Control and stabilization properties for a singular heat equation with an inverse square potential.

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Outline of the talk

Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, with $N \geq 3$, and $\omega \subset \Omega$.

The equation

We consider the control problem for the heat equation with an inverse square potential.

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = v \chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1}$$

Here, $\mu \in \mathbb{R}$.

(Not so precise) statement of the problem

Problem

How can we modify the state of the equation by using a control v localized in $\omega \times (0, T)$?

E.g., can we find a control v such that the solution u of (1) satisfies

$$u(T, v) = 0 \quad ?$$

Physical motivations

Consider the non linear elliptic system

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Appears in *combustion theory* and *stellar structures phenomena*[1].

When $\Omega = B(O,1)$, $U(x) = -2\log(|x|)$ is the unique singular radial solution[2].

$\rightsquigarrow$ Can we control the parabolic equation

$$\partial_t u - \Delta u = \lambda e^u + v\chi_\omega, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

around the stationary solution U ?

$\rightsquigarrow$ *"linearize" the equation.*

[1] Franck-Kamenetskii 1969 and Chandrasekhar, 1985

[2] Mignot & Puel 1997

The equation

The formally linearized equation :

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Remarks on the Cauchy problem

Careful !

• The Cauchy problem is well-posed[3] if

$$\mu \leq \mu^*(N) = \left(\frac{N-2}{2}\right)^2.$$

• If $\mu > \mu^*(N)$ and if $u_0 \geq 0$ and $v \geq 0$, then no solution exists[4], even locally in time !

The Hardy inequality

Strongly related to the Hardy inequality (Recall $N \geq 3$)

$$\forall u \in H_0^1(\Omega), \quad \mu^*(N) \int_\Omega \frac{|u|^2}{|x|^2}\,\mathrm{dx} \leq \int_\Omega |\nabla u|^2\,\mathrm{dx}.$$

[3] Vazquez and Zuazua, JFA, 2000

[4] Baras & Goldstein, 1984

The Hardy inequality

The Hardy inequality

$$\forall u \in H_0^1(\Omega), \quad \mu^*(N) \int_\Omega \frac{|u|^2}{|x|^2} \, \mathrm{dx} \leq \int_\Omega |\nabla u|^2 \, \mathrm{dx}.$$

- $\mu^*(N)$ is independent of Ω $\rightsquigarrow$ *Scaling invariance.*
- $u \in H_0^1(\Omega) \Longrightarrow u/|x|^2 \in H^{-1}$.
- The equality is not attained ! $\longrightarrow$ A natural norm is

$$\|u\|_{\mathcal{H}}^2 = \int |\nabla u|^2 \, \mathrm{dx} - \mu^*(N) \int \frac{|u|^2}{|x|^2} \, \mathrm{dx}$$

If $\mu \leq \mu^*(N)$, the solution belongs to $C([0,T]; L^2) \cap L^2(0,T; \mathcal{H})$.

Outline of the talk

The control properties of (1) *change completely depending on the value of* μ*:*

- When $\mu \leq \mu^*(N)$, null controllability property holds: For any open set ω, any $T > 0$ and any initial data $u_0 \in L^2(\Omega)$, there exists a control function $v \in L^2((0, T) \times \Omega)$ such that $u(T, v) = 0$.
- When $\mu > \mu * (N)$, stabilization properties fail.

The subcritical case $\mu \leq \mu^*(N)$

Here $\mu \leq \mu^*(N)$.

Theorem (SE 2008)

Given any non-empty open set $\omega \subset \Omega$, for any $T > 0$ and $u_0 \in L^2(\Omega)$, there exists a control $v \in L^2((0,T) \times \omega)$ such that the solution of (1) satisfies

$$u(T, v) = 0.$$

Besides, there exists a constant C_T such that

$$\|v\|_{L^2((0,T)\times\omega)} \leq C_T \|u_0\|_{L^2(\Omega)} .$$

Previous works

- **On the heat equations:**
  • Fursikov & Imanuvilov, *Controllability of evolution equations* (1996),
  • Lebeau & Robbiano, *Contrôle exact de l'équation de la chaleur* (1995),
  • Fernandez-Cara & Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equation* (2000),
- **Heat equations with singular coefficients:**
  • Martinez & Vancostenoble, *Carleman estimates for one-dimensional degenerate heat equations* (2007)
  • Vancostenoble & Zuazua, *Null-controllability for the heat equation with singular inverse-square potentials* (2008).

