Inverse Scattering on Noncompact Manifolds

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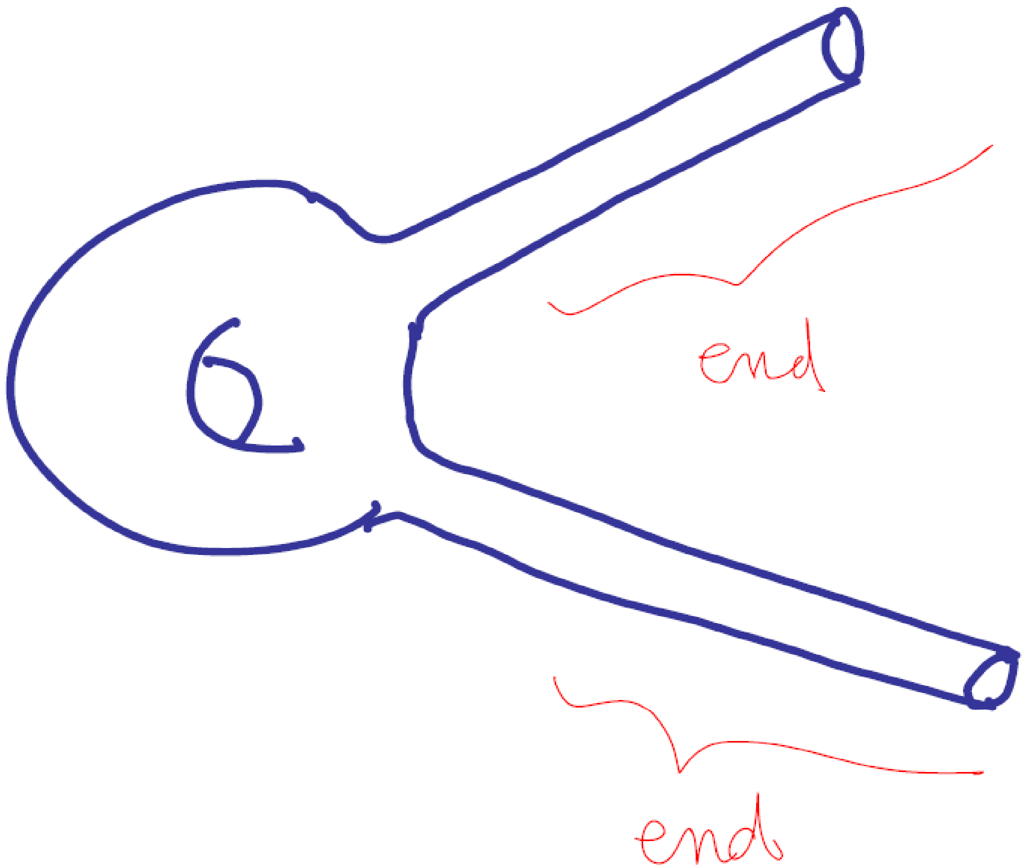
The manifold we are going to discuss consists of

compact part (arbitrary)

\+ non compact part (end)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Two important examples

- Waveguides = cylindrical ends with asymptotically Euclidean metric
- General 2-dim. Hyperbolic manifolds

The results

- Suppose :

  (a) The S-matrix corresponding to one end coincides, and

  (b) two metrics coincide on that end.

Then :

two metrics are globally isometric.

More precisely

Fix one manifold.

Consider two metrics on it.

Pick up one end.

Send waves from that end, and observe the scattered waves on the same end.

Suppose two metrics coincide on that end.

Then, these two Riemannian metrics are globally isometric.

The 1st part : Waveguide problem

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N$$

$$\mathcal{M}_i \simeq M_i \times (0, \infty)$$

M_i is a compact manifold equiped with metric $h_i(x, dx)$ with or without boundary. If it has a boundary, we impose Dirichlet, or Neumann boundary condition.

Daily life example

- Settings of optical and electric cables
- Oil, gas, water pipe lines

Assumptions on the metric

The metric on $\mathcal{M}_i$ behaves like

$$ds^2 \simeq (dy)^2 + h_i(x, dx), \quad x \in M_i, \quad y \in (0, \infty)$$

in the short-range sense. i.e.

$$ds^2 - (dy)^2 - h_i(x, dx) = O(y^{-1-\epsilon_0}), \quad \epsilon_0 > 0,$$

together with its derivatives.

Unperturbed Laplace-Beltrami operator on the end

To fix the idea, we consider the case in which M_i has a boundary assuming Neumann boundary condition. Let

$$H_j^{(0)} = -\partial_y^2 - \Delta_{h_j}$$

be the associated Laplace-Beltrami operator on $M_i \times (0, \infty)$ with Neumann boundary condition.

