About the controllability of systems of parabolic equations

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Main control problems

Let $\Omega \subset \mathbf{R}^N$ open and smooth set. Let $\omega, \mathcal{O} \subset \Omega$ be a nonempty subsets and $Q = \Omega \times (0, T); \Sigma = \partial\Omega \times (0, T)$, $\nu > 0$.

Main control problems

We consider

$$\begin{cases} y_t - \Delta y + a(x,t)y = h\chi_\omega; & q_t - \nu\Delta q + b(x,t)q = y\chi_{\mathcal{O}} & \text{in } Q, \\ y = 0; & q = 0 & \text{on } \Sigma, \\ y(0) = y^0; & q(0) = q^0 & \text{in } \Omega \end{cases}$$

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Control problem: For every $y^0, q^0 \in L^2(\Omega)$ and $T > 0$ does there exists $h \in L^2(Q)$ such that simultaneously $y(T) = q(T) = 0$?

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or, for any $\varepsilon > 0$ and y^1, q^1 does there exits h such that

$$\|q(T) - q^1\| + \|y(T) - y^1\| \leq \varepsilon$$

Main control problems

Insensitizing controls:
Consider

$$\begin{cases} y_t - \Delta y + a(x,t)y = h\chi_\omega + \xi; & -q_t - \Delta q + b(x,t)q = y\chi_{\mathcal{O}} & \text{in } Q, \\ y = 0; & q = 0 & \text{on } \Sigma, \\ y(0) = y^0; & q(T) = 0 & \text{in } \Omega \end{cases}$$

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$$\|q(0)\| \leq \varepsilon$$

Main control problem

Boundary control:

$$\begin{cases} y_t - \Delta y = 0; & q_t - \nu\Delta q = y & \text{in } Q, \\ y = h; & q = 0 & \text{on } \Gamma_0 \times (0,T), \\ y = 0; & q = 0 & \text{on } \partial\Omega\backslash\Gamma_0 \times (0,T), \\ y(0) = y^0; & q(0) = q^0 & \text{in } \Omega \end{cases}$$

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Does there exists h such that

$$y(T) = q(T) = 0?$$

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or, for any $\varepsilon > 0$ and y^1, q^1 does there exits h such that

$$\|q(T) - q^1\| + \|y(T) - y^1\| \leq \varepsilon$$

Previous results

Recall what do we know for a single equation

Previous results

$$(2) \qquad \begin{cases} w_t - \Delta w = h\chi_\omega \text{ in } Q; \\ w = 0; \text{ on } \Sigma, \\ w(0) = w^0; \text{ in } \Omega \end{cases}$$

Previous results

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Theorem (Fabre-Puel-Zuazua)

For every non-empty set $\omega \subset \Omega$, for every $T > 0$, for every $w^0, w^1 \in L^2(\Omega)$ and $\varepsilon > 0$ there exists $h \in L^2(\omega \times (0,T))$ such that $\|w(T) - w^1\| \leq \varepsilon$.

Previous results

Key point in the proof: Take the adjoint system to (2)

$$(3) \qquad \begin{cases} -\varphi_t - \Delta\varphi = 0 \text{ in } Q; \\ \varphi = 0; \text{ on } \Sigma, \\ \varphi(T) = \varphi^0; \text{ in } \Omega \end{cases}$$

Then the approximate controllability of (2) is equivalent to the following unique continuation property for (3):

$$\varphi = 0 \text{ in } \omega \times (0,T) \Rightarrow \varphi^0 \equiv 0(\text{ and then} \varphi \equiv 0 \text{ in } Q).$$

Previous results

Theorem (Lebeau-Robbiano, Fursikov-Imanuvilov)

For every non-empty set $\omega \subset \Omega$, for every $T > 0$ and for every $w^0 \in L^2(\Omega)$ there exists $h \in L^2(\omega \times (0,T))$ such that $w(T) = 0$. (Similar results for boundary control)

Previous results

Key point for both proofs: Take the adjoint system to (2)

$$(3) \qquad \begin{cases} -\varphi_t - \Delta\varphi = 0 \text{ in } Q; \\ \varphi = 0; \text{ on } \Sigma, \\ \varphi(T) = \varphi^0; \text{ in } \Omega \end{cases}$$

