1 - \int_0^t k(s)\,ds)\|A^{1/2}u(t)\|^2 - F(u(t)) + \frac{1}{2}\int_0^t k(t-s)\|A^{1/2}u(t) - A^{1/2}u(s)\|^2\,ds$$

Notice that

(i) guarantees $E_u \geq 0$ even in presence of source

(ii) controls growth of $F(u(t))$

# Dissipation

Memory damping produces dissipation, i.e. :

**Lemma**

$$E_u'(t) = -\frac{1}{2}k(t)\|A^{1/2}u(t)\|^2 + \frac{1}{2}\int_0^t k'(t)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2\,ds \leq 0$$

# Polynomial and exponential cases

- kernel $k$: $\exists k > 0$ and $p \in (1, +\infty]$ such that

$$k'(t) \leq -ck^{1+1/p}(t) \qquad \forall t \geq 0$$

  This corresponds to polynomial and exponential case

- source term $F$:

$$\begin{cases} F(0) = \nabla F(0) = 0 \\ |\langle \nabla F(x)\,,\, x\rangle| \leq \psi(\|A^{1/2}x\|)\|A^{1/2}x\| \quad \forall\, x \in D(A^{1/2}) \end{cases}$$

assumptions satisfied by nonlinear wave equation with kernel decaying exponentially ($p = \infty$) or polynomially ($1 < p < \infty$)

# Main theorem

- $(u_0, u_1) \in D(A^{1/2}) \times X$ such that $||A^{1/2}u_0|| + ||u_1|| < \rho$
- $u$ mild solution of our abstract equation on $[0, \infty)$

Then $\exists c > 0$ independent of data such that

$p = \infty$ (exponentially decaying kernel)

$$E_u(t) \leq E_u(0)e^{1-c\,t} \qquad \forall\, t \geq 0$$

$1 < p < \infty$ (polynomially decaying kernel)

$$E_u(t) \leq E_u(0)\Big(\frac{c(1+p)}{t+p\,c}\Big)^p \qquad \forall\, t \geq 0$$

$F \equiv 0$ (no source term case) same decay for arbitrary initial data

# Memory energy

As in the frictional case : one controls a weighted integral of the energy by the corresponding weighted integral of the memory energy, i.e. :

$$\int_0^t k(t-s)||A^{1/2}u(t) - A^{1/2}u(s)||^2\, ds\,,$$

and by terms of the form $C\, E_u^{1/p}(0)E_u(S)$ for some $C$ by the choice of appropriate multipliers. By this way, one can control the three first terms of the energy.

# Integral inequality ($p = \infty$ )

if $p = \infty$ then estimate for the memory energy is easy: it relies on condition

$$k'(t) \leq -ck(t)$$

to conclude

$$\frac{1}{2}\int_S^T \int_0^t k(t-s)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2 \, ds \, dt$$

$$\leq -\frac{1}{2c}\int_S^T \int_0^t k'(t-s)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2 \, ds \, dt$$

$$\leq -\frac{1}{c}\int_S^T E_u'(t) \, dt \leq \frac{1}{c}E_u(S)$$

# Technical lemma for the case $p \in (2, \infty)$

when $p \in (2, \infty)$. One has to proceed differently and to use a technique by Cavalcanti and Oquendo 2003

- define for all $m \geq 1$

$$\varphi_m(t) = \int_0^t k^{1-1/m}(t-s)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2\, ds$$

**Lemma**

*If $\varphi_m(t)$ bounded, then*

$$\int_S^T E_u^{1+m/p}(t)\, dt \leq C\Big(E_u^{m/p}(0) + \|\varphi_m\|_\infty^{m/p}\Big) E_u(S)$$

*for all* $S \geq S_0 > 0$

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# Integral inequality for $p \in (2, \infty)$

obtain integral inequality with $\alpha = 1/p$ in four steps

- since $\|\varphi_2\|_\infty \leq CE_u(0)$, above lemma with $m = 2$ yields

$$\int_S^\infty E_u^{1+\frac{2}{p}}(t)\,dt \leq CE_u^{\frac{2}{p}}(0)E_u(S) \qquad \forall\, S \geq S_0 \qquad (6)$$

- integral inequality for $E_u$ with $\alpha = 2/p$

$$E_u(t) \leq E_u(0)\left(\frac{C(p, S_0)}{2t + C(p, S_0)}\right)^{\frac{p}{2}} \qquad \forall t \geq 0 \qquad (7)$$

# Integral inequality for $p \in (2, \infty)$

- observe $|\varphi_1(t)| \leq C\big(\int_0^t E_u(s)\,ds + tE_u(t)\big) \leq CE_u(0)$ since $p > 2$
- apply technical lemma with $m = 1$

$$\int_S^\infty E_u^{1+\frac{1}{p}}(t)\,dt \leq CE_u^{\frac{1}{p}}(0)E_u(S) \qquad \forall\, S \geq S_0$$