Cosine transform on the end

Let $\lambda_{j,1} < \lambda_{j,2} \leq \cdots$ be the eigenvalues of $-\Delta_{h_j}$ and $\varphi_{j,n}(x)$ the associated normalized eigenvectors. We define

$$\mathcal{F}_j^{(0)}(\lambda)f = \sum_{n=1}^{\infty} c_{\lambda_{j,n}}(\lambda)\mathcal{F}_{j,n}^{(0)}(\lambda)f,$$

where $c_{\lambda_{j,n}}(\lambda)$ is the characteristic function of $(\lambda_{j,n}, \infty)$

$$\mathcal{F}_{j,n}^{(0)}(\lambda)f(x) = \mathcal{F}_{cos}(\lambda - \lambda_{j,n}^{(0)})P_{j,n}f(x),$$

$$\mathcal{F}_{cos}(\lambda)f(x) = \pi^{-1/2}\lambda^{-1/4}\int_0^{\infty}\cos(y\sqrt{\lambda})f(x,y)dy,$$

$$P_{j,n}f(x) = \langle f(\cdot,y), \varphi_{j,n}\rangle\varphi_{j,n}(x).$$

Unitarity of the cosine transforms

$\mathcal{F}_j^{(0)} : L^2(\Omega_j) \to \widehat{\mathcal{H}}_j$ is unitary, where

$$\widehat{H}_j = \sum_{n=1}^{\infty} L^2((\lambda_{j,n}, \infty); d\lambda) \otimes \varphi_{j,n}(x)$$

$$= \left\{ \sum_{n=1}^{\infty} f_n(\lambda)\varphi_{n,j}(x); f_n \in L^2((\lambda_{j,n}, \infty)) \right\}$$

Diagonalization of free Laplacians

We also put

$$\mathcal{F}^{(0)} = \left(\mathcal{F}_1^{(0)}, \cdots, \mathcal{F}_N^{(0)}\right)$$

This diagonalizes free Laplacians

$$\mathcal{F}^{(0)} : (H_1^{(0)}, \cdots, H_N^{(0)}) \to (\lambda, \cdots, \lambda).$$

Partition of unity

Let $\chi_j \in C^\infty(\mathcal{M})$ be such that $\chi_j(y) = 1$ on $\mathcal{M}_j$ if $y > 2$ and $\chi_j(y) = 0$ if $0 < y < 1$. Let $\chi_0 = 1 - \sum_{j=1}^{N} \chi_j$. Then $\{\chi_j\}_{j=0}^{N}$ is a partition of unity on $\mathcal{M}$.

Wave operators

Let H be the Laplace-Beltrami operator on $\mathcal{M}$. Then the wave operators

$$W_{\pm} = \mathrm{s} - \lim_{t \to \pm\infty} \sum_{j=1}^{N} e^{it\sqrt{H}} \chi_j e^{-it\sqrt{H_j^{(0)}}}$$

exist and satisfy

$$\mathrm{Ran}(W_{\pm}) = \mathcal{H}_{pp}(H)^{\perp} = \mathcal{H}_{ac}(H),$$

where $\mathcal{H}_{pp}(H)$ is the closure of the linear hull of all eigenvectors of H, and $\mathcal{H}_{ac}(H)$ is the absolutely continuous subspace for H.

Asymptotically free waves

It means that for any $f \in \mathcal{H}_{ac}(H)$ as $t \to \pm\infty$

$$e^{-it\sqrt{H}} f \simeq \sum_{j=1}^{N} \chi_j e^{-it\sqrt{H_j^{(0)}}} f_{j\pm}^{(0)}.$$

On each end the waves behave like free waves.

Scattering operator

The scattering operator is defined by

$$S = (W_+)^* W_-.$$

It maps the wave pattern of the remote past to that of the remote future:

$$S : \begin{pmatrix} f_{1-}^{(0)} \\ \cdot \\ \cdot \\ \cdot \\ f_{N-}^{(0)} \end{pmatrix} \longrightarrow \begin{pmatrix} f_{1+}^{(0)} \\ \cdot \\ \cdot \\ \cdot \\ f_{N+}^{(0)} \end{pmatrix}$$

S-matrix

We put

$$\widehat{S} = \mathcal{F}^{(0)} S (\mathcal{F}^{(0)})^*.$$

Then for any $\lambda > 0$, there exists

$$\widehat{S}(\lambda) \in \mathbf{B}\big(\oplus_{j=1}^{N} L^2(M_j); \oplus_{j=1}^{N} L^2(M_j)\big)$$

such that

$$\big(\widehat{S} f\big)(\lambda) = \widehat{S}(\lambda) f(\lambda), \quad \forall f \in \oplus_{j=1}^{N} \widehat{H}_j.$$

The structure of the S-matrix

$$\widehat{S}(\lambda) = \left(\widehat{S}_{jk}(\lambda)\right)_{1\leq j,k\leq N}$$

This is an $N \times N$ (operator-valued) matrix. Each entry $\widehat{S}_{jk}(\lambda)$ is a finite size matrix, whose size $\to \infty$ as $\lambda \to \infty$.

