Some questions concerning geometric inverse problems

Anna DOUBOVA

Dpto. E.D.A.N. - Univ. of Sevilla

joint work with

E. Fernández Cara - Univ. of Sevilla

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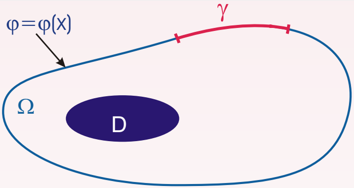
Outline

1. Laplace equation
   - Uniqueness
   - Partial identification of D (an algorithm)
   - Open questions
   - Stationary elasticity systems
2. Navier-Stokes and Boussinesq
   - Uniqueness
   - Partial identification of D
   - Open questions
3. Wave equations
   - N-dimensional
   - Lamé and elasticity



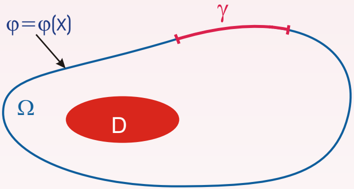
Direct problem

- Given data: D, Ω, φ, $\gamma \subset \partial\Omega$
- Find u

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ u = \varphi & \text{on} \quad \partial\Omega \\ u = 0 & \text{on} \quad \partial D \end{cases}$$

- Observation: $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_{\gamma}$

Geometric inverse problem for Laplace equation



Inverse problem

- Given data: Ω, φ, $\gamma \subset \partial\Omega$
- Additional data (observation): $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_{\gamma}$
- Find: D (and then the solution u)

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ u = \varphi & \text{on} \quad \partial\Omega \\ u = 0 & \text{on} \quad \partial D \end{cases}$$

Uniqueness

Assume: D^i simply connected, ∂D^i Lipshchitz, $\varphi \neq 0$

$$\begin{cases} -\Delta u^i = 0 & \text{in} \quad \Omega \setminus \overline{D^i}, \quad i = 0, 1 \\ u^i = \varphi & \text{on} \quad \partial\Omega \\ u^i = 0 & \text{on} \quad \partial D^i \end{cases}$$

Theorem

$$\frac{\partial u^0}{\partial n} = \frac{\partial u^1}{\partial n} \quad on\, \gamma \subset \partial\Omega \quad \Longrightarrow \quad D^0 = D^1$$

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Consequence of **unique continuation property**

Uniqueness

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Consequence of **unique continuation property**

Some previous works, several authors...

- Alessandrini & al.
- Andrieux, Abda & Jaoua
- Isakov
- Kavian

Partial identification of D

Assume: D is known

$$D + m = \{x + m(x) : x \in D\}, \quad \alpha^m = \frac{\partial u^m}{\partial n}\Big|_\gamma \text{ is known}$$

Find m ?

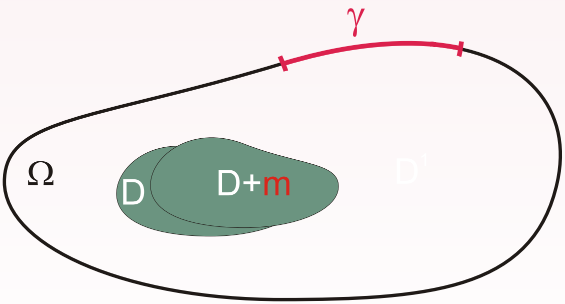

Figure: Deformations of D

Partial identification of D

Assume: $m = 0$ near $\partial\Omega$

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{D} \\ u = \varphi & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial D \end{cases} \qquad \begin{cases} -\Delta u^m = 0 & \text{in } \Omega \setminus (\overline{D + m}) \\ u^m = \varphi & \text{on } \partial\Omega \\ u^m = 0 & \text{on } \partial(D + m) \end{cases}$$

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Theorem

Assume: $\varphi \in H^{3/2}(\partial\Omega)$, $\varphi \neq 0$, $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_\gamma$, $\alpha^m = \left.\dfrac{\partial u^m}{\partial n}\right|_\gamma$

Also, assume: $(m \cdot n)|_{\partial D} \in M$, $\mathbf{dim}\, M < \infty$, $M \subset W^{1,\infty}(\partial D)$

Then: $\exists\, G : L^2(\gamma) \mapsto M$ *computable such that*

$$(m \cdot n)|_{\partial D} = G(\alpha^m - \alpha) + o(m) \quad \textit{for small } m$$