# General decaying kernels

The proof of the above results rely strongly on the exponential or polynomial structure of the kernel and in particular in Cavalcanti-Oquendo's argument. It no longer works if for instance

$$k(t) = c(t+1)^{-p}(\ln(t+e))^{-q}\,, \qquad t \geq 0\,, p > 1, q > 1\,.$$

# General decaying kernels

One has to find an appropriate link between

$$\int_0^t (t-s+1)^{-p}(\ln(t-s+e))^{-q}\int_\Omega |\nabla u(t)-\nabla u(s)|^2\,dx\,ds$$

and

$$\int_0^t (t-s+1)^{-p-1}(\ln(t-s+e))^{-q-1}\left[p\ln(t+e)+ q(t+1)(t+e)^{-1}\right]\int_\Omega |\nabla u(t)-\nabla u(s)|^2\,dx\,ds$$

# General kernels

Extension to general decaying kernels have been considered in Cavalcanti-Martinez (Nonlinear Anal. 2008) for a wave equation with both frictional and memory dampings. Their method combines multiplier arguments to a sharp lemma about convergent and divergent series and uses both frictional and memory dampings.

# General decaying kernels

We prove that

**Theorem (A.-B.-Cannarsa)**

*Assume that the convolution kernel* $k : [0, \infty) \to [0, \infty)$ *is a locally absolutely continuous function, positive which decays at infinity and satisfies some additional assumptions. Then, the energy of solutions satisfies the decay estimate*

$$E_u(t) \leq \kappa_1(u)k(t) \;, \quad \forall\, t \geq T_1$$

# General decaying kernels

We can associate to a general kernel $k$ two numbers $\mu_1(k)$ and $\mu_2(k)$ with $\mu_1(k) \leq \mu_2(k)$. There exist two critical values (of different nature) $0 < \alpha < \beta$, which do not depend on $k$. We can show that $\mu_2(k) \leq \beta$ always holds. Moreover, The lower bound is critical for well-posedness, that is the condition $\mu_1(k) \geq \alpha$ is necessary for the evolution equation to be well-posed. The upper condition measures in some way how fast $k$ goes to 0 at $\infty$.

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# General decaying kernels

The assumption in the above theorem is of the form:

$$\alpha < \mu_1(k) \leq \mu_2(k) < \beta$$

The upper bound can be reached, we still have a decay in this situation, but weaker. The same situation holds if the lower bound is reached, but the estimate is different in this latter case. For our previous example:

$$k(t) = c(t+1)^{-p}(\ln(t+e))^{-q}, \qquad t \geq 0 .$$

# General decaying kernels

The assumption of the above theorem is valid, so that the energy of solutions decays at least as fast as $k$ at $\infty$.

We recover also as a peculiar case the polynomial case. Slower exponential decay of the kernel and more complex decay are also treated by this way.

# General decaying kernels

- The decay rates that we obtain are optimal in the sense that we show that the energy decays at least as fast as the kernel $k$ when critical values are not reached. We obtain quasi-optimal decay rates in case of the upper bound.
- The method is uniform, meaning that we treat all these cases by the same way and it includes known results.

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# Singular boundary memory dampings

We consider a wave equation subjected to a memory boundary damping of the form

- $$k * u_t$$

  on a part of the boundary.
- This generates two difficulties
- no frictional damping is considered, so that in general no uniform decay is expected.
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# Singular boundary memory dampings

- Typical examples of kernels, we consider for instance (but our results apply to a wider class) are :

$$k(t) = g_{1-\alpha} \exp^{-\gamma\, t}\,, \quad t > 0\,,$$

- where $g_\delta$ stands for the Riemann-Liouville kernel, i.e.

$$g_\delta(t) = \frac{t^{\delta-1}}{\Gamma(\delta)}\,.$$

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$$\begin{cases} u_{tt} - \Delta u = 0, \quad t > 0,\ x \in \Omega, \\ \partial_\nu u + \eta(x) u + k * u_t + \sigma(x) u_t = 0, \quad t > 0,\ x \in \Gamma_0, \\ u = 0, \quad t > 0,\ x \in \Gamma_1, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u_t(0,x) = u_1(x), \quad x \in \Omega. \end{cases}$$

Here $\nu$ denotes the outer unit normal of $\Gamma$, $k \in L_{1,loc}(\mathbb{R}_+)$ is a nonnegative singular kernel.
Under assumptions on $k$, covered by the case of Liouville type singular kernels, we prove that the energy decays polynomially at infinity.

# Open problems

- Localized and boundary memory feedbacks
- Optimality and critical cases. In fact more can be said on optimality, but results are still very limited.
- Proof of the nonuniform stability in case of singular boundary memory feedback.
- Understand competition between frictional and memory damping when both are present.

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Thanks for your attention