On the global controllability of a nonlinear Korteweg-de Vries equation

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Let T and $L > 0$.

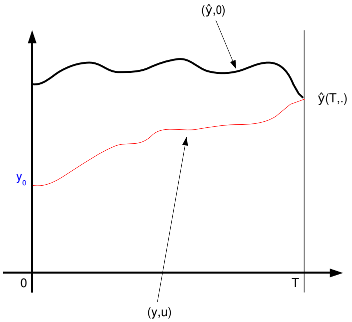
We are interested in the following control-system:

$$(\Sigma) \begin{cases} y_t + y_{xxx} + y_x + yy_x = u(t), \\ y(t,0) = v_1(t),\ y(t,L) = v_2(t), \\ y_x(t,L) = 0, \end{cases}$$

where, at time $t \in [0, T]$,
-the state is $y(t,.) \in L^2(0, L)$,
-the controls are $v_1(t)$, $v_2(t)$ and $u(t) \in \mathbb{R}$.

Question

Global controllability to the trajectories ?



Global exact controllability of (KdV)

Theorem

$\forall T, L > 0$, $\forall y_0 \in L^2(0, L)$, $\forall \hat{y} \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H_0^1(0, L))$ which satisfies

$$\begin{cases} \hat{y}_t + \hat{y}_{xxx} + \hat{y}_x + \hat{y}\hat{y}_x = 0, \\ \hat{y}(t, 0) = \hat{y}(t, L) = \hat{y}_x(t, L) = 0, \end{cases}$$

$\exists u \in C^\infty([0, T])$ vanishing on a neighborhood of 0 and T, $\exists v_1$ and $v_2 \in \cap_{\epsilon \in (0,1/2)} H^{1/2-\epsilon}(0, L)$ such that there exists a unique

$$y \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

solution of (KdV) with initial condition

$$y(0, x) = y_0(x)$$

and which satisfies

$$y(T, x) = \hat{y}(T, x).$$

Well-posedness without external force

$$(*)\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = h_1(t),\ y(t,L) = h_2(t),\ y_x(t,L) = h_3(t), \\ y(0,x) = \tilde{y}(x). \end{cases}$$

$$\tilde{y}_k(0) = h_1^{(k)}(0), \quad \tilde{y}_k(L) = h_2^{(k)}(0), \quad \tilde{y}_k'(L) = h_3^{(k)}(0)$$

for $k \in \mathbb{N}$ and where

$$\begin{cases} \tilde{y}_0(x) = \tilde{y}(x), \\ \tilde{y}_k(x) = -(\tilde{y}_{k-1}^{(3)}(x) + \tilde{y}_{k-1}'(x) + \dfrac{1}{2}\displaystyle\sum_{j=0}^{k-1}\binom{k-1}{j}(\tilde{y}_j(x)\tilde{y}_{k-j-1}(x))') \end{cases}$$

for $k \in \mathbb{N}^*$.

Definition

Let $T > 0$ and $s \geq 0$. A four-tuple $(\tilde{y}, \vec{h}) = (\tilde{y}, h_1, h_2, h_3) \in H^s(0, L) \times H^{(s+1)/3}(0, T) \times H^{(s+1)/3}(0, T) \times H^{s/3}(0, T)$ is said to be s-compatible if

$$\tilde{y}_k(0) = h_1^{(k)}(0), \quad \tilde{y}_k(L) = h_2^{(k)}(0) \tag{1}$$

hold for
- $k = 0, \ldots, [s/3] - 1$ when $s - 3[s/3] \leq 1/2$
- $k = 0, \ldots, [s/3]$ when $s - 3[s/3] > 1/2$

and

$$\tilde{y}_k'(L) = h_3^{(k)}(0) \tag{2}$$

holds for
- $k = 0, \ldots, [s/3] - 1$ when $s - 3[s/3] \leq 3/2$
- $k = 0, \ldots, [s/3]$ when $s - 3[s/3] > 3/2$.

Theorem (2003, Bona-Sun-Zhang)

For any $s \geq 0$, for any T, $L > 0$ and for any s-compatible $(\tilde{y}, \vec{h}) \in H^s(0, L) \times H^{\mu_1(s)}(0, T) \times H^{\mu_1(s)}(0, T) \times H^{\mu_2(s)}(0, T)$, (*) is well-posed in

$$C^0([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+1}(0, L)),$$

and

$$\mu_1(s) := \begin{cases} \epsilon + (5s+9)/18 & \text{if } 0 \leq s < 3, \\ (s+1)/3 & \text{if } s \geq 3; \end{cases}$$

$$\mu_2(s) := \begin{cases} \epsilon + (5s+3)/18 & \text{if } 0 \leq s < 3, \\ s/3 & \text{if } s \geq 3, \end{cases}$$

where ϵ is any positive constant.

