

On the linearized local Calderón problem: a Watermelon approach

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- 1 Introduction
- 2 Harmonic exponentials
- 3 The Watermelon approach

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The Calderón problem

In a seminal paper of 1980, A. Calderón asked whether it was possible to determine the electrical conductivity of a body by making current and voltage measurements at the **boundary**.

Mathematical formulation: let Ω be a bounded open set in \mathbf{R}^n , the electrical conductivity is represented by a positive bounded function γ . Given a potential f on the boundary, the induced potential on Ω satisfies

$$\operatorname{div}(\gamma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

The voltage to current map is given by

$$\Lambda_\gamma f = (\gamma \partial_\nu u)|_{\partial\Omega}$$

where ν is the exterior unit normal. The question raised by Calderón is whether the map $\gamma \mapsto \Lambda_\gamma$ is **injective**.

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The linearized Calderón problem

In fact, Calderón dealt with the **linearized** problem. We have

$$Q_\gamma(f, g) = \int_{\partial\Omega} \Lambda_\gamma f g \, ds = \int_{\Omega} \gamma \nabla u \nabla v \, dx$$

if u and v are solutions to the former Dirichlet problem with boundary data f, g .

Then the differential of the map $\gamma \mapsto Q_\gamma$ at $\gamma = 1$ is given by

$$D_\gamma Q|_{\gamma=1}(\delta\gamma)(f, g) = \int_{\Omega} \delta\gamma \nabla u \nabla v \, dx$$

if u and v are **harmonic** functions with trace f, g at the boundary.

The **linearized problem** is the injectivity of the former differential at $\gamma = 1$.

The linearized Calderón problem

The linearized problem can be reformulated in these terms: does the cancellation of the integral

$$\int_{\Omega} \delta\gamma \nabla u \nabla v \, dx = 0$$

for all couple of **harmonic functions** (u, v) imply $\delta\gamma = 0$?

The answer can easily be seen to be **yes**: take u and v to be two conjugate harmonic exponentials

$$e^{-ix \cdot \zeta}, \quad \zeta \in \mathbf{C}^n, \quad \zeta^2 = 0$$

and one obtains

$$0 = |\zeta|^2 \int_{\Omega} \delta\gamma e^{-ix \cdot (\zeta + \bar{\zeta})} \, dx = |\zeta|^2 \widehat{1_{\Omega} \delta\gamma}(2 \operatorname{Re} \zeta)$$

hence $\delta\gamma = 0$ since any vector $\xi \in \mathbf{R}^n$ is the real part of a $\zeta \in \mathbf{C}^n$ such that $\zeta^2 = 0$.

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Inverse problem for the Schrödinger equation

We have

$$\operatorname{div}(\gamma \nabla u) = \gamma \Delta u + \nabla \gamma \cdot \nabla u = \frac{1}{\sqrt{\gamma}} (\Delta - q)(\sqrt{\gamma} u)$$

where $q = \Delta \sqrt{\gamma} / \sqrt{\gamma}$. When γ is smooth enough, (using boundary determination) the Calderón problem is a particular case of the following inverse problem on the Schrödinger equation: does the equality of the Dirichlet-to-Neumann maps $\Lambda_{q_1}^s = \Lambda_{q_2}^s$ imply $q_1 = q_2$?

The Dirichlet-to-Neumann map Λ_q^s associated to the Schrödinger equation is

$$\Lambda_q^s : H^{\frac{1}{2}}(\partial\Omega) \ni f \mapsto \partial_\nu u$$

where u is a solution to the Dirichlet problem

$$-\Delta u + qu = 0, \quad u|_{\partial\Omega} = f.$$

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The linearized version

The linearized version of the inverse problem for the Schrödinger equation can be reformulated in these terms: does the cancellation of the integral

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Local problem

The **local** problem or the problem with **partial data** are also of great interest.