Known results

Heat equation with a potential

Controllability of

$$\begin{cases} \partial_t u - \Delta u + a(x) u = v \chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Null controllabilty results[5] for

$$a \in L^p(\Omega), \quad p > 2N/3.$$

Here,

$$\frac{1}{|x|^2} \in L^{N/2-\varepsilon}(\Omega), \quad \varepsilon > 0 \quad !$$

[5] Imanuvilov & Yamamoto (2003)

Known results

Remark and open problem

- Null- controllability does not hold for

$$a \in L^p, \quad p < N/2.$$

In this case, unique continuation fails.[a]

- We do not know what happens when

$$a \in L^p(\Omega), \quad p \in (N/2, 2N/3].$$

cf Koch Tataru 2001 for unique continuation results.

[a] Koch-Tataru 2002

Anyway, with the inverse square potential, we cannot apply the existing theory !

Known results

Controllability of the heat equation with a singular inverse-square potential holds[6] under some geometric assumptions on the control region ω, namely :

Geometric Assumption

ω contains an annular set which surrrounds the singularity.

Keynote of the proof: Use spherical harmonics to work with radial equations near the singularity, and a Carleman estimate on a 1-d heat equation with a singular potential.

Our goal: Remove the geometrical assumption on ω.

From now, we assume $\bar{\omega} \cap \bar{B}(O, 1) = \phi$. (*scaling argument*)

[6] Vancostenoble & Zuazua, JFA, 2008

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Controllability result

Theorem (SE 2008)

Null-controllability holds for (1) for $\mu \leq \mu^*(N)$ without any geometric assumption !

Difficulties :

- We cannot use spherical harmonics.
- Choosing a weight conveniently in a Carleman estimate.

Sketch of the proof

Step 1

Use HUM, and hence rather study the observability of the adjoint system, that is the inequality

$$\int_\Omega |w(x,0)|^2 \, \mathrm{dx} \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2 \, \mathrm{dxdt}$$

for any solution of

$$\begin{cases} \partial_t w + \Delta w + \dfrac{\mu}{|x|^2} w = 0, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,T) = w_T(x), & x \in \Omega. \end{cases}$$

Sketch of the proof

Step 2

Derive a Carleman estimate.

Choose ψ such that:

$$\begin{cases} \psi(x) = \ln(|x|), & x \in B(O,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \backslash \bar{B}(O,1), \\ |\nabla \psi(x)| \geq \delta > 0, & x \in \bar{\Omega} \backslash \omega. \end{cases}$$

Define $\sigma(t,x)$ as the weight function

$$\sigma(t,x) = s\theta(t)\Big(e^{2\lambda \sup \psi} - \frac{1}{2}|x|^2 - \phi(x)\Big),$$

where s and λ are large coefficients, and θ and ϕ are

$$\theta(t) = \Big(\frac{1}{t(T-t)}\Big)^3, \qquad \phi(x) = e^{\lambda\psi(x)}.$$

Carleman estimate

Theorem (SE 2008)

$\exists K > 0, \exists \lambda_0 > 0, \forall \lambda \geq \lambda_0, \exists s_0(\lambda), \forall s \geq s_0$, any w solution of the adjoint equation satisfies

$$s\lambda^2 \iint_{\Omega\backslash B(O,1)} \theta\phi e^{-2\sigma}|\nabla w|^2 + s^3\lambda^4 \iint_{\Omega\backslash B(O,1)} \theta^3\phi^3 e^{-2\sigma}|w|^2$$

$$+ s \iint_{\Omega\times(0,T)} \theta e^{-2\sigma}\frac{|w|^2}{|x|} + s^3 \iint_{\Omega\times(0,T)} \theta^3 e^{-2\sigma}|x|^2|w|^2$$

$$\leq K\left(s\lambda^2 \iint_{\omega\times(0,T)} \theta\phi e^{-2\sigma}|\nabla w|^2 + s^3\lambda^4 \iint_{\omega\times(0,T)} \theta^3\phi^3 e^{-2\sigma}|w|^2\right)$$

Comments

About the weight function

$$\sigma(t,x) = s\theta(t)\Big(e^{2\lambda \sup \psi} - \frac{1}{2}|x|^2 - e^{\lambda\psi}\Big),$$

- In the unit ball

$$\sigma \simeq s\theta(t)(C - |x|^2),$$

  as in Vancostenoble & Zuazua for radial singular heat equations. (Recall $\psi = \ln(|x|)$ in the unit ball)

- Outside the ball

$$\sigma \simeq s\theta\Big(e^{2\lambda \sup \psi} - e^{\lambda\psi}\Big),$$

  as the one in Fursikov & Imanuvilov for the observability of the heat equation.