Smoothing effect:

Let $y^0 \in L^2(0,L)$. Let $y \in C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$ be the solution of

$$\left\{ \begin{array}{l} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = 0,\ y(t,L) = 0,\ y_x(t,L) = 0, \\ y(0,x) = y^0(x). \end{array} \right.$$

Then, for any $s \in \mathbb{N}$, there exists $\eta_s > 0$ such that

$$y \in C^0([\eta_s, T]; H^s(0,L)) \cap L^2(\eta_s, T; H^{s+1}(0,L)).$$

Well-posedness with an external force

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = f(t,x), \\ y(t,0) = h_1(t),\ y(t,L) = h_2(t),\ y_x(t,L) = h_3(t), \\ y(0,x) = \phi(x). \end{cases}$$

Theorem

For any T, $L > 0$, for any $\phi \in L^2(0,L)$, for any $(h_1, h_2, h_3) \in H^1(0,T) \times H^1(0,T) \times L^2(0,T)$ and for any $f \in L^1(0,T; L^2(0,L))$, the previous Cauchy problem is well-posed in

$$C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

and its solution moreover satisfies

$$y_x(.,0) \in L^2(0,T).$$

Only one control is acting over the system: local results

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \; y_x(t,L) = w(t). \end{cases}$$

- Rosier (97)

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}; k, \; l \in \mathcal{N}^* \right\}$$

$$L \notin \mathcal{N}$$

$$\begin{cases} y_t + y_{xxx} + y_x = 0, \\ y(t,0) = y(t,L) = 0, \; y_x(t,L) = w(t). \end{cases}$$

$$L \in \mathcal{N} : \exists M \subset L^2(0,L), \, \dim M < +\infty,$$

$$M = \{\text{unreachable states}\}.$$

Only one control is acting over the system: local results

- Coron - Crépeau (04)

$$L = 2k\pi,\ k \in \mathbb{N}^*,\ \forall T > 0$$

- Cerpa (06)

$$\dim\ M = 2,\ \ T \geq T_0$$

- Cerpa - Crépeau (07)

$$\forall L \in \mathcal{N},\ T \geq T_0$$

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \\ y(t,0) = v_1(t),\ y(t,L) = 0, \\ y_x(t,L) = 0. \end{cases}$$

- Rosier (04) $\forall T > 0$

- Glass - Guerrero (07) $\forall T > 0$

With two controls: local results

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \\ y(t,0) = 0,\ y(t,L) = v_2(t), \\ y_x(t,L) = w(t). \end{cases}$$

- Rosier (97)

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \\ y(t,0) = v_1(t),\ y(t,L) = v_2(t), \\ y_x(t,L) = 0. \end{cases}$$

- Glass - Guerrero (07)

With three controls

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \\ y(t,0) = v_1(t),\ y(t,L) = v_2(t), \\ y_x(t,L) = w(t), \end{cases}$$

- Rosier (99): global result for large time

- Zhang (99): neighborhood of smooth trajectories, $T > 0$

First remark:
only boundary controls $\Rightarrow$local results in small time or global results in large time $\Rightarrow$introduction of an internal control $u(t)$ to obtain global results in small times.

First remark:
only boundary controls ⇒local results in small time or global results in large time ⇒introduction of an internal control $u(t)$ to obtain global results in small times.

Second remark:
- G. and G.: two boundary controls ⇒ linearized system controllable ⇒ local exact controllability of the nonlinear one (fixed point)

- Global controllability ⇒ a key term : yy_x ⇒ What about the controllability of the nonviscous Burgers equation?

- Idea : we reduce the proof of our glob. cont. result to the proof of an approximate controllability result.

- To this aim, we use the controllability of the nonviscous Burgers equation.

Main result

Let T and $L > 0$.

We are interested in the following control system:

$$(Bnv)\left\{ \; y_t + yy_x = u(t), \right.$$

where, at time $t \in [0, T]$,
-the state is $y(t, .) \in C^1([0, L])$,
-the controls are $y(t, 0)$, $y(t, L)$ and $u(t) \in \mathbb{R}$.