Partial data:

Does the Dirichlet-to-Neumann map measured only on **one part** of the boundary uniquely determine the conductivity? More precisely, if

$$\Lambda_{\gamma_1} f|_{\Sigma} = \Lambda_{\gamma_2} f|_{\Sigma} \quad \text{for all } f \in H^{\frac{1}{2}}(\partial\Omega)$$

where Σ is, say, an open neighbourhood of a point x_0 of the boundary, do we have $\gamma_1 = \gamma_2$?

Local problem:

If for all functions $f \in H^{\frac{1}{2}}(\partial\Omega)$ **supported** in Σ we have

$$\Lambda_{\gamma_1} f|_{\Sigma} = \Lambda_{\gamma_2} f|_{\Sigma}$$

where Σ is, say, an open neighbourhood of a point x_0 of the boundary, do we have $\gamma_1 = \gamma_2$?

Some references

A few dates:

- 1980 Calderon's seminal paper: linearization of the problem
- 1985 Kohn and Vogelius: boundary determination and real analytic case
- 1987 Sylvester and Uhlmann: resolution of the identifiability problem in dimension $n \geq 3$
- 1988 Nachmann: uniqueness in the 2D case
- 2002 Bukgheim and Uhlmann: partial data on big subsets of the boundary
- 2004 Kenig, Sjöstrand and Uhlmann: partial data on possibly small subsets
- 2006 Astala, Päivärinta: uniqueness for L^∞ conductivities
- 2008 Imanuvilov, Uhlmann and Yamamoto: partial data and local problem in 2D

Main Theorem

We investigate the **linearized version** of the local problem (on the Schrödinger equation, but the method should also work for the conductivity problem): does the cancellation of the integral

$$\int_{\Omega} f uv \, dx = 0$$

for all couple of **harmonic functions** (u, v) **vanishing** on some open subset Γ of the boundary imply $f = 0$?

Theorem

Let Ω be a connected bounded open set in \mathbf{R}^n , $n \geq 2$, with smooth boundary. The set of products of harmonic functions on Ω which vanish on a closed proper subset $\Gamma \subsetneq \partial\Omega$ of the boundary is dense in $L^1(\Omega)$.

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From local to global

In fact, it is enough to prove **local uniqueness**.

Theorem

Let Ω be a bounded open set in \mathbf{R}^n , $n \geq 2$, with smooth boundary. Let $x_0 \in \partial\Omega$ assume that we have the cancellation

$$\int_{\Omega} f uv \, dx = 0$$

for any couple of harmonic functions u and v vanishing on the complementary Γ of an open neighbourhood of x_0 . Then there exists $\delta > 0$ such that f vanishes on $B(x_0, \delta) \cap \Omega$.

Using a **conformal transformation**, one can suppose without loss of generality that Ω is on one side of the tangent hyperplane to Ω at x_0 .

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The setting

Our setting will therefore be as follows: $x_0 = 0$, $T_{x_0}(\Omega) : x_1 = 0$ and

$$\Omega \subset \{x \in \mathbf{R}^n : |x + e_1| < 1\}, \quad \Gamma = \{x \in \partial\Omega : x_1 \leq -2c\}.$$

We assume

$$\int_{\Omega} fuv \, dx = 0$$

for any couple of harmonic functions u and v on Ω satisfying

$$u|_{\Gamma} = v|_{\Gamma} = 0$$

The Laplacian on \mathbf{R}^n has $p(\xi) = \xi^2$ as a principal symbol, we consider the (complex) characteristic set

$$p^{-1}(0) = \{\zeta \in \mathbf{C}^n : \zeta^2 = 0\}.$$

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Decomposition of the frequencies