Consider $u'(m)$:

$$\begin{cases} -\Delta u'(m) = 0 & \text{in } \Omega \setminus \overline{D} \\ u'(m) = 0 & \text{on } \partial\Omega \\ u'(m) = -(m \cdot n)\dfrac{\partial u}{\partial n} & \text{on } \partial D \end{cases}$$

Then:

$$\frac{\partial u'(m)}{\partial n} = \alpha^m - \alpha + o(m)$$

The task: compute $(m \cdot n)|_{\partial D}$ from $\left.\dfrac{\partial u'(m)}{\partial n}\right|_\gamma$ ("known")

Goal: find $(m \cdot n)|_{\partial D}$ from $\left.\frac{\partial u'(m)}{\partial n}\right|_{\gamma}$

Compute $(m \cdot n)|_{\partial D} \in M \Leftrightarrow$ compute $\displaystyle\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n}\, h \, ds, \ \forall h \in M$

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Compute $(m \cdot n)|_{\partial D} \in M \Leftrightarrow$ compute $\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n} h \, ds, \ \forall h \in M$

$(m \cdot n)|_{\partial D} \in M$, $\dim M < \infty$, $\Longrightarrow \frac{\partial u'(m)}{\partial n}|_{\partial D} \in E$, $\dim E < \infty$.

Assume the following control problem is solvable:

$$\begin{cases} -\Delta \theta_h = 0 & \text{in } \Omega \setminus \overline{D} \\ \theta_h = v 1_\gamma & \text{on } \partial\Omega \\ \dfrac{\partial \theta_h}{\partial n} = h & \text{on } \partial D \end{cases}$$

$P_E(\theta_h|_{\partial D}) = 0 \quad P_E : L^2(\partial\Omega) \mapsto E$ ortogonal projector

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$$-\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n} h \, ds = \int_{\gamma} \frac{\partial u'(m)}{\partial n} v \, ds \quad \forall h \in M$$

Sketch of the proof

Step 2: Data assimilation approach (J.-P. Puel)

$$\begin{cases} -\Delta u'(m) = 0 & \text{in } \Omega \setminus \overline{D} \\ u'(m) = 0 & \text{on } \partial\Omega \\ u'(m) = -(m \cdot n)\dfrac{\partial u}{\partial n} & \text{on } \partial D \end{cases} \qquad \begin{cases} -\Delta \theta_h = 0 & \text{in } \Omega \setminus \overline{D} \\ \theta_h = v 1_\gamma & \text{on } \partial\Omega \\ \dfrac{\partial \theta_h}{\partial n} = h & \text{on } \partial D \end{cases}$$

$$P_E(\theta_h|_{\partial D}) = 0 \quad P_E : L^2(\partial\Omega) \mapsto E \text{ ortogonal projector}$$

$$\begin{aligned} -\int_{\partial D} (m \cdot n)\frac{\partial u}{\partial n} h \, ds &= \int_{\partial D \cup \partial\Omega} u'(m) \frac{\partial \theta_h}{\partial n} \, ds \\ &= \int_{\partial D} \frac{\partial u'(m)}{\partial n} P_E(\theta_h|_{\partial D}) \, ds + \int_{\partial\Omega} \frac{\partial u'(m)}{\partial n} \theta_h \, ds \\ &= \int_\gamma \frac{\partial u'(m)}{\partial n} v \, ds \qquad \forall h \in M \end{aligned}$$

$$\begin{cases} -\Delta\theta_h = 0 & \text{in } \Omega \setminus \overline{D} \\ \theta_h = v 1_\gamma & \text{on } \partial\Omega \\ \dfrac{\partial\theta_h}{\partial n} = h & \text{on } \partial D \end{cases}$$

$$P_E(\theta_h|_{\partial D}) = 0$$

An exact finite-controllability problem

Unique continuation $\Longrightarrow$ Existence

In fact: $\forall\, \varepsilon > 0\ \exists\, v$ such that

$$P_E(\theta_h|_{\partial D}) = 0, \quad ||\theta_h|_{\partial D}||_{L^2} \le \varepsilon$$

(E. Zuazua)

Algorithm: partial identification of D up to $o(m)$

Assume:

- $\mathcal{I} = \mathbf{dim}\, M, \quad \{h^1, \dots, h^{\mathcal{I}}\}$ basis of M,

$$(m \cdot n)|_{\partial D} = \sum_{i=1}^{\mathcal{I}} \lambda_i h^i$$

- v^i, $i = 1, \dots, \mathcal{I}$ controls with $h = h^i$

Then: computation of $\lambda_i \iff$ resolution of a linear system:

$$\sum_{i=1}^{\mathcal{I}} H_{ij} \lambda_i = \int_\gamma (\alpha^m - \alpha)\, v^j \, ds, \quad 1 \le j \le \mathcal{I}$$

where

$$H_{ij} = \int_{\partial D} \left(-\frac{\partial u}{\partial n} \right) h^i \, h^j \, ds$$

A first open question

Consider a penalized problem:

$$\inf_{m\in\mathcal{M}_{ad}} \frac{1}{2}||\alpha^m - \alpha||^2_{L^2(\gamma)} + \frac{\varepsilon}{2}||m||^2_M$$

Optimality system:

$$\begin{cases} -\Delta u^\varepsilon = 0 & \text{in } \Omega \setminus \overline{(D+m^\varepsilon)} \\ u^\varepsilon = \widetilde{\varphi} & \text{on } \partial\Omega \\ u^\varepsilon = 0 & \text{on } \partial(D+m^\varepsilon) \end{cases} \qquad \begin{cases} -\Delta \varphi^\varepsilon = 0 \\ \varphi^\varepsilon = (\alpha^{m^\varepsilon} - \alpha)1_\gamma \\ \varphi^\varepsilon = 0 \end{cases}$$

$$\begin{cases} \displaystyle\int_{\partial(D+m^\varepsilon)} \left(-\frac{\partial u^\varepsilon}{\partial n}\right)\left(\frac{\partial \varphi^\varepsilon}{\partial n}\right)(p-m^\varepsilon)\cdot n + \varepsilon(m^\varepsilon, p-m^\varepsilon)_M \geq 0 \\ \forall\, p \in \mathcal{M}_{ad}, \quad m^\varepsilon \in \mathcal{M}_{ad} \end{cases}$$

A penalized problem

Estimates:

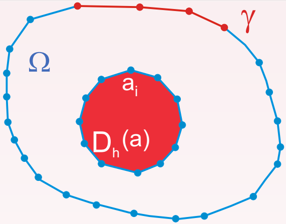
$$||\alpha^{m^\varepsilon}||_{L^2(\gamma)} \leq C, \quad ||\varphi^{m^\varepsilon}||_{H^r(\gamma)} \leq C$$

Question

Do we have

$$m^\varepsilon \to m^*, \quad \alpha^{m^\varepsilon} \to \alpha^* \quad \text{with} \quad \alpha^* = \alpha\,?$$

(under appropriate hypotheses)



A second open question

Consider the discretized inverse pb.

$\Omega = \cup_{K \in \mathcal{T}_h} K$, $\mathcal{T}_h$ triangulation

$a = \{a_i \in \partial D : 1 \leq i \leq \mathcal{I}_0\}$

A second open question

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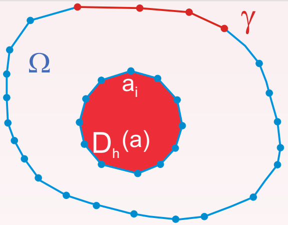


Inverse problem

Given: $\varphi_h \in W_h(a)$, $\alpha_h \in A_h$. Find: a

$$A_h = \{\delta_h \in C^0(\overline{\gamma}) : \delta_h|_\ell \in \mathbb{P}_1(\ell), \forall \text{ segment } \ell \subset \gamma\}$$

$$\mathcal{U}_h(a) = \{q_h \in W_h(a) : q_h = 0 \text{ at the nodes } \notin \gamma\}, \quad A_h \cong \mathcal{U}_h(a)$$

$$\int_\gamma \alpha_h q_h \, ds = \int_{\Omega \setminus D_h(a)} \nabla u_h \cdot \nabla q_h \, dx \quad \forall q_h \in \mathcal{U}_h(a), \, \alpha_h \in A_h$$

Question

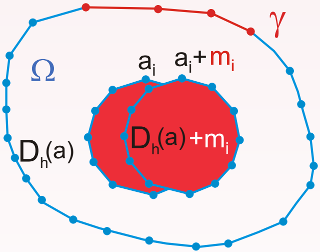
Conditions on $\mathcal{T}_h$ for uniqueness ?

Partial identification of a

Assume: a is known

$$a + m = \{a_i + m_i : 1 \leq i \leq \mathcal{I}_0\}$$

$$\alpha_h^m \simeq \frac{\partial u_h(m)}{\partial n}|_\gamma$$



Find m?