Why this structure appears?

$H_j^{(0)} = -\partial_y^2 - \Delta_{h_j}$ has generalized eigenfunctions

$$\Psi_{j,n}(\lambda; x, y) = \cos(\sqrt{\lambda - \lambda_{j,n}}\; y)\varphi_{j,n}(x).$$

This means that if the total energy λ is fixed, physicaly, you will see only waves f the form

$$\Psi_{j,n}(\lambda; x, y), \quad \lambda_{j,n} \le \lambda.$$

So, you will see only those wave patterns $\Psi_{j,n}(\lambda)$ such that $\lambda_{j,n} \le \lambda$.

Important assumption

Assume on $\mathcal{M}_1$

$$ds^2 = (dy)^2 + h_1(x, dx)$$

(exactly equal).

Result (Y.Kurylev-M.Lassas-H.I)

Suppose we are given 2 such metrics, and suppose furthermore

$$\widehat{S}_{11}^{(1)}(\lambda) = \widehat{S}_{11}^{(2)}(\lambda), \quad \forall \lambda.$$

Then two manifolds are isometric.

Eigenfunctions

On the end Ω_1, $-\partial_y^2 - \Delta_{h_1}$ has the eigenfunction
$\Psi_{1,n}^{(0)}(\lambda) = \pi^{-1/2}(\lambda - \lambda_{1,n})^{-1/4}\cos(y\sqrt{\lambda - \lambda_{1,n}})\varphi_{1,n}(x)$.
$H = -\Delta_G$ has the eigenfunction $\Psi_{1,n}$ satisfying

$$(H - \lambda)\Psi_{1,n} = 0$$

and also

$$\Psi_{1,n} \simeq \Psi_{1,n}^{(0)}, \quad \text{as} \quad y \longrightarrow \infty$$

on Ω_1. Moreover

$$P_{1,m}\left(\Psi_{1,n} - \Psi_{1,n}\right) \sim C_{1m,1n}(\lambda)A_{1m,1n}(\lambda),$$

as $y \longrightarrow \infty$ on Ω_1.

Analytic continuation of the S-matrix

$$\widehat{S}_{11}(\lambda) = 1 - 2\pi i \sum_{\lambda_{1,m}<\lambda, \lambda_{1,n}<\lambda} A_{1m,1n}(\lambda)$$

$A_{1m,1n}(\lambda)$ is defined for $\lambda > \max\{\lambda_{1,m}, \lambda_{1,n}\}$. It has an analytic continuation to $\mathbf{C}_+$, aned then to the half-real axis $\{\lambda \in \mathbf{R}\ ;\ \lambda < \max\{\lambda_{1,m}, \lambda_{1,n}\}\}$, which is denoted by $A^{(nph)}_{1m,1n}(\lambda)$.

