Analysis of the HUM Control Operator
Exact Controllabilty for the Semilinear Waves

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Exact controllability for the waves

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{in} \quad]0, +\infty[\times M \\ (u(0), \partial_t u(0)) = U_0 = (u_0, u_1) \in H^1 \times L^2 \end{cases}$$

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ω open subset of M and $T > 0$ (suitable)

The Goal

$$(v_0, v_1) \in E = H^1 \times L^2$$

Find a source $f(t,x) \in L^2_{loc}([0,+\infty[, L^2)$, $\text{supp} f \subset \omega$

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$$A : (u_0, u_1) \longrightarrow f$$

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$$\int_M (\overline{g_0}\, \partial_t u(T) - \overline{g_1}\, u(T)) dx = \int_0^T \int_M \chi(x) \left| g \right|^2 dxdt$$

$$\langle G_0, SG_0 \rangle_{E_{-1}, E} = \int_0^T \int_M \chi(x) \left| g \right|^2 dxdt$$

Observation

$$\int_0^T \int_M \chi(x) \left|g\right|^2 dxdt \geq C \left\| G_0 \right\|_{E_{-1}}^2$$

$$\left\| SG_0 \right\|_E \geq C \left\| G_0 \right\|_{E_{-1}} \quad \text{and} \quad S \quad : E_{-1} \leftrightarrow E$$

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Remark: This HUM control is solution of a variational problem: it's the only one that minimizes

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Definition: We will call HUM control operator the operator Λ defined from E into E_{-1} by

$$G_0 = \Lambda U_0$$

Two problems

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b) Treatment of the frequencies

$\longrightarrow$ Does the control ΛU_0 load the frequencies carrying the data ?
$\longrightarrow$ If U_0 has only low frequencies, how are the high frequencies of ΛU_0 ?
$\longrightarrow$ Does it handle individually the frequencies of the data U_0 ?

Control of nonlinear equations

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$\longrightarrow f$ subcritical: stabilization $\longrightarrow$ small datas....: non uniform control time

(Dehman-Lebeau-Zuazua 03') et (Dehman-Gérard-Lebeau 06' for Schrödinger).

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Program

$\longrightarrow$ Sharp analysis of the control operator $\Lambda : E \longrightarrow E_{-1}$
$\longrightarrow$ Quantification of Λ.

An abstract control

→ $\exp(itA)$, $t \geq 0$, a semi-group of contractions on a hilbert space H .
→ B bounded operator on H and $g \in L^1([0,T],H)$

$$(\partial_t - iA)f = Bg, \qquad f(0) = 0$$

$$f(T) = \int_0^T e^{i(T-t)A} Bg(t)dt$$

We choose g solution of

$$(\partial_t - iA^*)g = 0, \qquad g(T) = g_0$$

Observation

$$\int_0^T \left\| B^* e^{-itA^*} h \right\|_H^2 dt \geq C \left\| h \right\|_H^2$$

The HUM optimal control is given by

$$g(t) = B^* e^{-i(T-t)A^*} g_0$$

$$f_0 = f(T) = M_T g_0$$

$$M_T = \int_0^T e^{i(T-t)A} BB^* e^{-i(T-t)A^*} dt = \int_0^T e^{itA} BB^* e^{-itA^*} dt$$

If A is self-adjoint

$$M_T = \int_0^T e^{itA} BB^* e^{-itA} dt = \Lambda^{-1}$$

$$m(t) = e^{itA} BB^* e^{-itA}$$

Notations

$M = \Omega$ open bounded, connected and smooth subset of $\mathbb{R}^d$.

$(e_j, \omega_j^2)_{j\geq 1}$ the spectral elements of Ω.

$$-\Delta e_j = \omega_j^2 e_j, \qquad e_{j|\partial\Omega=0}, \qquad \|e_j\|_{L^2(\Omega)} = 1$$

$$H^s(\Delta) = \{u = \sum_j a_j e_j, \quad \sum_j (1+\omega_j^2)^s |a_j|^2 < \infty\} = \mathcal{D}((-\Delta_D)^{s/2})$$

$$\lambda = \lambda(x, D_x) = \sqrt{|\Delta|}$$

$$\lambda(x, D_x) \sum_j a_j e_j = \sum_j \omega_j a_j e_j$$

If M is a compact manifold without boundary, it is a pseudo-differential operator of order 1 (Helffer-Sjöstrand formula).

