Exact controllability of a one-dimensional wave problem with a unilateral constraint at the boundary

Control

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The control problem

Consider the system

$$\begin{cases} y'' - y_{xx} = 0, & (0,T)\times(0,1), \\ y(t,0) = u(t), & t\in(0,T), \\ y(t,1) \geq \psi(t),\ y_x(t,1) \geq 0, & t\in(0,T), \\ (y(t,1)-\psi(t))y_x(t,1) = 0, & t\in(0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x\in(0,1). \end{cases}$$

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with $\left(y^0, y^1\right) \in H^1 \times L^2(0,1)$, $u \in L^2(0,T), \psi \in H^1(0,T)$.
We look for a $T_0 \geq 0$ such that for any $\left(z^0, z^1\right) \in H^1 \times L^2(0,1)$:

$$\forall T > T_0, \forall \left(y^0, y^1\right) \in H^1 \times L^2(0,1),\ \exists u \in L^2(Q_T)^m,$$
$$\left(y, y'\right)(T) = \left(z^0, z^1\right) \text{ in } (0,1).$$

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- **Asymptotic behaviour**: Rivera - Oquendo (Funk. Ekvac., 1999) with an internal dissipation.

Controls for a linear problem

In the sequel: $\psi \leq 0$ and $z^0 = z^1 = 0$.

$$\begin{cases} \phi'' - \phi_{xx} = 0, & (t,x) \in Q_T, \\ \phi(t,0) = u(t), & t \in (0,T), \\ \phi(t,1) = f(t), & t \in (0,T), \\ \phi(0,x) = \phi^0(x), \quad \phi'(0,x) = \phi^1(x), & x \in (0,1) \end{cases}$$

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- **Step 1**: $\forall T > 2$, we give an explicit description of the set

$$U = U\left(\phi^0, \phi^1, f\right) = \left\{u; \ \left(\phi, \phi'\right)(T) = 0\right\}.$$

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- **Step 2**: For any $u \in U$, we compute the Dirichlet-Neumann map

$$f \longmapsto Af = \phi_x(.,1)$$

Controls for a linear problem

The change of variables

$$p = \phi' - \phi_x, \;\; q = \phi' + \phi_x$$

transforms the system into

$$\begin{cases} p' + p_x = q' - q_x = 0, & Q_T, \\ (p+q)(.,0) = 2u', & (0,T), \\ (p+q)(.,1) = 2f' & (0,T), \\ p^0 = \phi^1 - \phi^0_x, \; q^0 = \phi^1 + \phi^0_x, & (0,1). \end{cases}$$

We solve this system with the characteristics method.

Step 1

Proposition

Let $T \in (2,3)$. *Then* $\forall \; \left(p^0, q^0, f\right) \in L^2(0,1)^2 \times H^1(0,T)$, *we have*

$(p,q)\,(T) = 0$ *in* $(0,1)$ *if and only if*

$$\begin{cases} u'(\,t\,) = f'(t+1) + \frac{1}{2}q^0(\,t\,) & T-2 < t < 1 \\ u'(t) = \; f'(t+1) + \; f'(t-1) - \frac{1}{2}p^0(2-t) & 1 < t < T-1 \\ u'(t) = f'(t-1) - \frac{1}{2}p^0(2-t) & T-1 < t < 2 \\ u'(t) + \; u'(\,t-2\,) = \; f'(t-1) + \frac{1}{2}q^0(\,t-2\,) & 2 < t < T \end{cases}$$

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PROOF: We solve the system with the characteristics method and find u' in such a way that $p(T) = q(T) = 0$.

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We denote by U the set of u satisfying these relations.

Step 2

Proposition

Let $T \in (2,3)$, $(\phi^0, \phi^1) \in H^1(0,1) \times L^2(0,1)$ *and* $u \in U$. *Let* $A : H^1(0,T) \longrightarrow L^2(0,T)$ *the Dirichlet-Neumann map defined by* $Af = \phi_x(.,1)$. *Then*

$$
\begin{aligned}
Af(t) &= \begin{cases} f'(t) - p^0(1-t) & a.e.\, 0 < t < 1 \\ f'(t) - 2u'(t-1) - q^0(t-1) & a.e.\, 1 < t < T-1 \\ -f'(t) & a.e.\ T-1 < t < T \end{cases} \\
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$$= \begin{cases} f'(t) - v(t) & a.e.\, 0 < t < T-1 \\ -f'(t) & a.e.\ T-1 < t < T \end{cases}$$