THEOREM

(2) is null controllable at time T with controls

$$\|h\|_{L^2} \leq C\|w^0\|_{L^2}$$

if and only if, there exits a constant $C > 0$ such that, for every $\varphi^0 \in L^2(\Omega)$, the corresponding solution to (3) satisfies the observability inequality:

$$(4) \qquad \int_\Omega |\varphi(0)|^2 dx \leq C \int_0^T \int_\omega |\varphi|^2 dxdt.$$

Idea of the proof:

$\Rightarrow$) We mutiply (3) by w and integrate:

$$-\int_0^T\int_\Omega \varphi_t w - \int_0^T\int_\Omega \Delta\varphi w = 0$$

Integrating by parts we get:

$$-\int_\Omega \varphi(T)w(T) + \int_\Omega \varphi(0)w^0 = \int_0^T\int_\omega h\varphi$$

$$\int_\Omega \varphi(0)w^0 \leq C\|w^0\|_{L^2}(\int_0^T\int_\omega \varphi^2)^{1/2}$$

Previous results

$\Leftarrow$)(Idea of the proof)
The observability inequality helps to define a functional that REACHES a MINIMUM. We obtain a control from this minimum...

Previous results

How to prove the observability inequality (4)?

PROOF (F-I " CARLEMAN INEQUALITY")

$$\left\{ \begin{array}{c} s\lambda^2 \iint_Q e^{-2s\alpha}\rho|\nabla\varphi|^2 + s^3\lambda^4 \iint_Q e^{-2s\alpha}\rho^3|\varphi|^2 \leq \\ C_1\left(s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\rho^3|\varphi|^2 \right), \end{array} \right.$$

where

$$\alpha(x,t) = \frac{e^{\lambda m||\eta^0||_\infty} - e^{\lambda\left(m||\eta^0||_\infty + \eta^0(x)\right)}}{t(T-t)}, \quad \rho(x,t) = \frac{e^{\lambda\left(m||\eta^0||_\infty + \eta^0(x)\right)}}{t(T-t)}$$

with s and λ positive reals and $m > 1$ fixed. η^0 is a particular function constructed for $(\omega;\Omega)$.

Proof (Carleman inequality implies the observability inequality)

Observe that

$$e^{-2s\alpha}\rho^3 \geq e^{3\lambda_1 m||\eta^0||_\infty}\frac{2^{12}}{3^3}T^{-6}e^{2^5 M_0 s/3T^2} \quad \forall (x,t)\in\overline{\Omega}\times[T/4,3T/4],$$

and from Carleman we obtain

$$\iint_{\Omega\times[T/4,3T/4]}|\varphi|^2 \leq C\iint_{\omega\times(0,T)}|\varphi|^2$$

Energy estimates for φ imply

$$||\varphi(0)||^2_{L^2(\Omega)} \leq ||\varphi(T/4)||^2_{L^2(\Omega)} \leq \frac{T}{2}\iint_{\Omega\times(T/4,3T/4)}|\varphi(x,t)|^2.$$

Boundary control for one equation

For one equation internal controllability implies boundary controllability.

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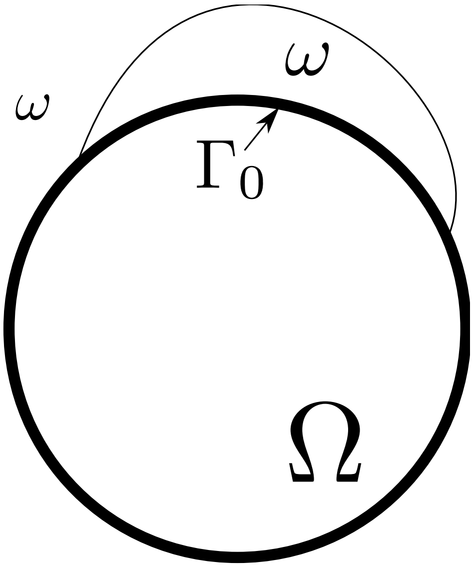
In fact:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } Q, \\ y = h & \text{on } \Gamma_0 \times (0,T) \\ y = 0 & \text{on } \partial\Omega \backslash \Gamma_0 \times (0,T) \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases}$$

$$\tilde{\Omega} = \Omega \cup \omega$$



Extension

Solve the problem

$$\begin{cases} \widetilde{y}_t - \Delta \widetilde{y} = h 1_\omega & \text{in } \widetilde{\Omega} \times (0,T), \\ \widetilde{y} = 0 & \text{on } \partial\widetilde{\Omega} \times (0,T) \\ \widetilde{y}(x,0) = y^0(x) 1_\Omega & \text{in } \widetilde{\Omega}, \\ \widetilde{y}(T) = 0 \end{cases}$$