Littlewood-Paley Decomposition

Let $\varphi \in C_0^\infty(\mathbb{R})$, and $\psi \in C_0^\infty(\mathbb{R}^*)$ such that

$$\varphi(s) + \sum_{k=1}^{\infty} \psi(2^{-k}s) = 1, \qquad s \in \mathbb{R}$$

$$\begin{cases} \psi_0(s) = \varphi(s) & \\ \psi_k(s) = \psi(2^{-k}s), & k \geq 1 \end{cases}$$

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Spectral localization operators

$k \in \mathbb{N}$, $\quad u = \sum_j a_j e_j$,

$$\psi_k(D)u = \sum_j \psi_k(\omega_j) a_j e_j = \sum_j \psi(2^{-k}\omega_j) a_j e_j$$

$$S_k(D) = \sum_{j=0}^{k} \psi_j(D) = \psi_0(2^{-k}D), \qquad k \geq 0$$

$\longrightarrow$ If M is compact, these are pdo of order 0

$\longrightarrow$ They commute to the laplacian.

Time dependant control

$$\begin{cases} \partial_t^2 u - \Delta u = \chi g \\ U_0 = (0,0) \end{cases}$$

$$\chi(t,x) = \varphi(t)\chi_0(x) \text{ in } C^\infty(\mathbb{R}\times\overline{\Omega}),$$

$$\varphi \text{ flat at } t = 0, T \text{ and } \varphi > 0 \text{ in }]0, T[.$$

$$\omega = \{x \in \Omega, \quad \chi_0(x) \neq 0\}$$

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Geometric Control Condition (GCC)

ω satisfies (GCC) at time T if every geodesic ray of Ω travelling with speed 1 and starting at $t = 0$, enters the open set ω in a time $t < T$.

Identification

$$E = H_0^1 \times L^2 \leftrightarrow L^2 \otimes L^2$$

$$u_0 = \lambda^{-1}(h_+ + h_-), \qquad u_1 = i(h_+ - h_-)$$

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad B = \frac{1}{2i}\begin{pmatrix} \chi & \chi \\ -\chi & -\chi \end{pmatrix}$$

$$(\partial_t - iA)f = Bg, \qquad f(0) = 0$$

$$M_T = \frac{1}{2}\int_0^T \begin{pmatrix} e^{it\lambda}\chi^2 e^{-it\lambda} & -e^{it\lambda}\chi^2 e^{it\lambda} \\ -e^{-it\lambda}\chi^2 e^{-it\lambda} & e^{-it\lambda}\chi^2 e^{it\lambda} \end{pmatrix} dt = \frac{1}{2}\begin{pmatrix} Q_+ & T_+ \\ T_- & Q_- \end{pmatrix}$$

$$Q_\pm = \int_0^T e^{\pm it\lambda}\chi^2 e^{\mp it\lambda} dt, \qquad T_\pm = -\int_0^T e^{\pm it\lambda}\chi^2 e^{\pm it\lambda} dt$$

Theorem 1: Quantification of Λ

Under (GCC), $Q_{\pm}$ is an isomorphism on $H^s(\Delta)$ for every $s \geq 0$ and $T_{\pm}$ is smoothing ($T_{\pm} : L^2 \rightarrow H^{\sigma}(\Delta)$, for every σ).
Moreover, with $L_{\pm} = Q_{\pm}^{-1}$

$$\Lambda = \begin{pmatrix} 2L_+ & 0 \\ 0 & 2L_- \end{pmatrix} + R$$

where R is smoothing.
In particular Λ is an isomorphism on $H^s(\Delta) \otimes H^s(\Delta)$.