In dimension $n = 2$, this set is the union of two complex lines

$$p^{-1}(0) = \mathbf{C}\gamma \cup \mathbf{C}\bar{\gamma}$$

where $\gamma = ie_1 + e_2 = (i, 1) \in \mathbf{C}^2$. The differential of the map

$$\begin{aligned} s : p^{-1}(0) \times p^{-1}(0) &\rightarrow \mathbf{C}^n \\ (\zeta, \eta) &\mapsto \zeta + \eta \end{aligned}$$

at $(\gamma, -\bar{\gamma})$ is surjective: if $\varepsilon > 0$ is small enough, all $w \in \mathbf{C}^n$, $|z - 2ie_1| < 2\varepsilon$ may be decomposed under the form

$$z = \zeta + \eta, \quad \text{with } \zeta, \eta \in p^{-1}(0), |\zeta - \gamma| \lesssim \varepsilon, |\eta + \bar{\gamma}| \lesssim \varepsilon.$$

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Harmonic exponentials

The exponentials with linear weights

$$e^{-\frac{i}{h}x \cdot \zeta}, \quad \zeta \in p^{-1}(0)$$

are **harmonic functions**. We need to add a correction term in order to obtain harmonic functions u satisfying the boundary requirement $u|_{\Gamma} = 0$. Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be a cutoff function which equals 1 on Γ , we consider the solution w to the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w|_{\partial\Omega} = -(e^{-\frac{i}{h}x \cdot \zeta} \chi)|_{\partial\Omega}. \end{cases}$$

The function

$$u(x, \zeta) = e^{-\frac{i}{h}x \cdot \zeta} + w(x, \zeta)$$

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Exponential estimates

We have the following bound on w :

$$\begin{aligned} \|w\|_{H^1(\Omega)} &\lesssim \|e^{-\frac{i}{h}x \cdot \zeta} \chi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq \mathcal{O}(h^{-\frac{1}{2}} |\zeta|^{\frac{1}{2}}) e^{\frac{1}{h} H_K(\operatorname{Im} \zeta)} \end{aligned} \quad (1)$$

where H_K is the **supporting function** of the compact subset $K = \operatorname{supp} \chi \cap \partial\Omega$ of the boundary

$$H_K(\xi) = \sup_{x \in K} x \cdot \xi, \quad \xi \in \mathbf{R}^n.$$

In particular, if we take χ to be supported in $x_1 \leq -c$ and equal to 1 on $x_1 \leq -2c$ then the bound (1) becomes

$$\|w\|_{H^1(\Omega)} \leq \mathcal{O}(h^{-\frac{1}{2}} |\zeta|^{\frac{1}{2}}) e^{-\frac{c}{h} \operatorname{Im} \zeta_1} e^{\frac{1}{h} |\operatorname{Im} \zeta'|} \quad \text{when } \operatorname{Im} \zeta_1 \geq 0. \quad (2)$$

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Exponential estimates

Using those exponential solutions, we obtain from the cancellation of the integral the following bound

$$\left| \int_{\Omega} f(x) e^{-\frac{i}{h} x \cdot (\zeta + \eta)} dx \right| \leq \|f\|_{L^\infty(\Omega)} \left(\|e^{-\frac{i}{h} x \cdot \zeta}\|_{L^2(\Omega)} \|w(x, \eta)\|_{L^2(\Omega)} \right. \\ \left. + \|e^{-\frac{i}{h} x \cdot \eta}\|_{L^2(\Omega)} \|w(x, \zeta)\|_{L^2(\Omega)} + \|w(x, \eta)\|_{L^2(\Omega)} \|w(x, \zeta)\|_{L^2(\Omega)} \right)$$

thus when $\text{Im } \zeta_1 \geq 0$, $\text{Im } \eta_1 \geq 0$ and $\zeta, \eta \in p^{-1}(0)$

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In particular if $|z - 2iae_1| < \varepsilon a$ then

$$\left| \int_{\Omega} f(x) e^{-\frac{i}{h} x \cdot z} dx \right| \leq \mathcal{O}(h^{-1}) e^{-\frac{ca}{2h}} e^{\frac{2\varepsilon a}{h}}$$

We need to extrapolate the exponential decay.