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$\widetilde{y}|_\Omega = y$ solves

$$\begin{cases} y_t - \Delta y = 0 & \text{in } \Omega \times (0,T), \\ y = \widetilde{y} & \text{on } \Gamma_0 \times (0,T) \\ y = 0 & \text{on } \partial\Omega \backslash \Gamma_0 \times (0,T) \\ y(x,0) = y^0(x) & \text{in } \widetilde{\Omega}, \\ y(T) = 0 & \end{cases}$$

Two equations

Applying the same technique we get

$$\begin{cases} \widetilde{y}_t - \nu\Delta\widetilde{y} = h1_\omega; & \widetilde{q}_t - \Delta\widetilde{q} = \widetilde{y} \quad \text{in } \widetilde{\Omega}\times(0,T), \\ \widetilde{y} = 0; & \widetilde{q} = 0 \quad \text{on } \partial\widetilde{\Omega}\times(0,T), \\ \widetilde{y}(0) = y^0 1_\Omega; & \widetilde{q}(0) = q^0 1_\Omega \quad \text{in } \widetilde{\Omega} \\ \widetilde{y}(T) = 0 & \widetilde{q}(T) = 0 \quad \text{in } \widetilde{\Omega} \end{cases}$$

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and then

$$\begin{cases} y_t - \nu\Delta y = 0; & q_t - \Delta q = y & \text{in } \Omega\times(0,T), \\ y = \widetilde{y}; & q = \widetilde{q} & \text{on } \Gamma_0\times(0,T), \\ y = 0; & q = 0 & \text{on } \partial\Omega\backslash\Gamma_0\times(0,T), \\ y(0) = y^0; & q(0) = q^0 & \text{in } \Omega \\ y(T) = 0 & q(T) = 0 & \text{in } \Omega \end{cases}$$

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So we get TWO controls i.e. $\widetilde{y}, \widetilde{q}$!!

Results for systems

- Boundary Control results (Fdez-Cara-G-Burgos-deT).
- Distributed control: $\mathcal{O} \cap \omega \neq \emptyset$.
  Insensitizing controls (deT, deT-Zuazua) and null controllability (deT, G-Burgos-deT)
- Distributed control: Approximate Controllability. $\mathcal{O} \cap \omega = \emptyset$ (Kavian-deT)
- Further Results and open problems. (Benabdallah et all, Guerrero, Fdez-Cara et all etc.)

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One dimensional framework

Consider

$$\begin{cases} y_t - \nu y_{xx} = 0 & q_t - q_{xx} = y & \text{in } \; Q = (0,T) \times (0,1)\,, \\ y(t,0) = h & q(t,0) = 0 & \text{in } (0,T)\,, \\ y(t,1) = 0 & q(t,1) = 0 & \text{in } (0,T) \\ y(.,0) = y^0 & q(.,0) = q^0 & \text{in } \; (0,1)\,, \end{cases}$$

The approximate controllability is equivalent to a unique continuation property for the adjoint system

$$\begin{cases} -\tilde{\varphi}_t - \nu \tilde{\varphi}_{xx} = \tilde{\psi} & -\tilde{\psi}_t - \tilde{\psi}_{xx} = 0 & \text{in } \; Q, \\ \tilde{\varphi}(t,0) = \tilde{\varphi}(t,1) = 0 & \tilde{\psi}(t,0) = \tilde{\psi}(t,1) = 0 & t \in (0,T), \\ \tilde{\varphi}(.,T) = \tilde{\varphi}^0 & \tilde{\psi}(.,T) = \tilde{\psi}^0 & \text{in } \; (0,1)\,, \end{cases}$$

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Does $\tilde{\varphi}_x|_{x=0} = 0$ implies $\tilde{\psi} \equiv \tilde{\varphi} \equiv 0$?

