Boundary stabilization of wave equation by means of a general multiplier technique

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Control and Inverse Problems in PDE : Theoretical and Numerical Aspects

If Ω is a regular bounded open set of $\mathbb{R}^n$ ($n \geq 2$), we consider the following wave problem :

$$(S) \begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -(m.\nu) g(u') & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u^0 & \text{in } \Omega, \\ u'(0) = u^1 & \text{in } \Omega, \end{cases}$$

with $(u_0, u_1) \in H_D^1(\Omega) \times L^2(\Omega) := H$ where

$$H_D^1(\Omega) = \{v \in H^1(\Omega);\ v = 0 \text{ on } \partial\Omega_D\}.$$

We assume :

- $\partial\Omega_N = \{x \in \partial\Omega, m(x).\nu(x) > 0\}$,
- $\partial\Omega_D = \{x \in \partial\Omega, m(x).\nu(x) \leq 0\}$ satisfies $\mathcal{H}^{n-1}(\partial\Omega_D) > 0$,
- g is a non-decreasing function such that, for some $p \geq 1$ and $k_+ \geq k_- > 0$:

$$\forall s \in \mathbb{R}, \quad k_+|s| \geq |g(s)| \geq k_- \min(|s|, |s|^p),$$

- m is a $\mathcal{C}^1$ vector field on $\bar{\Omega}$ such that

$$\inf_{\bar{\Omega}} \mathrm{div}(m) > \sup_{\bar{\Omega}} \left(\mathrm{div}(m) - 2\lambda_m\right).$$

where λ_m denotes the smallest eigenvalue of the symmetric part of the gradient $(\nabla m)^s$.

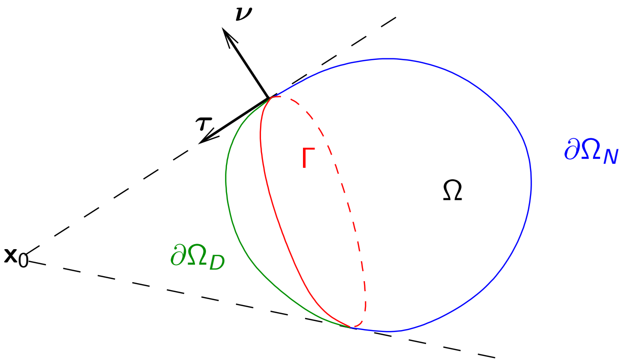

FIG.: Picture with $m(x) = x - x_0$.

Basic definitions

1. Let us define the energy function

$$E(t,u) = \frac{1}{2}\int_{\Omega} (u'(t))^2 + |\nabla u(t)|^2 dx.$$

   Our purpose is to obtain some decay estimates of $E(t,u)$ with respect to time.

2. Formally, the time derivative of E is

$$E'(t,u) = -\int_{\partial\Omega_N} (m.\nu) g(u'(t)) u'(t) d\sigma$$

3. The domain $\mathcal{D}$ associated to the wave operator is

$$\left\{(u,v) \in \left(H^1_D(\Omega)\right)^2 ;\ \Delta u \in L^2(\Omega),\ \partial_\nu u = -(m.\nu) g(v) \text{ on } \partial\Omega_N\right\}$$

Some references

- [KZ] Komornik,V., Zuazua, E., 1990, J. Math. pures et appli.,
- [BLM] Bey, R., Lohéac, J.-P., Moussaoui, M., 1999, J. Math. pures et appli.,
- [O] Osses, A., 2001, SIAM J. Control Optim.,
- [CLO] Cornilleau, P., Lohéac, J.-P., Osses, A.. To appear in J.D.C.S., 2009.