Partial identification of a

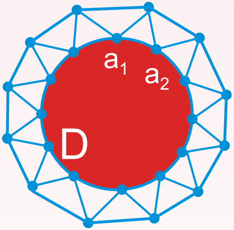
We find:

$$\alpha_h^m = \alpha_h + L_h(m) + o(m)$$

m can be computed from $L_h(m)$ up to $o(m)$:

for example for the particular triangulation,

we have to solve



$$\begin{bmatrix} \otimes & \otimes & \otimes & \otimes & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \otimes & \otimes & \otimes & \otimes & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & 0 & \cdots & \cdots & 0 & \otimes & \otimes \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \\ \vdots \\ m_{\mathcal{I},1} \\ m_{\mathcal{I},2} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ \vdots \\ k_{\mathcal{I}-1} \\ k_{\mathcal{I}} \end{bmatrix}$$

Stationary elasticity systems

$$\begin{cases} -\nabla \cdot (\mu(x)(\nabla u + \nabla u^t)) - \nabla(\lambda(x)\nabla \cdot u) = 0 & \text{in } \Omega \setminus \overline{D} \\ u = \varphi & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial D \end{cases}$$

Observation: $\alpha = \mu(\nabla u + \nabla u^t) \cdot n + \lambda(\nabla \cdot u)n$ on γ

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Key point: **Unique continuation property**

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Several results: Dehman, Robbiano, Yamamoto, Weck...

- Lin & Wang (2005): $N = 2$, λ, μ Lipschitz
- Escauriaza (2005): $N = 2$, μ Lipschitz, $\lambda \in L^\infty$
- Alessandrini & Morasi (2001): $N \geq 2$, $\lambda, \mu \in C^{1,1}$

In that case: **uniqueness** and **reconstruction** (with $m \cdot n \in M$, $\dim M < +\infty$)

Stationary anisotropic elasticity systems

$$\begin{cases} \nabla \cdot \sigma(u) = 0, \quad \sigma_{kl}(u) = \sum_{i,j=1}^{3} a_{ijkl}\varepsilon_{ij}(u) \\ \varepsilon_{kl}(u) = \frac{1}{2}(\partial_k u_l + \partial_l u_k) \\ \dots \end{cases}$$

Observation: $\alpha = \sigma(u) \cdot n$ on γ

Key point: **Unique continuation property**

Again: **uniqueness**, **reconstruction** ($m \cdot n \in M$, $\dim M < +\infty$)

Nakamura & Wang (2006): $N = 2$, $a_{ijkl} \in W^{1,\infty}$

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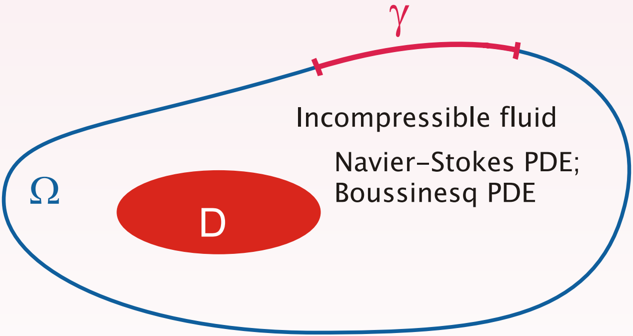
Question

Conditions on a_{ijkl} for $N \geq 3$?

Geometric inverse problem for fluids



Inverse problem

Find a rigid body D

Find D from Ω, (φ, ψ), (α, β) with

$$\begin{cases} -\nu\Delta u + (u \cdot \nabla)u + \nabla p = \theta g, \quad \nabla \cdot u = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ -\kappa\Delta\theta + u \cdot \nabla\theta = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ u = \varphi, \quad \theta = \psi & \text{on} \quad \partial\Omega \\ u = 0, \quad \theta = 0 & \text{on} \quad \partial D \end{cases}$$

$(u, p, \theta) =$ (velocity, pressure, temperature)
$\nu =$kinematic viscosity; $\kappa =$thermal conductivity; $g =$gravity

Inverse problem for stationary Boussinesq system

Find D from Ω, (φ, ψ), (α, β) with

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ -\kappa\Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ u = \varphi, \quad \theta = \psi & \text{on} \quad \partial\Omega \\ u = 0, \quad \theta = 0 & \text{on} \quad \partial D \end{cases}$$