One dimensional framework

Theorem

Suppose that $\nu \neq 1$ then, the unique continuation property holds true if and only if $\sqrt{\nu} \notin \mathbb{Q}$. In other words, for $\nu \neq 1$, system is approximately controllable at time $T > 0$ if and only if $\sqrt{\nu} \notin \mathbb{Q}$.

Simplify notation

Define $\varphi(t) = \tilde{\varphi}(T-t), \quad \psi(t) = \tilde{\psi}(T-t)$, then

$$\left\{ \begin{array}{lll} \varphi_t - \nu\varphi_{xx} = \psi & \psi_t - \psi_{xx} = 0 & \text{in } Q, \\ \varphi(t,0) = \varphi(t,1) = 0 & \psi(t,0) = \psi(t,1) = 0 & t \in (0,T), \\ \varphi(.,0) = \tilde{\varphi}^0 & \psi(.,0) = \tilde{\psi}^0 & \text{in } (0,1)\,, \end{array} \right.$$

The counterexample (Fdez-Cara, González-Burgos, deT)

Let $w_j(x) = \sin(\pi j x)$ denote the eigenfunctions of the Dirichlet Laplacian in $(0,1)$, for the eigenvalue $\lambda_j = \pi^2 j^2$
Then

$$\varphi(x,t) = \sum_{j\geq 1} (a_j - \frac{b_j}{(\nu - 1)\lambda_j}) e^{-\nu\lambda_j t} w_j(x) + \sum_{j\geq 1} \frac{b_j}{(\nu - 1)\lambda_j} e^{-\lambda_j t} w_j(x),$$

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with $b_j = \int_0^1 \tilde{\psi}^0(x)\sin(\pi j x)dx$, $a_j = \int_0^1 \tilde{\varphi}^0(x)\sin(\pi j x)dx$.

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$$\varphi_x(0,t) = \sum_{j\geq 1}(j\pi)\left((a_j - \frac{b_j}{(\nu-1)\lambda_j})e^{-\nu\lambda_j t} + \sum_{j\geq 1}\frac{b_j}{(\nu-1)\lambda_j}e^{-\lambda_j t}\right)$$

The counterexample (Fdez-Cara, González-Burgos, deT)

Suppose now that $\sqrt{\nu} \in \mathbb{Q}$. That means that $\nu = \frac{j_0^2}{i_0^2}$ and then

$$\nu i_0^2 = j_0^2, \qquad \nu \lambda_{i_0} = \lambda_{j_0}.$$

Take now

$$b_{j_0} = 0, a_{j_0} = 1$$

and

$$b_{i_0} = (\nu - 1)\pi^2 i_0^2, \quad a_{i_0} = \frac{b_{i_0}}{(\nu - 1)\lambda_{i_0}}.$$

Then, $\varphi_x(0,t) = 0$ in $(0,T)$ but $\varphi \neq 0, \psi \neq 0$.

Null controllability

THEOREM (FDEZ-CARA, GLEZ-BURGOS, DET)

Suppose that $\nu = 1$ then, system

$$\begin{cases} y_t - y_{xx} = 0 & q_t - q_{xx} = y & \text{in } (0,T)\times(0,1), \\ y(t,0) = h & q(t,0) = 0 & \text{in } (0,T), \\ y(t,1) = 0 & q(t,1) = 0 & \text{in } (0,T) \\ y(.,0) = y^0 & q(.,0) = q^0 & \text{in } (0,1), \end{cases}$$

is null controllable at time $T > 0$.

Null controllability

Theorem (Fdez-Cara, Glez-Burgos, deT)

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is null controllable at time $T > 0$.

Obtain the observability inequality using Fattorini-Russell technique.