Statement of the results

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Theorem 2: Under (GCC),

$$\begin{aligned} \|\psi_k(D)\Lambda - \Lambda\psi_k(D)\|_{L^2} &\leq c2^{-k} \\ \|S_k(D)\Lambda - \Lambda S_k(D)\|_{L^2} &\leq c2^{-k} \end{aligned}$$

Remarks

1. Finally, the HUM control Λ preserves automatically the regularity of the data to be controlled.
2. Λ acts almost individually on each frequency block of the solution.
3. Λ has a pseudo-differential behavior.
4. $(e_n, e_n) \longrightarrow (e_{n+1}, e_{n+1})$: the control vector and the controlled solution live at frequency ω_n , for n large enough.
5. Numerical experiments: G.Lebeau-M.Nodet (08').
6. The regularity of the control function $\chi(t, x)$ plays a key role.

Case of a compact manifold without boundary

$(e_j, \omega_j^2)_{j\geq 0}$ the spectral elements of M.

$$\Pi_+ \sum_{j\geq 0} a_j e_j = \sum_{j\geq 1} a_j e_j$$

For $(x,\xi) \in T^*M\backslash 0$, we denote

$$\gamma(s) = \gamma_{(x,\xi)}(s), \quad s \in \mathbb{R}$$

the bicharacteristic of the wave operator starting from (x,ξ).

$$\alpha(x,\xi) = \left(\int_0^T \chi^2(\gamma(s))ds\right)^{-1}$$

$$\beta(x,\xi) = \alpha(x,-\xi)$$

Under (GCC), these are elliptic pseudo- differential symbols of order 0.

Theorem 3

Under (GCC), Λ is an elliptic pseudo- differential operator of order 0,

$$\Lambda = \Pi_+ \begin{pmatrix} 2\alpha(x,D) & 0 \\ 0 & 2\beta(x,D) \end{pmatrix} \Pi_+ + R$$

where R is a smoothing pdo.

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Corollary *:* With the same assumptions, if $U_0 = (u_0, u_1)$ and $G_0 = (g_0, g_1) = \Lambda U_0$, $s \geq 0$,

$$WF_{s+1}(u_0) \cup WF_s(u_1) = WF_s(g_0) \cup WF_{s-1}(g_1)$$

$\rightarrow$ Λ is microlocal.
$\rightarrow$ Analogous result on a domain Ω of $\mathbb{R}^d$ by G.Lebeau-J.Rauch (work in progress).

Semilinear wave equation

$M = \Omega$ open, bounded and smooth subset of $\mathbb{R}^3$, $\quad 1 \leq p < 5$
$f : \mathbb{R} \to \mathbb{R}$, of class C^3, $\quad f(0) = 0, \quad sf(s) \geq 0 \qquad \forall s \in \mathbb{R}$

$$\left| f^{(j)}(s) \right| \leq C(1 + |s|)^{p-j}, \qquad \text{for} \quad j = 1, 2, 3$$

$$\begin{cases} \Box u + f(u) = \chi g \quad \text{in } \Omega \\ u_{|\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = U_0 \in H_0^1 \times L^2 \end{cases}$$

$$U(0) = U_0 \to U(T) = U_1$$

Strichartz Inequalities

Theorem (*Burq-Lebeau-Planchon 07'***):** For every $T > 0$, $E_0 > 0$ and $g \in L^1(0, T; L^2(\Omega))$, the system

$$\begin{cases} \Box u + f(u) = g & \text{in} \quad]0, T[\times\Omega \\ u = 0 & \text{on} \quad]0, +\infty[\times\partial\Omega \\ \|u(0)\|_{H_0^1} + \|\partial_t u(0)\|_{L^2} \leq E_0 \end{cases}$$

admits a unique solution u in the class $C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ satisfying

$$\|u\|_{L^5(0,T;W^{\frac{3}{10},5})} \leq C$$

for some $C = C(T, E_0, \|g\|_{L^1(0,T;L^2)})$.
In particular, for every $q \in [5, +\infty]$, $\exists C' > 0$,

$$\|u\|_{L^q(0,T;L^{3r})} \leq C'$$

with $1/q + 1/r = 1/2$.

Examples: $L^\infty(L^6)$, $L^5(L^{10})$, $L^8(L^8)$, ...