Rellich's relations

Theorem

Let $n \geq 2$ and $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$, let $u \in H^1(\Omega)$ such that

$$\Delta u \in L^2(\Omega), \quad u_{/\partial\Omega_D} \in H^{3/2}(\partial\Omega_D), \quad \partial_\nu u_{/\partial\Omega_N} \in H^{1/2}(\partial\Omega_N).$$

Then, $2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2 \in L^1(\partial\Omega)$ and there exists $\zeta \in H^{\frac{1}{2}}(\Gamma)$ such that

$$2\int_\Omega \Delta u(m.\nabla u)dx = \int_\Omega (\mathrm{div}(m)I - 2(\nabla m)^s)(\nabla u, \nabla u)dx$$
$$+ \int_{\partial\Omega} (2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2)d\sigma + \int_\Gamma (m.\tau)|\zeta|^2 d\gamma$$

Remark

The last term has to be understood in the summation over the discrete set Γ if $n = 2$.

Rellich's relations

Remark

- One shows that under these hypotheses, u can be written as the sum of a part which belongs to $H^2(\Omega)$ and the solution of a mixed problem :

$$\begin{cases} -\Delta u_S = f & \text{in } \Omega, \\ u_S = 0 & \text{on } \partial\Omega_D, \\ \partial_\nu u_S = 0 & \text{on } \partial\Omega_N. \end{cases}$$

  for some $f \in L^2(\Omega)$.
  This solution is basically a coefficient ζ times a singular function, which brings the last term in Rellich's identity.

- For $n = 2$, the singular function is the well-known Shamir function on the domain $\Omega_0 = \{(r, \theta), 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi\}$:

$$U_s(r, \theta) = \rho(r)\sqrt{r}\sin\left(\frac{\theta}{2}\right).$$

Stabilization results

For the stabilization result, we only need

$$2\int_\Omega \Delta u(m.\nabla u)dx \leq \int_\Omega (\mathrm{div}(m)I - 2(\nabla m)^s)(\nabla u, \nabla u)dx + \int_{\partial\Omega} (2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2)d\sigma$$

Theorem

Assuming that $m.\tau \leq 0$ on Γ; then, for every $(u_0, u_1) \in H$, there exists $C, T > 0$ such that the energy of the solution u of (S) satisfies :

$$\text{if} \quad p = 1, \quad E(u,t) \leq E(u,0)\exp\left(1 - \frac{t}{C}\right) \quad \forall t > T,$$

$$\text{if} \quad p > 1, \quad E(u,t) \leq Ct^{2/(1-p)} \quad \forall t > T.$$

Remarks

- C depends on the initial energy $E(u, 0)$ in the second case, not in the first,

Remarks

- this result extends previous ones :

BLM $m(x) = (x - x_0)$,

CLO $m(x) = (dI + A)(x - x_0)$ with A a skew-symmetric matrix,

Remarks

- new cases are given by this framework :
  1. $m(x) = (A_1 + A_2)(x - x_0)$ with A_1 some positive definite matrix and A_2 any skew symmetric matrix,
  2. $m(x) = (dI + A)(x - x_0) + F(x)$ with $\|(\nabla F)^s\|_\infty < \frac{d}{n}$ and A a skew-symmetric matrix.

Sketch of proof for Rellich's relations

Away from the interface Γ, the solution is regular. Hence, with $\Omega_\varepsilon = \{x \in \Omega;\ d(x,\Gamma) > \varepsilon\}$, we have the following identity :

Rellich identity in regular case

$$2\int_{\Omega_\varepsilon} \Delta u (m.\nabla u) dx = \int_{\Omega_\varepsilon} (\mathrm{div}(m) I - 2(\nabla m)^s)(\nabla u, \nabla u) dx + \int_{\partial\Omega_\varepsilon} (2\partial_\nu u (m.\nabla u) - (m.\nu)|\nabla u|^2) d\sigma.$$

Sketch of proof for Rellich's relations

Let's define : $\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega$, $\partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$.