Insensitizing controls

Our problem, control $q(0)$ solution to

$$\begin{cases} y_t - \Delta y = h\chi_\omega + \xi; & -q_t - \Delta q = y\chi_{\mathcal{O}} & \text{in } Q, \\ y = 0; & q = 0 & \text{on } \Sigma, \\ y(0) = y^0; & q(T) = 0 & \text{in } \Omega \end{cases}$$

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Adjoint system:

$$\begin{cases} -z_t - \Delta z = p\chi_{\mathcal{O}}; & p_t - \Delta p = 0 & \text{in } Q, \\ z = 0; & p = 0 & \text{on } \Sigma, \\ z(x,T) = 0; & p(x,0) = p^0 & \text{in } \Omega. \end{cases}$$

Adjoint system:

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Does there exist $C > 0$ such that

$$\int_\Omega |z(0)|^2 dx \leq C \iint_{\omega\times(0,T)} |z|^2 dxdt? \tag{5}$$

THEOREM (DET,99)

Inequality (5) is not valid when $\mathcal{O} = \Omega$ (the must simple case) and $\Omega \backslash \bar{\omega} \neq \emptyset$.

Main results I

Theorem (deT,99)

Inequality (5) is not valid when $\mathcal{O} = \Omega$ (the must simple case) and $\Omega \backslash \bar{\omega} \neq \emptyset$.

Main point

One equation is FORWARD and the other is BACKWARD. We have the following result when $\mathcal{O} \cap \omega \neq \emptyset$ (González-Burgos, deT. [2008])

$$\begin{cases} -z_t - \Delta z = p\chi_{\mathcal{O}}; & -p_t - \Delta p = 0 & \text{in } Q, \\ z = 0; & p = 0 & \text{on } \Sigma, \\ z(x,T) = z^0; & p(x,T) = p^0 & \text{in } \Omega. \end{cases}$$

Then, there exist $C > 0$ such that

$$\int_\Omega |z(0)|^2 dx + \int_\Omega |p(0)|^2 dx \leq C \iint_{\omega \times (0,T)} |z|^2 dxdt$$

Main results I

Theorem (deT,99)

Inequality (5) is not valid when $\mathcal{O} = \Omega$ (the must simple case) and $\Omega \backslash \bar{\omega} \neq \emptyset$.

Proof

Without lost of generality suppose that $0 \in \Omega$ but $0 \notin \omega$.

$$\begin{cases} -z_t - \Delta z = p; & p_t - \Delta p = 0 & \text{in } Q, \\ z = 0; & p = 0 & \text{on } \Sigma, \\ z(x,T) = 0; & p(x,0) = \Delta \delta_0 & \text{in } \Omega. \end{cases}$$

$$z(0) \notin L^2(\Omega)!!$$

but

$$\int_0^T \int_\omega z^2 dxdt < \infty.$$

Main results I

Take $p_n(0) = p(x, 1/n)$. Then

$$\lim_{n\to\infty} \frac{\int_\Omega z_n^2(0)dx}{\int_0^T \int_\omega z_n^2 dxdt} = \infty$$

.

Main results I

THEOREM (DET, ZUAZUA 2009)

Sea $\mathcal{O} = \Omega$, then there exist $B, C > 0$ such that for every z solution to the adjoint system verifies

$$\sum_{j=1}^{\infty} e^{-B\sqrt{\lambda_j}} |z_{j,0}|^2 \leq C \int_0^T \int_\omega z^2 dxdt$$

where λ_j are the eigenvalues corresponding to eigenvectors φ_j of the Dirichlet Laplacian.

Main results I

What happens if $\mathcal{O} \neq \Omega$?

Main results I

Theorem (deT,Zuazua 2009)

Let Ω be a bounded open interval of ***R****. There exist non empty subdomains $\mathcal{O} \neq \Omega$, $\mathcal{O} \not\subset \omega$ such that the spectral inequality*

$$| \int_{\Omega} z(x,0)\varphi_1(x)dx|^2 \leq C \int_0^T \int_{\omega} z^2 \, dx \, dt,$$

fails. Here φ_1 stands for the first eigenfunction of the Dirichlet Laplacian in Ω. In other words,

$$\sup_{(z,p) \text{ solutions of the adjoint}} \frac{| \int_{\Omega} z(x,0)\varphi_1(x)dx|^2}{\int_0^T \int_{\omega} z^2 \, dx \, dt} = \infty.$$

Main results I

THEOREM (DET,99)