PROOF: With the assumption on u, it suffices to note that

$$\phi_x(t,1) = \frac{q(t,1) - p(t,1)}{2}$$

We consider the penalized system:

$$\begin{cases} y_\varepsilon'' - y_{\varepsilon,xx} = 0, & (0,T)\times(0,1) \\ y_\varepsilon(t,0) = u_\varepsilon(t) & t\in(0,T) \\ y_{\varepsilon,x}(t,1) = \varepsilon^{-1}\left[y_\varepsilon(t,1)+l\right]^- & t\in(0,T) \\ y_\varepsilon(0,x) = y^0(x),\ \ y_\varepsilon'(0,x) = y^1(x), & x\in(0,1) \end{cases}$$

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- Step 1: Reduce this problem to a fixed point problem.
- Step 2: uniform estimates of $(u_\varepsilon, y_\varepsilon, y'_\varepsilon, y''_\varepsilon)$ with respect to ε.
- Step 3: pass to the limit (as in Lebeau-Schatzman or Kim) to obtain a solution of our control problem.

- Start with the system

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with $u \in U$. We have: $\phi(T) = \phi'(T) = 0$.

- With these controls u (depending on f, ϕ^0, ϕ^1 only on $(T-2, T)$), we get a solution of our penalized problem by solving the equation

$$Af_\varepsilon = \varepsilon^{-1}\left[f_\varepsilon + l\right]^-$$

Indeed, for $u \in U$

$$Af(t) = \begin{cases} f'(t) - p^0(1-t) & a.e.\, 0 < t < 1 \\ f'(t) - 2u'(t-1) - q^0(t-1) & a.e.\, 1 < t < T-1 \\ -f'(t) & a.e.\ T-1 < t < T \end{cases}$$

And thus, the equation $Af_\varepsilon = \varepsilon^{-1}\left[f_\varepsilon + l\right]^-$ becomes the differential equation

$$f_\varepsilon'(t) = \begin{cases} \varepsilon^{-1}\left[f_\varepsilon(t) + l\right]^- + p^0(1-t) & a.e.\, 0 < t < 1 \\ \varepsilon^{-1}\left[f_\varepsilon(t) + l\right]^- + 2u'(t-1) + q^0(t-1) & a.e.\, 1 < t < T-1 \\ -\varepsilon^{-1}\left[f_\varepsilon(t) + l\right]^- & a.e.\ T-1 < t < T \end{cases}$$

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- The problem is to construct a uniformly bounded family of solution (f_ε) with respect to ε.
- This is done with a suitable choice of u on $(0, T-2)$.

Direct method

- Consider the system

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- With these controls u (depending on f, ϕ^0, ϕ^1 only on $(T-2, T)$), we get a solution of our problem by solving

$$f(t) - \psi(t) \geq 0, \; Af(t) \geq 0, \; (f(t) - \psi(t))\, Af(t) = 0, \quad t \in (0,T).$$

with $f(0) = \phi^0(1)$.

Direct method

These relations write

$$\begin{cases} f(t)-\psi(t)\geq 0, \\ f'(t)-v(t)\geq 0, \\ (f(t)-\psi(t))\,(f'(t)-v(t))=0 \\ f(0)=\phi^0(1) \end{cases}, \quad t\in(0,T-1),$$

and

$$\begin{cases} f(t)-\psi(t)\geq 0, \\ f'(t)\leq 0, \\ (f(t)-\psi(t))\,f'(t)=0 \end{cases}, \quad t\in(T-1,T),$$

Direct method

We solve this differential inequation using the following consequence of a result by Bénilan-Pierre (1978):

Proposition

Let $h \in H^1(0,T)$ *and* $\theta_0 \geq h(0)$. *Then*

$$\theta(t) = \max\left(\theta_0, \sup_{0 \leq s \leq t} h(s)\right), \; t \in [0,T[$$

is the unique solution in $H^1(0,T)$ *of the problem:*

$$\begin{cases} \theta \geq h & \text{in } (0,T) \\ \theta' \geq 0 & \text{in } (0,T) \\ \theta'(\theta - h) = 0 & \text{in } (0,T) \\ \theta(0) = \theta_0 & \end{cases}$$

Direct method

Setting:

$$V(t) = \int_0^t v(s)ds$$

we find

$$f(t) = V(t) + \max\left(y^0(1), \sup_{0\le s\le t}(\psi(s) - V(s))\right), \; t \in (0, T-1).$$

and

$$f(t) = \left[\sup_{t\le s\le T}\psi(s)\right]^+, \;\; T-1 < t < T$$

A suitable choice of u in $(0, T-2)$ will allow continuity of f in $t = T-1$.

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- What is about the higher dimension?

$$\begin{cases} y'' - \Delta y = 0 & (0, T) \times \Omega \\ y = u & (0, T) \times \Gamma_0 \\ \text{Constraints} & (0, T) \times \Gamma_1 \\ \text{Initial data} & \Omega \end{cases} \quad \textit{Data}$$

A "good" description of the Dirichlet-Neumann map shoud give a solution to this problem...