- We now analyze the behavior of each integral terms on Ω_ε or $\widetilde{\partial\Omega_\varepsilon}$ thanks to Lebesgue's theorems : their limits are the corresponding terms on Ω or $\partial\Omega$.
- The remaining term to treat is $I_\varepsilon(\nabla u) = \displaystyle\int_{\partial\Omega_\varepsilon^*} 2\partial_\nu u(m.\nabla u) - (m.\nu)|\nabla u|^2 d\sigma.$

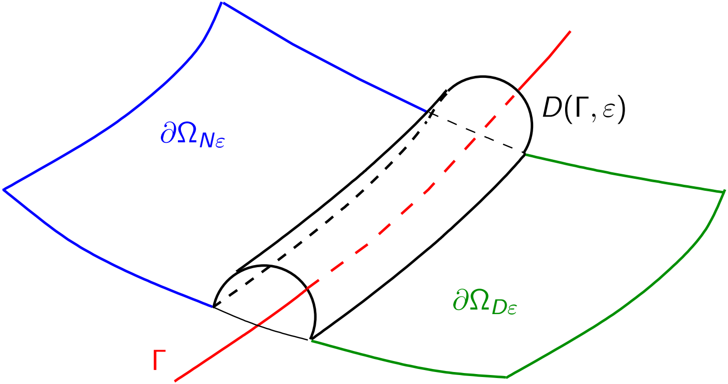

FIG.: Picture of $\partial\Omega_\varepsilon^*$ and $\widetilde{\partial\Omega_\varepsilon}$

Behavior of $I_\varepsilon(\nabla u)$

We first split,locally near any point $x^* \in \Gamma$, the gradient of u in its component along Γ and $< \nu^*, \tau^* >$:

$$\nabla u = \nabla_T u + \nabla_2 u.$$

Regularity result on $\nabla_T u$ shows that is suffices to study $I_\varepsilon(\nabla_2 u)$. We now split u in $u_R + \eta \otimes U_s^{x^*}$, its regular part and singular part. Using regularity of u_R, we only have to deal with the following term

$$\int_{C_\varepsilon(x^*)} 2(\nu.\nabla U_s^{x^*})(m.\nabla U_s^{x^*}) - (m.\nu)|\nabla U_s^{x^*}|^2 dl$$

and this can be one using the following elementary identity :

$$2\partial_\nu U_s^{x^*}(m.\nabla U_s^{x^*}) - (m.\nu)|\nabla U_s^{x^*}|^2 = \frac{1}{4\varepsilon} m.\tau^* \text{ on } C_\varepsilon(x^*).$$

Sketch of proof for stabilization

Proposition (Komornik)

Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ a non-increasing function such that there exists $q \geq 0$ and $C > 0$ which fulfill :

$$\forall t \geq 0, \int_t^\infty E^{q+1}(s)ds \leq CE(t).$$

Then, setting $T = CE^q(0)$, one has :

$$\text{if} \quad q = 0, \qquad E(t) \leq E(0) \exp\left(1 - \frac{t}{T}\right) \qquad \forall t \geq T,$$

$$\text{if} \quad q > 0, \qquad E(t) \leq E(0) \left(\frac{T + qT}{T + qt}\right)^{\frac{1}{q}} \quad \forall t \geq T.$$

▶We already know that, under the hypotheses on g and m, the energy is non increasing ; it remains to check the integral inequality.

The multiplier method

The main idea is thus to perform the multiplier method for

$$Mu := \alpha u + 2m.\nabla u,$$

which gives an energy estimate for strong solutions :

Lemma

There exists $c(\alpha) > 0$ such that, for any $0 \leq S < T < \infty$:

$$c(\alpha) \int_S^T E^{\frac{p-1}{2}} \left(\int_\Omega (u')^2 + |\nabla u|^2 dx \right) dt \leq$$

$$\left[E^{\frac{p-1}{2}} \left(\int_\Omega u' Mu dx \right) \right]_S^T + \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \left(\int_\Omega u' Mu dx \right) dt$$

$$+ \int_S^T E^{\frac{p-1}{2}} \left(\int_{\partial\Omega_N} (m.\nu)((u')^2 - |\nabla u|^2 - g(u')Mu) d\sigma \right) dt.$$