Suppose that $\omega \cap \mathcal{O} \neq \emptyset$ *then there exist* $M > 0$ *large enough and* $C > 0$*, such that every solution to the adjoint system satisfies*

$$\int_0^T \int_\Omega |p|^2 e^{\frac{-M}{t}} dxdt + \int_0^T \int_\Omega |z|^2 e^{\frac{-M}{t}} dxdt \leq C \int_0^T \int_\omega z^2 dxdt.$$

Main results I

Controllability result when $\mathcal{O} \cap \omega \neq \emptyset$

(deT99) If $\int_0^T\int_\Omega e^{M/t}\xi^2 dxdt < \infty$ then there exists a control $h \in L^2$ such that

$$\begin{cases} y_t - \Delta y = h\chi_\omega + \xi; & -q_t - \Delta q = y\chi_{\mathcal{O}} & \text{in } Q, \\ y = 0; & q = 0 & \text{on } \Sigma, \\ y(0) = 0; & q(T) = 0 & \text{in } \Omega \end{cases}$$

satisfies

$$q(0) = 0.$$

Moreover, (González-Burgos, Pérez-García 2004) if $\int_0^T\int_\Omega e^{M/t^3(T-t)^3}\xi^2 dxdt < \infty$ then there exists $h \in L^2$ such that $q(0) = 0$ and $y(T) = 0$. (Null control and SIMULTANEOUSLY insensitizing control)

Main results I

Proof (Idea of the proof)

- *Carleman estimates for z and p and $\omega_0 \subset \mathcal{O} \cap \omega$*

$$\iint_{\Omega\times(0,T)} s^3\lambda^4\rho^3 e^{-2s\alpha}|z|^2 dxdt + \iint_{\Omega\times(0,T)} s^3\lambda^4\rho^3 e^{-2s\alpha}|p|^2 dxdt$$
$$\leq Cs^3\lambda^4\left(\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\rho^3|p|^2 dxdt + \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\rho^3|z|^2\right)$$

- *Eliminate the term in p "growing" ω_0 to ω + local energy estimates.*
- *Energy estimates (on p) allow to change the "weight" to $e^{-M/t}$.*

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- *Eliminate the term in p "growing" ω_0 to ω + local energy estimates.*
- *Energy estimates (on p) allow to change the "weight" to $e^{-M/t}$.*

Approximate controllability is equivalent to a unique continuation property for the adjoint system. That is: $z \equiv 0$ in $\omega \times (0, T)$

$$\Rightarrow p^0 = 0 \text{ in } \Omega \text{ (and } p \text{ and } z \text{ in } Q)$$

for (p, z) solution to the adjoint system

$$\begin{cases} -z_t - \Delta z = p\chi_{\mathcal{O}}; & p_t - \Delta p = 0 & \text{in } Q, \\ z = 0; & p = 0 & \text{on } \Sigma, \ \ ? \\ z(x, T) = 0; & p(x, 0) = p_0 & \text{in } \Omega. \end{cases}$$

Some previous results

Carleman inequality implies unique continuation for ONE equation

$$\begin{cases} -u_t - \Delta u + a(x,t)u = 0 & \text{in } (0,T)\times\Omega = Q, \\ u = 0 & \text{on } (0,T)\times\partial\Omega \end{cases}$$

That is, if

$$u = 0 \text{ in } (0,T) \times \omega \subset Q$$

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Then

$$u \equiv 0 \text{ in } Q.$$

We also have "backward uniqueness" : if for $T > 0$ one has $u(T) = 0$ then $u \equiv 0 \equiv u_0$.

ε-insensitizing controls

We also have “backward uniqueness” : if for $T > 0$ one has $u(T) = 0$ then $u \equiv 0 \equiv u_0$. This is a consequence of the convexity of: $t \mapsto \log \|u(t)\|^2$ that implies

$$\forall t \in (0,T), \quad \|u(t)\| \leq \|u_0\|^{(T-t)/T} \|u(T)\|^{t/T}.$$

Go back to our problem

Unique continuation for cascade systems

Unique continuation for the adjoint system: $z \equiv 0$ in $\omega \times (0,T)$

$$\Rightarrow p^0 = 0 \text{ in } \Omega \text{ (and } p \text{ and } z \text{ in } Q\text{)}$$

for (p,z) solution to the adjoint system

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Unique continuation for cascade systems

Theorem (Bodart-Fabre, 1995)

Suppose that $\mathcal{O} \cap \omega \neq \emptyset$, *then* $p_0 = 0$, $z, p \equiv 0$.