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Composition theorem

For $p \in [4, 5[$, if $v(t, x) \in L^5(0, T; W^{\frac{3}{10},5}(\Omega))$ then

$$f(v) \in L^1(0, T; H^\mu(\Omega)),$$

with $\mu = \frac{3}{10}(5 - p)$.

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with $\mu = \frac{3}{10}(5 - p)$.

$\rightarrow f(v) = f(S_0 v) + \sum_{q\geq 0}[f(S_{q+1} v) - f(S_q v)] = f(S_0 v) + \sum_{q\geq 0} m_q v_q$

$\rightarrow$ Multipliers Lemma (Meyer, Paradifferential calculus).

Theorem 4: Semilinear waves

Under condition (GCC), for every $C_0 > 0$, there exist $r > 0$ and $k \in \mathbb{N}$, s.t for every U_0 , U_1 satisfying

$$\|U_0\|_{H^1_0\times L^2} \le C_0, \quad \|U_1\|_{H^1_0\times L^2} \le C_0$$
$$\|S_k(D)U_0\|_{H^1_0\times L^2} \le r, \quad \|S_k(D)U_1\|_{H^1_0\times L^2} \le r$$

there exists $g \in L^1(0, T; L^2)$ which exactly controls the semilinear wave equation at time T, namely, the unique solution of system

$$\begin{cases} \Box u + f(u) = \chi g \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = U_0 \end{cases}$$

satisfies $U(T) = U_1$.

$\rightarrow$ Condition on k and r : $\quad 2^{-k\alpha} + r \le A(1 + C_0)^{-B}$, where $\alpha = \frac{5-p}{4p}$

and A and B depend only on Ω and T.

Remarks

1- The low frequencies determine the nonlinear behavior of the equation.
2- For high frequencies, we have a "linear behavior".
3- The control process is achieved in uniform time: the one of linear control.
4- Condition on LF energy ? Open problem.
5- Method of proof: Fixed point process used near the linear control.
6- Main tools: Strichartz estimates and analysis of LF of the linear control.

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6- Main tools: Strichartz estimates and analysis of LF of the linear control.

Key lemma : Regularity of the nonlinear term

Remarks

1- The low frequencies determine the nonlinear behavior of the equation.
2- For high frequencies, we have a "linear behavior".
3- The control process is achieved in uniform time: the one of linear control.
4- Condition on LF energy ? Open problem.
5- Method of proof: Fixed point process used near the linear control.
6- Main tools: Strichartz estimates and analysis of LF of the linear control.

Key lemma : Regularity of the nonlinear term

Corollary: In the same framework, if (U_0^n) and (U_1^n) are two sequences of data weakly converging to zero in $H_0^1 \times L^2$, one can drive one of them to the other, by the semilinear wave evolution, at geometric control time.
In particular, one can drive (U_0^n) to 0.

Proof of Theorem 1

$$(\Lambda^{-1}) = M_T = \frac{1}{2}\begin{pmatrix} Q_+ & T_+ \\ T_- & Q_- \end{pmatrix}$$

$$Q_+ = \int_0^T e^{it\lambda}\chi^2 e^{-it\lambda}dt, \quad T_+ = -\int_0^T e^{it\lambda}\chi^2 e^{it\lambda}dt$$

Difficulty : χ^2 does not operate on the Sobolev spaces $H^s(\Delta)$

1. Q_+ is an isomorphism on L^2 : (GC) +Observation estimate

2. Q_+ is bounded on $H^s(\Delta)$

$$Q_+ = \sum_{i,j} C_{ij}, \quad C_{ij} = \int_0^T e^{it\lambda}\psi_i(D)\chi^2\psi_j(D)e^{-it\lambda}dt$$

The matrix $(2^{is}\,\|C_{ij}\|_{L^2\to L^2}\,2^{-js})$ is bounded on $l^2(\mathbb{N})$.

$$C_{ij} = \int_0^T e^{it\lambda}\lambda^{-1}\psi_i(D)(i\partial_t\chi^2 + \chi^2\lambda)\psi_j(D)e^{-it\lambda}dt$$

$$2^{is}\left\|C_{ij}\right\|_{L^2\to L^2} 2^{-js} \le C_N 2^{-(N-s)|i-j|}, \quad N \ge 1$$