$(u, p, \theta) =$ (velocity, pressure, temperature)
$\nu =$kinematic viscosity; $\kappa =$thermal conductivity; $g =$gravity

$$\left.\begin{array}{l} \sigma(u,p)\cdot n := (-p\,\mathrm{Id} + \nu(\nabla u + {}^t\nabla u))\cdot n = \alpha \\ \kappa\dfrac{\partial\theta}{\partial n} = \beta \end{array}\right\} \quad \text{on } \gamma\subset\Omega$$

Some previous works

Fluids:

- Alvarez, Conca, Friz, Kavian, & Ortega.
- Heck, Uhlmann & Wang
- AD, Fernández-Cara, González-Burgos & Ortega

Uniqueness result

Assume: $(\varphi, \psi) \neq (0,0)$ given; $\partial\Omega$, $\partial D^i \in C^{1,1}$, $i = 0, 1$

$$\begin{cases} -\nu\Delta u^i + (u^i \cdot \nabla)u^i + \nabla p^i = \theta^i g, \quad \nabla \cdot u^i = 0 & \text{in} \quad \Omega \setminus \overline{D^i} \\ -\kappa\Delta\theta^i + u^i \cdot \nabla\theta^i = 0 & \text{in} \quad \Omega \setminus \overline{D^i} \\ u^i = \varphi, \quad \theta^i = \psi & \text{on} \quad \partial\Omega \\ u^i = 0, \quad \theta^i = 0 & \text{on} \quad \partial D^i \end{cases}$$

and

$$\sigma(u^0, p^0) \cdot n = \sigma(u^1, p^1) \cdot n \quad \text{and} \quad \kappa\frac{\partial\theta^0}{\partial n} = \kappa\frac{\partial\theta^1}{\partial n} \quad \text{on } \gamma \subset \Omega$$

Theorem

$$D^0 = D^1$$

For the proof

Unique continuation property

$a, b \in L^\infty(G)^N, d \in L^\infty(G)$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = \eta g, \quad \nabla\cdot v = 0 & \text{in } G \\ -\kappa\Delta\eta + a\cdot\nabla\eta + v\cdot\nabla d = 0 & \text{in } G \end{cases}$$

(a generalization of a result by Fabre & Lebeau, 1996)

Theorem

Let $G \subset \mathbb{R}^N$, $\omega \subset G$ (open)

$$v = 0 \text{ and } \eta = 0 \text{ in } \omega \Longrightarrow v \equiv 0 \text{ and } \eta \equiv 0 \ (q \equiv Const.)$$

Corollary

Let $\Gamma \subset \partial G$ (open)

$$\left.\begin{array}{ll} v = 0, \quad \eta = 0 & \text{on } \Gamma \\ \sigma(v,q)\cdot n = 0, \quad \kappa\dfrac{\partial\eta}{\partial n} = 0 & \text{on } \Gamma \end{array}\right\} \Longrightarrow v \equiv 0 \text{ and } \eta \equiv 0$$

Some comments on uniqueness

- More complicated systems: OK (Appropriate cond. on f_i)

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_0(\theta,\eta), \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ -\kappa\Delta\theta + u\cdot\nabla\theta = f_1(\theta,\eta) & \text{in} \quad \Omega\setminus\overline{D} \\ -a\Delta\eta + u\cdot\nabla\eta = f_2(\theta,\eta) & \text{in} \quad \Omega\setminus\overline{D}\dots \end{cases}$$

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- Time-dependent: OK if $u|_{t=0} = 0$, $\theta|_{t=0} = 0$. Other dada?

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D}\times(0,T) \\ \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad \Omega\setminus\overline{D}\times(0,T) \\ u(x,0) = u^0(x), \quad u(x,0) = \theta^0(x) & \text{in} \quad \Omega\setminus\overline{D} \end{cases}$$

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$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_0(\theta,\eta), \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ -\kappa\Delta\theta + u\cdot\nabla\theta = f_1(\theta,\eta) & \text{in} \quad \Omega\setminus\overline{D} \\ -a\Delta\eta + u\cdot\nabla\eta = f_2(\theta,\eta) & \text{in} \quad \Omega\setminus\overline{D}\ldots \end{cases}$$

- Time-dependent: OK if $u|_{t=0} = 0$, $\theta|_{t=0} = 0$. Other dada?