Comments

- In the proof, we use the following Hardy inequality:

$$\mu^*(N) \int_\Omega \frac{|w|^2}{|x|^2} + \int_\Omega \frac{|w|^2}{|x|} \leq \int_\Omega |\nabla w|^2 + C \int_\Omega |w|^2.$$

- If $\mu < \mu^*(N)$, one can add in the left-hand side the following term

$$s(\mu^*(N) - \mu) \iint_{\Omega \times (0,T)} \theta e^{-2\sigma} \frac{|w|^2}{|x|^2}.$$

- For any $\gamma < 2$, replacing θ by $(t(T-t))^{-1-2/\gamma}$, one can obtain in the right hand-side

$$s \iint_{\Omega \times (0,T)} \theta e^{-2\sigma} \frac{|w|^2}{|x|^\gamma}.$$

To the observability inequality

Step 3

Using Cacciopoli's inequality, one easily obtains

$$\iint_{\Omega\times(\frac{T}{4},\frac{3T}{4})} |w(x,t)|^2 \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2.$$

Due to the dissipation properties of this singular heat equation, this implies the observability inequality

$$\int_{\Omega} |w(x,0)|^2 \,\mathrm{dx} \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2 \,\mathrm{dxdt}.$$

Further comments

Other singular potentials

Using the same Carleman estimate, one can prove the controllability of the heat equation with more general potentials

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u + \dfrac{m}{|x|^\gamma} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with $\gamma < 2$, and still $\mu \leq \mu^*(N)$.

Multipolar singular potentials

Multipolar case

Consider

$$\begin{cases} \partial_t u - \Delta u - \sum_i \frac{\mu_i}{|x - x_i|^2} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \text{on } \partial\Omega \end{cases}$$

where $\mu_i \leq \mu^*(N)$ for each i, and the set $\{x_i\}$ is finite.

Result

Null controllability in the multipolar case also holds !

Without loss of generality, we assume (*scaling*)

$$\inf\{|x_i - x_j|\} \geq 2, \qquad d(x_i, \partial\Omega) \geq 2.$$

Multipolar singular potentials

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Consider

$$\begin{cases} \partial_t u - \Delta u - \sum_i \frac{\mu_i}{|x - x_i|^2} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \text{on } \partial\Omega \end{cases}$$

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Result

Null controllability in the multipolar case also holds !

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Multipolar Carleman estimate

The weight function in the multipolar case

The following choice gives a “nice" Carleman inequality

$$\sigma(t,x) = s\theta\left(e^{2\lambda \sup \psi} - \frac{1}{2}\sum_i |x-x_i|^2\gamma(x-x_i) - e^{\lambda\psi(x)}\right),$$

where ψ satisfies

$$\begin{cases} \psi(x) = \ln(|x-x_i|), & x \in B(x_i,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \backslash \left(\cup_i \bar{B}(x_i,1)\right), \\ |\nabla\psi| \geq \delta > 0, & x \in \Omega\backslash\omega \end{cases}$$

and γ is such that

$$\gamma(x) = 1, \quad |x| \leq 1, \qquad \gamma(x) = 0, \quad |x| \geq 2.$$

Remark: A small difference in the multipolar case

From the Carleman estimate, any solution satisfies

$$\iint_{\Omega\times(\frac{T}{4},\frac{3T}{4})} |w(x,t)|^2 \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2.$$

But the system is not dissipative anymore !
Indeed, the quadratic form

$$Q(w) = \int_\Omega |\nabla w|^2 - \sum_i \int_\Omega \frac{\mu_i}{|x-x_i|^2}|w|^2$$

is not positive !

However, it is bounded from below, and this is enough to conclude!

The surcritical case $\mu > \mu^*(N)$

Known results

- Baras & Goldstein (1984): No solution for positive initial data and source, even localy in time!
- Vazquez & Zuazua (2000): Well-posed Cauchy problem in the ball when filtering.
- Vancostenoble & Zuazua (2008): Controllability in the ball for filtered initial data with a control on an annular set.

Problem

Control properties without filtering ? In any set ?

An optimal control approach

Question

Given $u_0 \in L^2(\Omega)$, can we find a control $f \in L^2((0,T); H^{-1}(\Omega))$ localized in ω such that there exists a solution $u \in L^2((0,T); H_0^1(\Omega))$ of (1) ?