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We need to **extrapolate** the exponential decay.

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The Segal-Bargmann transform

The Segal-Bargmann transform of an L^∞ function f on \mathbf{R}^n is given by the following formula

$$Tf(z) = \int_{\mathbf{R}^n} e^{-\frac{1}{2\hbar}(z-y)^2} f(y) dy$$

with $z = x + i\xi \in \mathbf{C}^n$.

The analytic wave front set $\text{WF}_a(f)$ of f is the complement of the set of all covectors $(x_0, \xi_0) \in T^*\mathbf{R}^n \setminus 0$ such that there exists a neighbourhood V_{z_0} of $z_0 = x_0 - i\xi_0$ in \mathbf{C}^n , $\chi \in C_0^\infty(\mathbf{R}^n)$ with $\chi(x_0) = 1$, and $c > 0$ and $C > 0$ such that

$$|T(\chi f)(z)| \leq C e^{-\frac{c}{\hbar} + \frac{1}{2\hbar} |\text{Im } z|^2}, \quad \forall z \in V_{z_0}, \quad \forall \hbar \in (0, 1].$$

The analytic wave front set $\text{WF}_a(f)$ is a closed conic set and its image by the first projection $T^*\mathbf{R}^n \rightarrow \mathbf{R}^n$ is the analytic singular support of f , i.e. the set of points $x_0 \in \mathbf{R}^n$ for which there is no neighbourhood on which f is a real analytic function.

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Kashiwara's Watermelon theorem

When a distribution f is supported on one side of a plane H then f cannot be analytic at $x_0 \in \partial H$, so the analytic wave front set of f cannot be empty. The following result gives explicitly a covector which is in the wave front set.

Theorem

Let f be a distribution supported in a half-space H , if $x_0 \in \partial H$ belongs to the support of f , then $(x_0, \pm\nu)$ belongs to the analytic wave front set of f where ν denotes a unit conormal to the hyperplane ∂H .

This is a consequence of a more general result due to Kashiwara

Watermelon Theorem

Let f be a distribution supported in a half-space H , if $x_0 \in \partial H$ and if (x_0, ξ_0) belongs to the analytic wave front set of f , then so does $(x_0, \xi_0 + t\nu)$ where ν denotes a unit conormal to the hyperplane ∂H provided $\xi_0 + t\nu \neq 0$.

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Segal-Bargmann and Fourier transform

Note that there is an *a priori* exponential bound

$$|Tf(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h}|\operatorname{Im} z|^2} \|f\|_{L^\infty}.$$

If f is supported in the half-space $x_1 \leq 0$ then the former estimate can be improved into

$$|Tf(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h}(|\operatorname{Im} z|^2 - |\operatorname{Re} z_1|^2)} \|f\|_{L^\infty}$$

when $\operatorname{Re} z_1 \geq 0$. The kernel of the Segal-Bargmann transform of a function $f \in L^\infty$ can be written as a linear superposition of exponentials with linear weights

$$e^{-\frac{1}{2h}(z-y)^2} = e^{-\frac{z^2}{2h}} (2\pi h)^{-\frac{n}{2}} \int e^{-\frac{t^2}{2h}} e^{-\frac{i}{h}y \cdot (t+iz)} dt$$

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$$Tf(z) = (2\pi h)^{-\frac{n}{2}} \iint e^{-\frac{1}{2h}(z^2+t^2)} e^{-\frac{i}{h}y \cdot (t+iz)} f(y) dt dy.$$

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The Watermelon approach

The idea of the proof of the Watermelon theorem is to **propagate the exponential decay** by use of the **maximum principle**. If f is supported in the half-space $x_1 \leq 0$, one works with the subharmonic function

$$\varphi(z_1) + \frac{1}{2}(\operatorname{Re} z_1)^2 - \frac{1}{2}(\operatorname{Im} z_1)^2 + h \log |Tf(z_0 + z_1 e_1)|$$

on a rectangle R .