Analytic continuation of eigenfunctions

$\Psi^{(0)}_{1,n}(\lambda)$ has an anlytic continuation

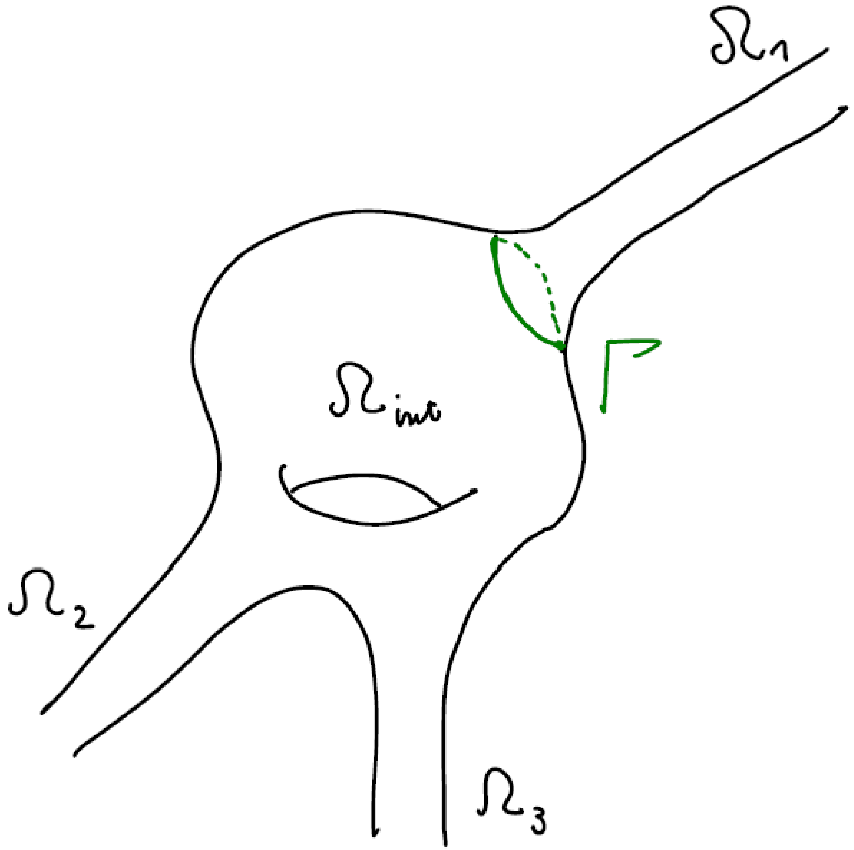
$$\Phi^{(0)}_{1,n}(\lambda) = \pi^{-1/2} e^{-\pi i/4} \big(\lambda_{1,n} - \lambda\big)^{-1/4} \cosh\left(y\sqrt{\lambda_{1,n} - \lambda}\right) \varphi_{1,n}(x).$$

$\Psi_{1,n}(\lambda)$ has an analytic continuation $\Phi_{1,n}(\lambda)$, which behaves like

$$P_{1,m}\left(\Phi_{1,n}(\lambda) - \Phi^{(0)}_{1,n}(\lambda)\right) \sim C'_{1m,1,n}(\lambda) A^{(nph)}_{1m,1n}(\lambda)$$

as $y \to \infty$ on Ω_1.

“Interior” boundary value problem



N-D map

Solve the following boundary value problem in Ω_{int} :

$$\begin{cases} (-\Delta_G - \lambda)u = 0 \quad \text{in} \quad \Omega_{int}, \\ \partial_\nu u = f \quad \text{on} \quad \Gamma, \\ \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega \cap \Omega_{int}, \\ u \text{ satisfies the radiation condition.} \end{cases}$$

N-D map $\Lambda_\Gamma(\lambda)$ is defined by

$$\Lambda_\Gamma(\lambda) : f \to u\big|_\Gamma.$$

From S-matrix to N-D map

Lemma The linear span of $\partial_\nu \Psi_{1,n}(\lambda)\big|_\Gamma$, $\partial_\nu \Phi_{1,n}(\lambda)(\lambda)$, $n = 1, 2, \cdots$, is dense in $L^2(\Gamma)$.

To prove this lemma, we introduced the *non-physical* (exponentially growing) eigenfunction $\Phi_{1,n}(\lambda)$, which is the analytic continuation of the *physical* eigenfunction $\Psi_{1,n}(\lambda)$.

Suppose two metrics $G^{(1)}$ and $G^{(2)}$ coincide on $\Gamma \setminus \Omega_{int}$, and $\widehat{S}_{11}^{(1)}(\lambda) = \widehat{S}_{11}^{(2)}(\lambda)$, $\forall\lambda$.

Theorem $\widehat{S}_{11}^{(1)}(\lambda) = \widehat{S}_{11}^{(2)}(\lambda), \ \forall\lambda,$

$$\Longrightarrow \Lambda_{\Gamma}^{(1)}(\lambda) = \Lambda_{\Gamma}^{(2)}(\lambda), \ \forall\lambda.$$

Brief look at the B-C method

Gel'fand Problem Let M be a compact manifold with boundary, and $-\Delta_g$ the Laplace-Beltrami operator with Neumann boundary condition. Let $\lambda_1 < \lambda_2 \leq \cdots$ be the eigenvalues, and $\varphi_1, \varphi_n, \cdots$ the associated eigenvectors. Then does the boundary spectral data (BSD)

$$\{(\lambda_j, \varphi_j|_{\partial M}; j = 1, 2, \cdots\}$$

determine the Riemannian metric?

- Gelfand problem is solved affirmatively by the boundary control method (BC method) due to M.I.Belishev (1988) and M.I.Belishev-Y.Kurylev (1992).