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D}\times(0,T) \\ \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad \Omega\setminus\overline{D}\times(0,T) \\ u(x,0) = u^0(x), \quad u(x,0) = \theta^0(x) & \text{in} \quad \Omega\setminus\overline{D} \end{cases}$$

- Only one observation $\sigma\cdot n$ on γ ?

The needed unique continuation property: $a, b \in L^\infty(G)^N$, $d \in L^\infty(G)$, $\nabla\cdot a = \nabla\cdot b = 0$ in G

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = \eta g, \quad \nabla\cdot v = 0 & \text{in} \quad G \\ -\kappa\Delta\eta + a\cdot\nabla\eta + v\cdot\nabla d = 0 & \text{in} \quad G \end{cases}$$

$v = 0$ and $\eta = 0$ in ∂G, $\sigma\cdot n = 0$ on $\gamma \implies v \equiv 0$ and $\eta \equiv 0$

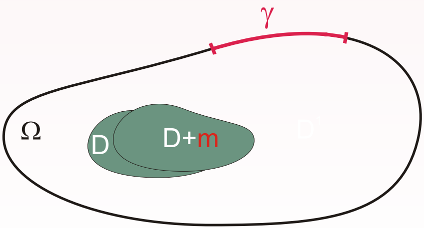
Partial identification of D

Assume: D is known

$$D + m = \{x + m : x \in D\}$$

$$(\alpha^m, \beta^m) = \left(\sigma(u^m, p^m) \cdot n|_\gamma \,,\, \kappa \frac{\partial \theta^m}{\partial n}\Big|_\gamma\right) \text{ is known}$$

Find m ?



Partial identification of D

Assume: $m = 0$ near $\partial\Omega$

(u^m, p^m, θ^m) sol. in $\Omega \setminus (\overline{D + m})$:

$$\begin{cases} -\nu\Delta u^m + (u^m \cdot \nabla)u^m + \nabla p^m = \theta^m g,\ \nabla \cdot u^m = 0 & \text{in } \Omega \setminus (\overline{D + m}) \\ -\kappa\Delta\theta + u^m \cdot \nabla\theta^m = 0 & \text{in } \Omega \setminus (\overline{D + m}) \\ u^m = \varphi, \quad \theta^m = \psi & \text{on } \partial\Omega \\ u^m = 0, \quad \theta^m = 0 & \text{on } \partial D \end{cases}$$

Partial identification of D

Assume: $m = 0$ near $\partial\Omega$

(u^m, p^m, θ^m) sol. in $\Omega \setminus (\overline{D+m})$:

$$\begin{cases} -\nu\Delta u^m + (u^m\cdot\nabla)u^m + \nabla p^m = \theta^m g,\ \nabla\cdot u^m = 0 & \text{in } \Omega\setminus(\overline{D+m}) \\ -\kappa\Delta\theta + u^m\cdot\nabla\theta^m = 0 & \text{in } \Omega\setminus(\overline{D+m}) \\ u^m = \varphi, \quad \theta^m = \psi & \text{on } \partial\Omega \\ u^m = 0, \quad \theta^m = 0 & \text{on } \partial D \end{cases}$$

Question

(φ,ψ), D, (u,p,θ) are known; $D+m$, (u^m,p^m,θ^m) are unknown

$$\begin{aligned} (\alpha^m,\beta^m) &= (\sigma(u^m,p^m)\cdot n, \kappa\frac{\partial\theta^m}{\partial n}) \quad \text{on } \gamma \\ (\alpha,\beta) &= (\sigma(u,p)\cdot n, \kappa\frac{\partial\theta}{\partial n}) \quad \text{on } \gamma \end{aligned}$$

Compute $(m\cdot n)|_{\partial D}$ from D, (α^m,β^m) and (α,β)

Partial identification of D

Theorem

Assume:

$$(\varphi, \psi) \neq (0,0), \quad (|\frac{\partial u}{\partial n}|, |\frac{\partial \theta}{\partial n}|) \neq 0 \text{ on } \partial D$$

$$(m \cdot n)|_{\partial D} \in M, \quad \mathbf{dim}\, M < \infty$$

Then:

$$\exists\, H : X(\gamma) \mapsto M \text{ computable such that}$$

$$(m \cdot n)|_{\partial D} = H(\alpha^m - \alpha, \beta^m - \beta) + o(m)$$

Proof

- Domain variation (F. Murat, J. Simon)
- Data assimilation (J.-P. Puel)