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = f\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

An optimal control approach

Consider then the functional

$$J_{u_0}(u,f) = \frac{1}{2}\iint_{\Omega\times(0,T)} |u(t,x)|^2 \,\mathrm{dxdt} + \frac{1}{2}\int_0^T \|f(t)\|^2_{H^{-1}(\Omega)}\,\mathrm{dt},$$

defined on the set

$$\mathcal{C}(u_0) = \Big\{(u,f) \in L^2((0,T);H_0^1(\Omega)) \times L^2((0,T);H^{-1}(\Omega)) \text{ such that } u \text{ satisfies (1) with } f \text{ supported in } \omega\Big\}.$$

We say that system (1) is stabilizable if there exists a constant C such that

$$\forall u_0 \in L^2(\Omega), \qquad \inf_{(u,f)\in\mathcal{C}(u_0)} J_{u_0}(u,f) \leq C\,\|u_0\|^2_{L^2(\Omega)}\,.$$

A regularized problem

Problem: The Cauchy problem is ill-posed !
$\leadsto$ Fernandez-Cara & Zuazua 2000, controllable semi-linear heat equations for which blow up occurs without control.

For $\varepsilon > 0$, introduce the regularized equations

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon - \dfrac{\mu}{|x|^2 + \varepsilon^2} u^\varepsilon = f\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad u^\varepsilon(x,0) = u_0(x). \end{cases} \tag{2}$$

and the functionals

$$J^\varepsilon_{u_0}(f) = \frac{1}{2} \iint_{\Omega\times(0,T)} |u^\varepsilon(x,t)|^2 \, \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \|f(t)\|^2_{H^{-1}(\Omega)} \, \mathrm{d}t,$$

for $f \in L^2(0,T; H^{-1}(\Omega))$ localized in ω, and u^ε solution of (2).

New formulation of the stabilization problem

Uniform stabilization property

Find a constant $C > 0$ such that for any $\varepsilon > 0$ and any $u_0 \in L^2(\Omega)$,

$$\inf_{\substack{f \in L^2(0,T;H^{-1}(\Omega)) \\ f \text{ localized in } \omega}} J^{\varepsilon}_{u_0}(f) \leq C \left\| u_0 \right\|^2_{L^2(\Omega)} .$$

Theorem (SE 2008)

The uniform stabilization property is false for any open subset ω such that $0 \notin \bar{\omega}$.

Almost sharp: if $0 \in \omega$, *the system obviously is controllable.*

Idea of the proof

Look at the most explosive mode Φ_0^ε such that

$$\left(-\Delta - \frac{\mu}{|x|^2+\varepsilon}\right)\Phi_0^\varepsilon = \lambda_0^\varepsilon \Phi_0^\varepsilon, \qquad \Phi_0^\varepsilon|_{\partial\Omega} = 0.$$

Since the Hardy inequality is not satisfied, we can prove that

$$\lim_{\varepsilon\to 0} \lambda_0^\varepsilon = -\infty, \quad \forall \alpha > 0, \ \lim_{\varepsilon\to 0} \|\phi_0^\varepsilon\|_{H^1(\Omega\setminus\bar{B}(0,\alpha))} = 0.$$

The control of this mode from ω will then require a lot of energy:

$$\inf_{\substack{f\in L^2((0,T);H^{-1}(\Omega))\\ f \text{ supported in } \omega}} J^\varepsilon_{\Phi_0^\varepsilon}(f) \underset{\varepsilon\to 0}{\longrightarrow} \infty.$$

Non linear parabolic equations

The operator

$$-\Delta - \frac{\mu}{|x|^2}$$

appears as the linearization of several nonlinear elliptic problems[7]

$$\begin{cases} -\Delta u_1 = \lambda_1 e^{u_1}, \quad -\Delta u_2 = \lambda_2 (1+u_2)^p \quad \text{in } B(O,1) \\ u = 0 \quad \text{on } S(0,1) \end{cases}$$

Problem

Can we control the corresponding nonlinear parabolic equation around the stationary state u_1, (respectively u_2)?

[7] Brézis & Vazquez, 1997

Wave equation with an inverse-square potential

Wave equation

Under which geometrical assumption can we control the following equation ?

$$\begin{cases} \partial_{tt}^2 u - \Delta u - \dfrac{\mu}{|x|^2} u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v\chi_\Gamma, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

•*Partial result in Vancostenoble-Zuazua 2009:*
Exact controllability holds under a multiplier condition $\{x.\vec{n} > 0\} \subset \Gamma$.

Thanks

Thank you for your attention !

More details available in

Control and stabilzation properties for a parabolic equation with an inverse square potential, CPDE, 2008

and by email at `sylvain.ervedoza@math.uvsq.fr`