Theorem

$\forall T, \ L > 0$, $\forall y^0, \ y^1 \in C^1([0, L])$, $\exists u \in C^\infty([0, T])$ vanishing on a neighborhood of 0 and T and $\exists y \in C^1([0, T] \times [0, L])$ which satisfies (Bnv) and $y(0, x) = y^0(x)$, $y(T, x) = y^1(x)$.

Previous results

They concern the controllability of (Bnv) by means of boundary control ($y(t,0)$).

- Ancona and Marson (98): you can not reach final constant states $C > 0$ from 0 if $T < L/C$.

- Horsin (98): you can go from 0 to C in time T if $T \geq L/|C|$.

A term of pressure

- Nonviscous Burgers equation:

  Analogue in 1D of the Euler equation for the incompressible inviscid fluids.

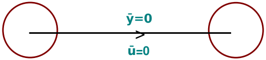
- Coron (93,96), Glass (97,00):

  The Euler equation for the incompressible inviscid fluids is controllable (even for small times).

- Conclusion:

The control $u(t)$ may play the role of the pressure.

Usual strategy

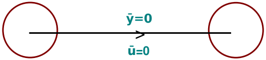


- Controllability of the linearized system around an equilibrium (here $(0, 0)$)

$$\Rightarrow$$

Local controllability of the nonlinear system around the equilibrium.

Usual strategy



- Controllability of the linearized system around an equilibrium (here $(0, 0)$)

$$\Rightarrow$$

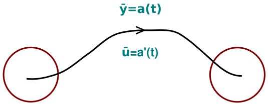
Local controllability of the nonlinear system around the equilibrium.

- Linearized system around the null solution:

$$y_t = u(t)$$

Not controllable.

Coron's return method



- Consists in finding a trajectory $(\bar{y}, \bar{u})$ going from 0 to 0 and such that the linearized system around $(\bar{y}, \bar{u})$ is controllable.

Coron's return method

Stabilisation: Coron (92)

Controllability of pde:

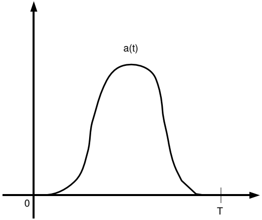
Coron: Euler (93), Navier-Stokes (96), Saint Venant (02)

Horsin: Burgers (98)

Fursikov, Imanuvilov: Navier-Stokes, Boussinesq (99)

Glass: Euler 3D (00), Vlasov Poisson (03), Euler isentropique 1D, Camassa-Holm (07)

Beauchard: Schrodinger (05)



a is assumed to be such that

$$\int_0^T a(t)\,dt > L.$$

Coron's return method

- We look at the linearized control system around $(\bar{y}, \bar{u})$ with

$$\bar{y}(t,x) = a(t)$$

and

$$\bar{u}(t) = a'(t).$$

- We look at the linearized control system around $(\bar{y}, \bar{u})$ with

$$\bar{y}(t,x) = a(t)$$

and

$$\bar{u}(t) = a'(t).$$

- Linearized control system:

$$y_t + a(t)y_x = v(t)$$

Null controllable (with $v \equiv 0$) if

$$\int_0^T a(t)\,dt > L$$

Consequence of the time-reversibility of (Bnv):

Theorem 1 holds if (Bnv) is null controllable,

i.e. if

$$\forall y^0,\ \exists u,\ \exists y \text{ such that } y(0,x) = y^0(x) \text{ and } y(T,x) = 0.$$

Local controllability of (Bnv)?

i.e.

$$|y^0|_{C^1([0,L])} \leq \epsilon \Rightarrow \exists u,\ \exists y\ /\ y(0,x) = y^0(x) \text{ and } y(T,x) = 0?$$

- Local controllability of (Bnv) $\Rightarrow$ Global controllability of (Bnv)

Proof:

$$|\lambda y^0|_{C^1([0,L])} \leq \epsilon \Rightarrow \exists u,\ y$$

We define

$$z(t,x) := \begin{cases} \frac{1}{\lambda} y(\frac{t}{\lambda}, x),\ t \leq \lambda T, \\ 0,\ t \geq \lambda T \end{cases}$$

and

$$v(t) := \begin{cases} \frac{1}{\lambda^2} u(\frac{t}{\lambda}),\ t \leq \lambda T, \\ 0,\ t \geq \lambda T. \end{cases}$$

Finally for $|y^0|_{C^1([0,L])}$ sufficiently small, we study the following problem:

$$(Bnv_2)\begin{cases} y_t + (a(t) + y)y_x = 0, \\ y(0,x) = y^0(x) \\ y(T,x) = 0. \end{cases}$$

Indeed:

If y satisfy (Bnv_2)

Then

$$(z := a(t) + y, u := a'(t))$$

satisfy (Bnv).