We obtain, using that $(u(t), u'(t)) \in \mathcal{D}$ for all t and Rellich's relations, an estimate of the integral on $[S, T]$ of the term

$$E^{\frac{p-1}{2}} \int_\Omega (\mathrm{div}(m) - \alpha)(u')^2 + ((\alpha - \mathrm{div}(m))I + 2(\nabla m)^s)(\nabla u, \nabla u) dx.$$

Let us choose α so that, uniformly on Ω,

$$\mathrm{div}(m) - \alpha > 0,$$
$$(\alpha - \mathrm{div}(m))I + 2(\nabla m)^s \text{ is positive definite.}$$

The optimal choice of α consequently gives

$$c = \frac{1}{2}\left(\inf_{\bar\Omega} \mathrm{div}(m) - \sup_{\bar\Omega}\left(\mathrm{div}(m) - \lambda_m\right)\right).$$

Sketch of proof for stabilization

Now, we estimate the right hand side terms using some routine inequalities :

- Young's is very useful in all this situation : it allows us to estimate the term $\int_{\partial\Omega_N} (m.\nu) g(u') M u d\sigma$,
- Poincaré's is used to estimate the term $\int_{\partial\Omega_N} u^2 m.\nu d\sigma$,
- Jensen's is used for the term $\int_{\partial\Omega_N} ((u')^2 + g(u')^2) m.\nu d\sigma$.

We finally obtain, for every $\varepsilon < \varepsilon_0$:

$$2c \int_S^T E^{\frac{p+1}{2}} dt \leq C(\varepsilon) E(S) + \varepsilon C \int_S^T E^{\frac{p+1}{2}} dt.$$

which gives the result for sufficiently small ε.

Numerical aspects

We choose $\Omega = (0,1)^2$ and the rotated multiplier $m(x) = R_\theta(x - x_0)$, where $x_0 \in \mathcal{D}_\theta$.

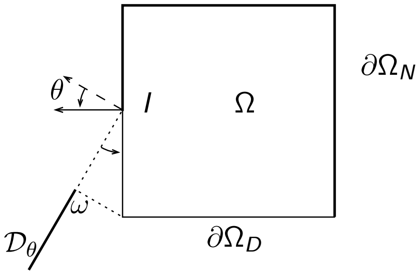

FIG.: When $\mathbf{x}_0$ belongs to $\mathcal{D}_\theta$, our geometrical condition is satisfied at $I = (0, 1/2)$.

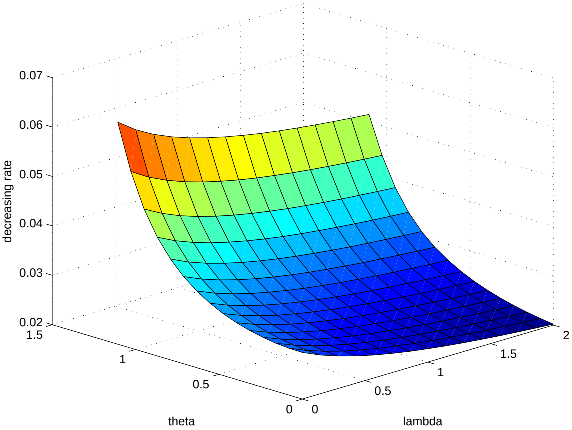

FIG.: Dependance of the decreasing rate with respect to $\lambda = |x_0 - \omega|, \ \theta$.

Comments and open problems

- Delayed wave equation We also obtained an exponential stabilization result in the case of a boundary feedback law

$$\partial_\nu u + (m.\nu)\mu_0 u'(t) + (m.\nu)\int_0^t u'(t-s)d\mu(s) = 0 \text{ on } \partial\Omega_N.$$

if, for some $\alpha > 0$, $\int_0^\infty e^{\alpha s} d|\mu|(s) < \mu_0$.

- A lot remains to be done...
  1. If $n = 2$, we also know that if $\partial\Omega$ is piecewise $\mathcal{C}^2$, Rellich's relations remains true if the angles on Γ are convex. What happens if $n \geq 3$?
  2. It seems that the condition $m.\tau \leq 0$ on Γ is unnecessary to get stabilization results, but we don't have any proof yet.