References

- M. I. Belishev, Boundary control in reconstruction of manifolds and metrics (the BC method), Inverse Problems 13 (1997), R1-R45.
- M. I. Belishev, Recent progress in the boundary control method, Inverse Problems 23 (2007), R1-R67.
- A. Katchalov, Y. Kurylev and M. Lassas, Inverse Boundary Spectral Problems, Chapman and Hall, 123 (2001)

Boundary spectral projection

- What is actually used in the BC method is the boundary spectral projection (BSP) :

$$\{(\lambda_j,\ \delta_S^* P_j \delta_S);\ j = 1,2,\ldots\},$$

$$\lambda_j = eigenvalue,$$

$$P_j = eigenprojection$$

- Here δ_S is the adjoint of the operator

$$\delta_S : \ f \in H^{-1/2}(S) \to u \in H^{-1}(M), \ \ S = \partial M,$$

which is the adjoint of the trace operator

$$\delta_S^* : H^1(M) \to tr\ u \in H^{1/2}(S)$$

such that

$$(\delta_S^* u, \psi)_S = (tr\ u, \psi)_S = (u, \delta_S \psi)_M$$

To give BSP is equivalent to give the N-D map :

$$\begin{cases} (-\Delta_g - z)u = 0, & in \quad M, \\ \partial_\nu = f, & on \;\; \partial M, \end{cases}$$

$$\Lambda(z) : f \to u\big|_{\partial M}$$

- To give the N-D map is equivalent to give

$$\delta_S^* (-\Delta_g - z)^{-1} \delta_S$$

- In this form, the Gel'fand problem can be extended to non-compact manifolds.

Modification of the BC-method

$-\Delta_G$ on Ω_{int} with Neumann boundary condition has the generalized Fourier transform $\mathcal{F}'(\lambda)$. It also has embedded eigenvalues μ_j in $[0, \infty)$ and the associated eigenprojection $P_j'(\lambda)$.

Theorem The N-D map $\Lambda_\Gamma(\lambda)$ determines BSP in Ω_{int} :

$$\mu_j, \quad \delta_\Gamma^* P_j'(\lambda)\delta_\Gamma, \quad \delta_\Gamma^* \mathcal{F}'(\lambda)^* \mathcal{F}'(\lambda)\delta_\Gamma.$$

Here δ_Γ is the operator

$$\delta_\Gamma : H^{-1/2}(\Gamma) \ni f \to u \in H^{-1}(M),$$

which is the adjoint of te trace operator

$$\delta_\Gamma^* : H^1(M) \ni u \to \operatorname{tr} u \in\to H^{1/2}(\Gamma)$$

such that

$$(\delta_\Gamma^* u, \psi)_\Gamma = (\operatorname{tr} u, \psi)_\Gamma.$$

BC method from local boundary

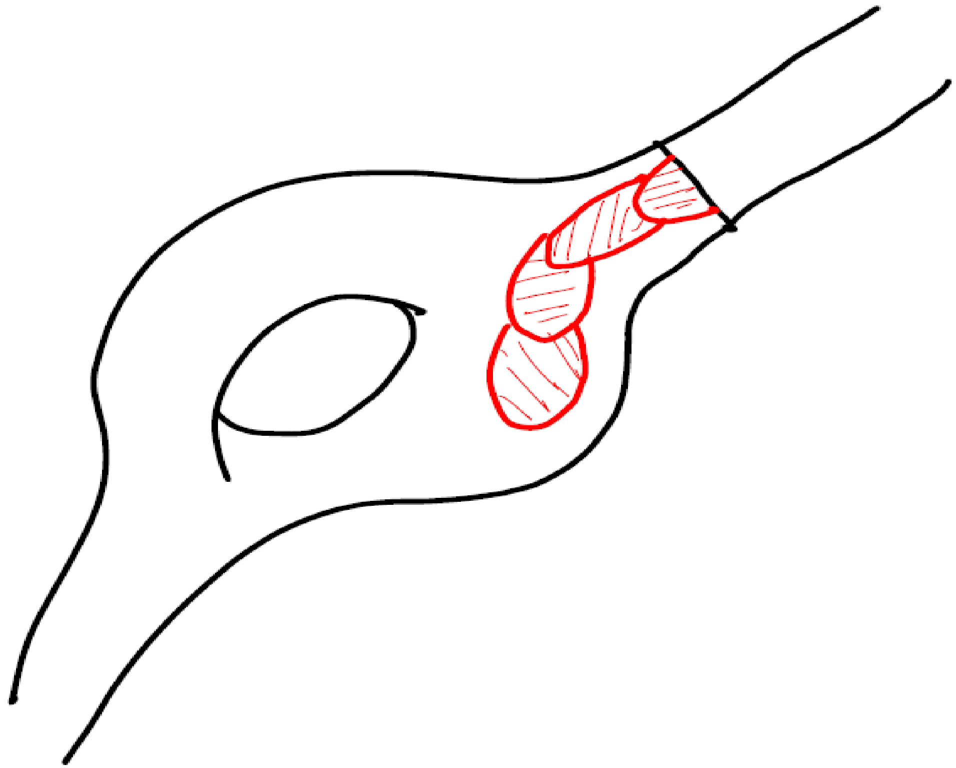
- A.Kachalov, Y.Kurylev and M.Lassas,
  Energy measurements and equivalence of boundary data for inverse problems on non-compact manifolds,
  IMA volumes in Mathematics and Applications (Springer Verlag) Geometric Methods in Inverse Problems and PDE Control,
  Eds., C. Croke, I.Lasiecka, G.Uhlmann, M.Vogelius (2004), 183-214.