3. Q_+ is an isomorphism on $H^s(\Delta)$

$$u \in L^2 \text{ and } Q_+ u \in H^s(\Delta) \Rightarrow u \in H^s(\Delta)$$

$$F_s = \{u \in L^2 \quad \text{s.t} \quad Q_+ u \in H^s(\Delta)\}$$

a) $H^s(\Delta)$ is closed in F_s

$$\|u\|_{H^s} \le c(\|Q_+ u\|_{H^s} + \|u\|_{L^2}) \quad \forall u \in H^s(\Delta)$$

b) $H^s(\Delta)$ is dense in F_s (regularization).

Lemma 1: For $a(x) \in C^\infty(\overline{\Omega})$,

$$\left\| [\psi_j(D), a(x)] \right\|_{L^2 \to L^2} \leq c 2^{-j}$$

Lemma 2:
i) The bracket $[Q_+, \lambda^{-s}]$ is bounded from L^2 to $H^{s+1}(\Delta)$ for $s \in [0, 2[$.
ii) The bracket $[Q_+, \lambda]$ is bounded on L^2.
iii) For every $s \geq 0$, the operator

$$A_s = \lambda^s Q_+ \lambda^{-s} - Q_+$$

is bounded from L^2 to $H^{1/2}(\Delta)$.
In particular, A_s is compact on L^2.

Proof of nonlinear control

$$f(u) = f'(0)u + \theta(u)$$

$$\begin{cases} \Box u + f(u) = \chi^2(g_1 + g_2) \\ U_0 \in H_0^1 \times L^2 \end{cases}$$

$$\Box g_1 + f'(0)g_1 = 0 \text{ and } G_1(0) = \Lambda(U_0, U_1)$$

$$u = v + w$$

$$\Box v + f'(0)v = \chi^2 g_1 \quad \text{and} \quad V(0) = U_0$$

then $V(T) = U_1$

$$\Box w + f'(0)w = -\theta(u) + \chi^2 g_2 \quad \text{and} \quad W(0) = 0$$

Let h be solution of

$$\Box h + f'(0)h = \theta(u) \text{ and } \quad H(T) = 0$$

The function $k = w + h$ is solution of

$$\Box k + f'(0)k = \chi^2 g_2 \text{ and } K(0) = H(0)$$

$\longrightarrow$ Now, the goal is to control this linear system

$$H(0) \longrightarrow 0$$

$\exists$? $G_2 \in L^2 \times H^{-1}$ such that

$$AG_2 = H(0) = K(0) = \Lambda^{-1} G_2$$

$\longrightarrow$ Fixed point in $L^2 \times H^{-1}$ for $L = \Lambda A$
$\longrightarrow$ L is compact and reproduces a small ball B_ρ centred at the origin of $L^2 \times H^{-1}$.

Egorov Theorem

$$\begin{cases} \partial_t u = iA(x; D_x)u & \text{in } \mathbb{R} \times M \\ u(0) = u_0 \end{cases}$$

$\rightarrow A = A_1 + A_0, \qquad A_1(x;\xi) \in S^1_{cl}$ real, $\quad A_0(x;\xi) \in S^0_{cl}$
$\rightarrow A_1(x;\xi)$ homogeneous in ξ for $|\xi| \geq 1$

$$u(t,x) = \mathcal{U}(t)u_0$$

$\mathcal{U}(t)$ is bounded on each $H^\sigma(M)$, with inverse $\mathcal{U}(-t)$.

Egorov Theorem

If $P_0 = p_0(x, D) \in OPS^m_{1,0}$, then for every t, the operator

$$P(t) = \mathcal{U}(t)P_0\mathcal{U}(-t)$$

belongs to $OPS^m_{1,0}$, modulo a smoothing operator. The principal symbol of $P(t)$ (mod $S^{m-1}_{1,0}$) at (x_0, ξ_0) is equal to $p_0(\gamma(t))$ where γ is the bicharacteristic of A_1 issued from (x_0, ξ_0).