Open questions

1. Only one additional observation: $\sigma(u,p)\cdot n$ on γ ?
2. Time-dependent systems ?

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \setminus \overline{D} \times (0, T) \\ u = \varphi_0 & \text{in } \partial\Omega \times (0, T) \\ u = 0 & \text{in } \partial D \times (0, T) \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

Inverse problem

Given data: Ω, T, φ_0, $\gamma \subset \partial\Omega$ (open), $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_{\gamma \times (0,T)}$

Find D

Uniqueness

$$\begin{cases} u^i_{tt} - \Delta u = 0 & \text{in } \Omega \setminus \overline{D^i} \times (0,T), \quad i = 0,1 \\ u^i = \varphi_0 & \text{in } \partial\Omega \times (0,T) \\ u^i = 0 & \text{in } \partial D \times (0,T) \\ u^i(x,0) = 0, \quad u^i_t(x,0) = 0 & \text{in } \Omega \setminus \overline{D^i} \end{cases}$$

Theorem

$$\left.\begin{array}{l} T > T_1(\Omega,\gamma) \\ \dfrac{\partial u^0}{\partial n} = \dfrac{\partial u^1}{\partial n} \quad on\, \gamma \times (0,T) \end{array}\right\} \Longrightarrow D^0 = D^1$$

Fundamental results: Hörmander, Lions

Attention: Weaker than the geometric condition (Only uniqueness, not observability!)

Partial identification of D

For simplicity: assume that the solution is a ball: $D = B(x^0, r)$
$\alpha = \frac{\partial u}{\partial n}|_{\gamma \times (0,T)}$, $\alpha^m = \frac{\partial u^m}{\partial n}|_{\gamma \times (0,T)}$ known

$$m = (d, s), \quad D + m = B(x^0 + d, r + s)$$

$$\alpha^m - \alpha = L(d, s) + \frac{1}{2} Q((d, s), (d, s)) + o(|(d, s)|^2)$$

$$L(d, s) = \frac{\partial z}{\partial n}|_{\gamma \times (0,T)}, \quad Q((d, s), (d, s)) = \frac{\partial w}{\partial n}|_{\gamma \times (0,T)},$$

$$\begin{cases} z_{tt} - \Delta z = 0 & \Omega \setminus \overline{B(x^0, r)} \times (0, T) \\ z|_{t=0} = z_t|_{t=0} & \Omega \setminus \overline{B(x^0, r)} \\ z = 0 & \partial\Omega \times (0, T) \\ z = -\frac{\partial u}{\partial n}(d \cdot n - s) & \partial B(x^0, r) \times (0, T) \end{cases}$$

$$\begin{cases} w_{tt} - \Delta w = 0 \\ w|_{t=0} = w_t|_{t=0} \\ w = 0 \\ w = -\nabla z \cdot (d - sn) - \frac{\partial z}{\partial n}(d \cdot n - s) - \frac{\partial^2 u}{\partial x_i \partial x_j}(d_j - s n_j) n_i (d \cdot n - s) \end{cases}$$

Partial identification of D

Algorithm to find $m = (d, s)$

Known data: α, $\widetilde{\alpha}$, L, Q

$$\inf_{(d,s)\in\mathcal{U}} ||(\widetilde{\alpha} - \alpha) - L(d, s) - \frac{1}{2} Q((d, s), (d, s))||^2$$

$$|| \cdot || = || \cdot ||_{L^2(0,T)}$$

An extremal problem in $\mathbb{R}^N \times \mathbb{R}$ for a polynomial of order 4

Evolution elasticity systems

Lamé - Uniqueness?

$$\begin{cases} u_{tt} - \nabla\cdot(\mu(x)(\nabla u + \nabla u^t)) - \nabla(\lambda(x)\nabla\cdot u) = 0 & \text{in } \Omega\setminus\overline{D}\times(0,T) \\ u = \varphi & \text{on } \partial\Omega\times(0,T) \\ u = 0 & \text{on } \partial D\times(0,T) \\ u(0) = 0, \quad u_t(0) = 0 & \text{in } \Omega\setminus\overline{D} \end{cases}$$

Observation: $\alpha = \mu(\nabla u + \nabla u^t)\cdot n + \lambda(\nabla\cdot u)n$ on $\gamma\times(0,T)$

Key point: **Unique continuation property**

- Uniqueness and partial identification are OK for constant or regular μ, λ
- Under consideration: for more general μ, λ