Local N-D map

Let $M^{(1)}$ and $M^{(2)}$ be Riemannian manifolds (not necessarily compact) with boundary. We equip $\partial M^{(r)}$ with the Riemannian metric induced from the metric from $M^{(r)}$. We say that $M^{(1)}$ and $M^{(2)}$ have a common part $\Gamma^{(1)} = \Gamma^{(2)}$ on the boundary if there exists an open set $\Gamma^{(r)} \subset \partial M^{(r)}$ and a diffeomorphism $\phi : \Gamma^{(1)} \to \Gamma^{(2)}$. Let $\Lambda^{(r)}(z)$ be the N-D map for the Laplace-Beltrami operator on $M^{(r)}$. Then we defeine

$$\Lambda^{(1)}(z)\Big|_{\Gamma^{(1)}} = \Lambda^{(2)}(z)\Big|_{\Gamma^{(2)}} \Longleftrightarrow \phi \circ \Lambda^{(1)}(z)\Big|_{\Gamma^{(1)}} = \Lambda^{(2)}(z)\Big|_{\Gamma^{(2)}} \circ \phi.$$

Step by step enlargement



Theorem

One can show that (with some additional assumptions) if $M^{(1)}$ and $M^{(2)}$ have Γ in common and the same N-D map (in the sense that the above equation holds for any $z \notin \mathbf{R}$), then $M^{(1)}$ and $M^{(2)}$ are isometric.

- The paper of Kachalov-Kurylev-Lassas (2004) is a conference report, and we need to add lots of arguments to implement it. Of course, it is possible.

The 2nd part

Asymptotically hyperbolic metric

Classification of 2-dim. Hyperbolic manifolds

On the upper-half space $\mathbf{C}_+ = \{z = x+iy \; ; \; y > 0\}$, we equip the hyperbolic metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

The infinity is

$$\partial \mathbf{C}_+ = \mathbf{R} \cup \{\infty\}.$$

Example of tessellation

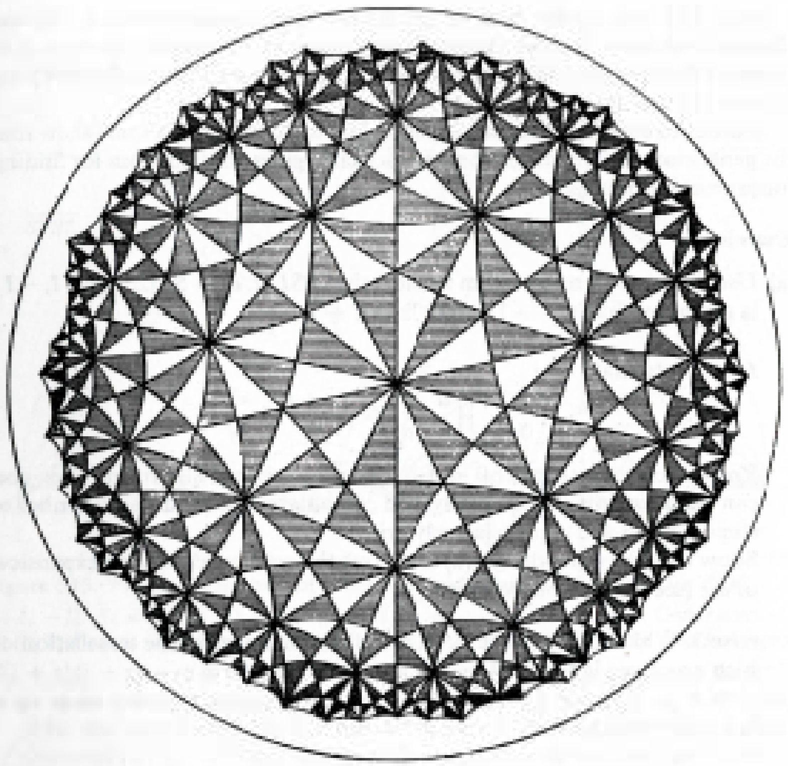


Figure 3.16. Tessellation of the unit disc. (From Klein and Fricke [1]. Reprinted by permission of Teubner.)