One of the edges of R is contained in the neighbourhood V_{z_0} where there is the **additional exponential decay** of the Segal-Bargmann transform and one chooses φ to be a **non-negative harmonic function** vanishing on the boundary of R except for the segment where there is the exponential decay. The fact that φ is positive on the interior of the rectangle R allows to propagate the exponential decay of the Segal-Bargmann transform and this translates into the propagation of singularities described in the Watermelon theorem.

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Estimates on the Segal-Bargmann transform

If $|t| < \varepsilon a$ and $|z - 2ae_1| < \varepsilon a$ with $\varepsilon \ll 1$, the decomposition of frequencies gives

$$t + iz = \zeta + \eta, \quad \zeta, \eta \in p^{-1}(0), \quad |\zeta - a\gamma| < \varepsilon a, \quad |\eta + a\bar{\gamma}| < \varepsilon a$$

in that setting the estimate that we have established reads

$$\left| \int_{\Omega} f(y) e^{-\frac{i}{h} y \cdot (t + iz)} dy \right| \leq \mathcal{O}(h^{-1}) e^{-\frac{ca}{2h}} e^{\frac{2\varepsilon a}{h}}$$

thus cutting in two the integral (in t) giving $Tf(z)$ as a linear superposition we get

$$|Tf(z)| \lesssim h^{-1} e^{\frac{1}{2h} (|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2)} e^{\frac{2\varepsilon a}{h}} \left(e^{-\frac{ca}{2h}} + e^{-\frac{\varepsilon^2 a^2}{4h}} \right)$$

provided $|z - 2ae_1| < \varepsilon a$ and $\operatorname{Re} z_1 \geq 0$. Now choosing $\varepsilon \ll 1$ and $a \lesssim \varepsilon$

$$|Tf(z)| \lesssim h^{-1} e^{\frac{1}{2h} (|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2 - \frac{ca}{2})}.$$

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The Watermelon approach

To sum-up we have obtained the following bounds on the Segal-Bargmann transform of f

$$e^{-\frac{\Phi(z_1)}{2h}} |Tf(z_1, x')| \lesssim h^{-1} \|f\|_{L^\infty(\Omega)} \begin{cases} 1 & \text{when } z_1 \in \mathbf{C} \\ e^{-\frac{a}{2h}} & \text{when } \operatorname{Re} z_1 = 2a, |\operatorname{Im} z_1|^2 + |x'|^2 \leq \varepsilon a \end{cases}$$

where the weight Φ is given by the following expression

$$\Phi(z_1) = \begin{cases} |\operatorname{Im} z_1|^2 & \text{when } \operatorname{Re} z_1 \leq 0 \\ |\operatorname{Im} z_1|^2 - |\operatorname{Re} z_1|^2 & \text{when } \operatorname{Re} z_1 \geq 0. \end{cases}$$

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The central lemma

Lemma

Let F be an entire function satisfying the following bounds

$$e^{-\frac{\Phi(s)}{2h}} |F(s)| \leq \begin{cases} 1 & \text{when } s \in \mathbf{C} \\ e^{-\frac{c}{2h}} & \text{when } |s - L| \leq b \end{cases}$$

then for all $r \geq 0$ there exist $c', \delta > 0$ such that F satisfies

$$|F(s)| \leq e^{-\frac{c'}{2h}}, \quad \text{when } |\operatorname{Re} s| \leq \delta \text{ and } |\operatorname{Im} s| \leq r.$$

Applying the former lemma we obtain

$$|Tf(x)| \lesssim e^{-\frac{c'}{2h}}$$

for all $x \in \Omega, |x_1| \leq \delta \ll 1$. Letting h tend to 0 we deduce

$$f(x) = 0, \quad \forall x \in \Omega, \quad 0 \geq x_1 \geq -\delta.$$

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