Idea: Schauder's fixed point theorem

Construction of a solution of the following system:

$$(\Sigma)\begin{cases} z_t + (a(t) + y)z_x = 0, \\ z(0,x) = y^0(x), \end{cases}$$

which depends continously on y.

Let

$$\begin{array}{rccc} \mathcal{F}: & C^1([0,T]\times[0,L]) & \rightarrow & C^1([0,T]\times[0,L]) \\ & y & \mapsto & z. \end{array}$$

If z satisfies (Σ) and $z(T,x)=0$, then

y is a fixed point of $\mathcal{F} \Rightarrow y$ satisfies (Bnv_2).

An approximate controllability result for KdV

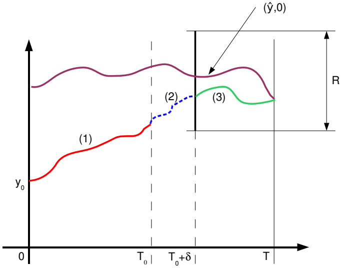
Question:

$$(Bnv)\left\{\; y_t + yy_x = u(t),\right.$$

Controllability results for the nonviscous B. equation
$\Rightarrow$
Controllability results for the nonlinear KdV equation?

$$(KdV)\left\{\begin{array}{l} y_t + y_{xxx} + y_x + yy_x = u(t), \\ y(t,0) = v_1(t),\ y(t,L) = v_2(t), \\ y_x(t,L) = 0, \end{array}\right.$$

Strategy



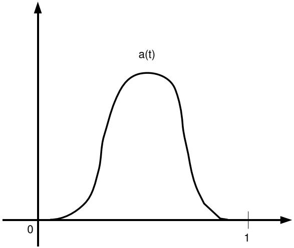
(1) Smoothing effect
(2) Approximate controllability
(3) Local exact controllability to the trajectories (Glass-Guerrero)

Where the global controllability of (Bnv) is used (1)

$|z^0|_{C^3([0,L])} \leq M$ and $|z^1|_{C^3([0,L])} \leq M$ being fixed.

- First step:
  Construction, for small times $\delta > 0$, of y^δ, a kind of approximation of the desired y.

Let



We introduce

$$a^{\delta} : t \longrightarrow \frac{1}{\delta} a(\frac{1}{\delta}).$$

Then, from the local exact controllability of the nv B. equation

$\exists y^{\delta}$ which satisfies

$$\begin{cases} y_t^{\delta} + (a^{\delta} + y^{\delta}) y_x^{\delta} = 0, \\ y^{\delta}(0, x) = z^0(x), \\ y^{\delta}(\delta, x) = z^1(x). \end{cases}$$

- Second step:

Estimation of $|y(\delta,.) - z^1(.)|_{H^1(0,L)}$ where y satisfies

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = (a^\delta)'(t), \\ y(t,0) = y^\delta(t,0) + a^\delta(t),\ y(t,L) = y^\delta(t,L) + a^\delta(t), \\ y_x(t,L) = 0, \\ y(0,x) = z^0(x). \end{cases}$$

Approximate controllability

Proposition

$\forall M > 0$, $\exists K > 0$, $\exists \delta_1 > 0$ /
$\forall 0 < \delta \leq \delta_1$,
$\forall |z^0|_{C^3([0,L])} \leq M$, $\forall |z^1|_{C^3([0,L])} \leq M$
$\exists v_1$ *and* $v_2 \in H^2(0,\delta)$ *and* $\exists u \in C^\infty([0,\delta])$ *vanishing on a neighborhood of* 0 *and* δ *such that*
the solution $y \in C^0([0,\delta]; L^2(0,L)) \cap L^2(0,\delta; H^1(0,L))$ *of*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = u(t), \\ y(t,0) = v_1(t),\ y(t,L) = v_2(t),\ y_x(t,L) = 0, \\ y(0,x) = z^0(x), \end{cases}$$

satisfies

$$|y(\delta,.) - z^1(.)|_{L^2(0,L)} \leq K\sqrt{\delta}.$$

- As a consequence of this result and the time reversibility of KdV equation, we get the global exact controllability of KdV with 4 controls (the fourth one is $y_x(t, L)$).

- Using the same strategy we can prove the controllability of the viscous Burgers equation.

Open question: can we remove the control $u(t)$ (existence of a minimal time for the controllability of KdV by means of 3 boundary controls ?)