Example of tessellation

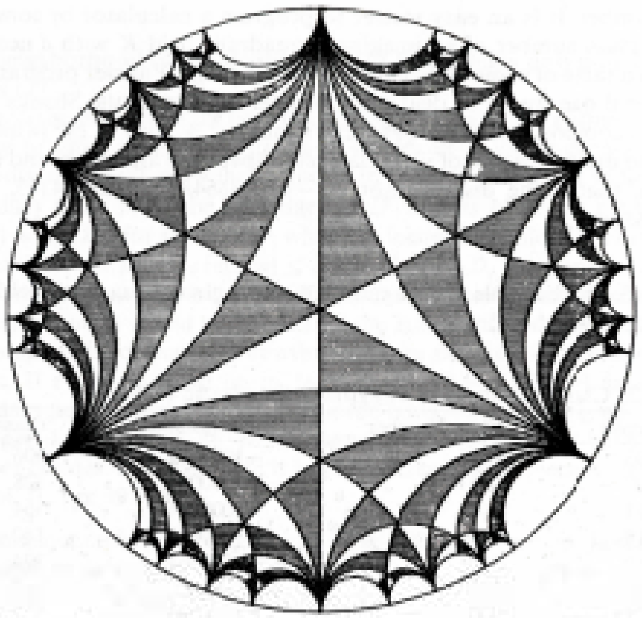


Figure 3.17. Another tessellation of the unit disc. (From Klein and Fricke [1]. Reprinted by permission of Teubner.)

Non-compact case

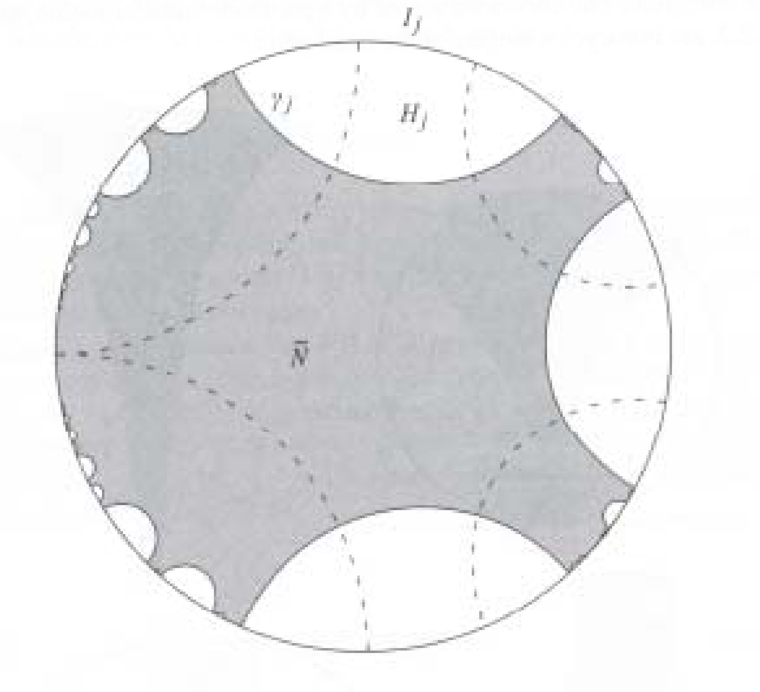


Fig. 2.8. Nielsen region.

The fundamental domain is surrounded by doted lines.

Moebius transformation

The action of

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}),$$

on $\mathbf{C}_+$ is defined by

$$\mathbf{C}_+ \ni z \to \gamma z := \frac{az + b}{cz + d}.$$

Classification of the action

$$
\begin{aligned}
\text{elliptic} &\iff \exists 1 \text{ fixed point } \in \mathbf{C}_+ \\
&\iff |\text{tr } \gamma| < 2,
\end{aligned}
$$

$$
\begin{aligned}
\text{parabolic} &\iff \exists 1 \text{ degenerate fixed point } \in \partial\mathbf{C}_+ \\
&\iff |\text{tr } \gamma| = 2,
\end{aligned}
$$

$$
\begin{aligned}
\text{hyperbolic} &\iff \exists 2 \text{ fixed points } \in \partial\mathbf{C}_+ \\
&\iff |\text{tr } \gamma| > 2.
\end{aligned}
$$