Unique continuation for cascade systems

Theorem (Bodart-Fabre, 1995)

Suppose that $\mathcal{O} \cap \omega \neq \emptyset$*, then* $p_0 = 0$*,* $z, p \equiv 0$*.*

Proof

$$z \equiv 0 \text{ in } \omega \times (0,T) \Rightarrow p \equiv 0 \text{ in } \omega \cap \mathcal{O} \times (0,T)$$

Classic unique continuation result for p *solution to*

$$\begin{cases} p_t - \Delta p = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x,0) = p^0 & \text{in } \Omega. \end{cases}$$

That implies

$$p \equiv 0 \text{ in } Q \text{ and then } p^0 = 0$$

Unique continuation for cascade systems

Theorem (General case (Kavian, deT (COCV 2009)))

$$z \equiv 0 \textit{ in } \omega \times (0,T) \Rightarrow p^0 = 0 \textit{ in } \Omega$$

Unique continuation for cascade systems

Same result for two FORWARD equations.

Idea of the proof

- Express the solution in its Fourier series.
- Obtain two equal series in $\omega \times (0, T)$
- Do an analytic extension in a particular point obtaining a complex exponential.
- Obtain an auxiliary funtional ($\Rightarrow p(T) = 0$). Backward uniqueness for p implies $p_0 = 0$...

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Further results: Boundary control

Take $y = (y_1, y_2)^*$. Consider system

$$\begin{cases} y_t - y_{xx} = Ay & \text{in } Q = (0,T)\times(0,1), \\ y(0,\cdot) = Bv, \quad y(1,\cdot) = 0 & \text{in } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,1), \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbf{R}^2$, $v \in L^2(0,T)$ is a control function.

THEOREM

Suppose that

$$(Kalman) \qquad \text{rank}\,[B\,|\,AB] = 2,$$

and $\pi^{-2}(\mu_1 - \mu_2) \neq j^2 - k^2$ *for every* $k, j \in \mathbb{N}$, *and* μ_1, μ_2 *the eigenvalues of* A. *Then, system is null controllable at time* $T > 0$.

Open Problems: Boundary control

- What happens in several dimensions?
- What happens in the non-linear case?
- That is, what happens for systems of the form

$$\begin{cases} y_t - y_{xx} + a(x,t)y = 0; & q_t - q_{xx} + b(x,t)q + c(x,t)y = y \\ y = h; & q = 0 \\ y = 0; & q = 0 \\ y(0) = y^0; & q(0) = q^0 \end{cases} \quad ?$$

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Other results and open problems

Unique continuation:

Other results and open problems

Unique continuation: The general proof is not valid in systems of the form:

$$\begin{cases} -z_t - \Delta z + a(x,t)z = p\chi_\omega; & p_t - \Delta p + b(x,t)p = 0 & \text{in } Q, \\ z = 0; & p = 0 & \text{on } \Sigma, \\ z(x,T) = 0; & p(x,0) = p_0 & \text{in } \Omega. \end{cases}$$

However in the case $\omega \cap \mathcal{O} \neq \emptyset$, the proof of Bodart-Fabre is valid.

Other results and open problems

A hope!! Hugo Leiva perturbation method maybe used to prove the approximate controllability for

$$\begin{cases} y_t - \Delta y + a(x,t)y = h\chi_\omega; & q_t - \Delta q + b(x,t)q = y\chi_{\mathcal{O}} & \text{in } Q, \\ y = 0; & q = 0 & \text{on } \Sigma, \\ y(x,0) = y^0; & q(x,0) = q^0 & \text{in } \Omega. \end{cases}$$

but it maybe require $\|a\|_\infty, \|b\|_\infty << 1$.

Non cascade systems: Benabdallah et all. Complicated. Kalman type condition.

Other results and open problems

Other coupling: Guerrero.

Other results and open problems

Insensitizing controls. For the wave equation. Dáger, Tebou.
For quasigeostrophic ocean model:Fdez-Cara,García, Osses

Other results and open problems

Other boundary conditions: Bodart,Glez-Burgos, Pérez-García.

Merci.
Thank you.
Gracias.