Discrete subgroup

$$SL(2,\mathbf{R}) \supset \Gamma \quad : \quad \text{discrete subgroup,}$$

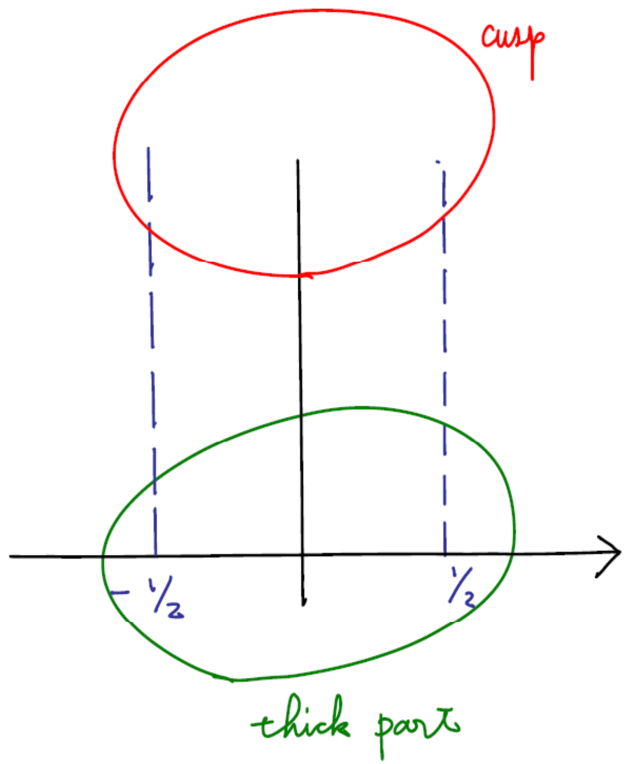
$$\mathcal{M} = \Gamma\backslash\mathbf{H}^2 \quad : \quad \text{fundamental domain.}$$

Γ is geometrically finite
$\iff M$ is a finite-sided convex polygon
$\iff \Gamma$ is finitely generated.

Example (translation)

Ex 1 $\Gamma: z \to z+1$

$$M = (-\frac{1}{2}, \frac{1}{2}) \times (0, \infty), \quad ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

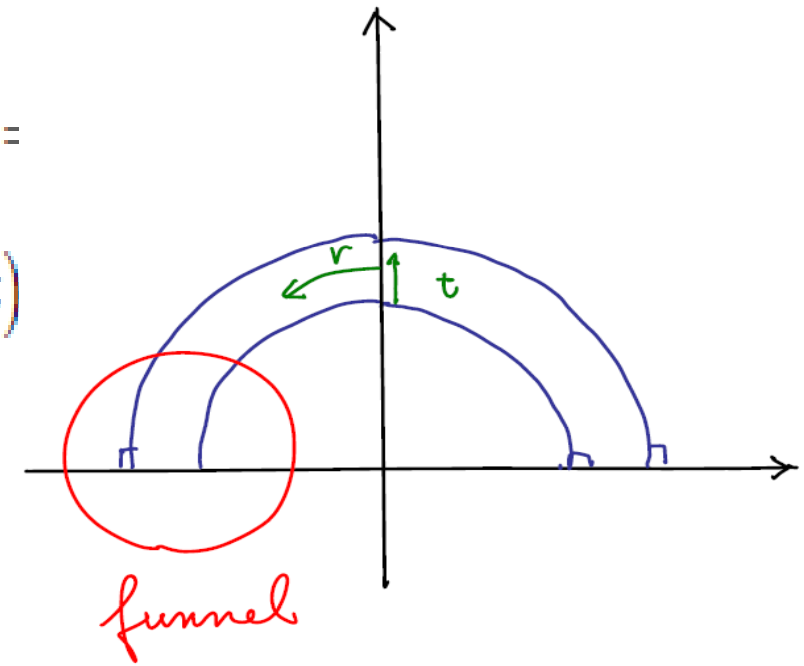


Example (dilation)

Ex 2 $\Gamma : z \to \lambda z \ (\lambda > 1)$

$$ds^2 = (dr)^2 + \cosh^2 r (dt)^2 \quad (y =$$

$$= \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} + \frac{y}{4}\right)^2 (dt)$$



Note that near the infinity of the funnel, the metric behaves like

$$ds^2 = \frac{(dy)^2 + (1 + O(y^2))(dx)^2}{y^2}.$$

These examples are called "elementary"

$$\Lambda(\Gamma) = \{\text{the limit points of the orbit } \gamma z; \gamma \in \Gamma\}$$

$$\Lambda(\Gamma) \subset \partial \mathbf{C}_+$$

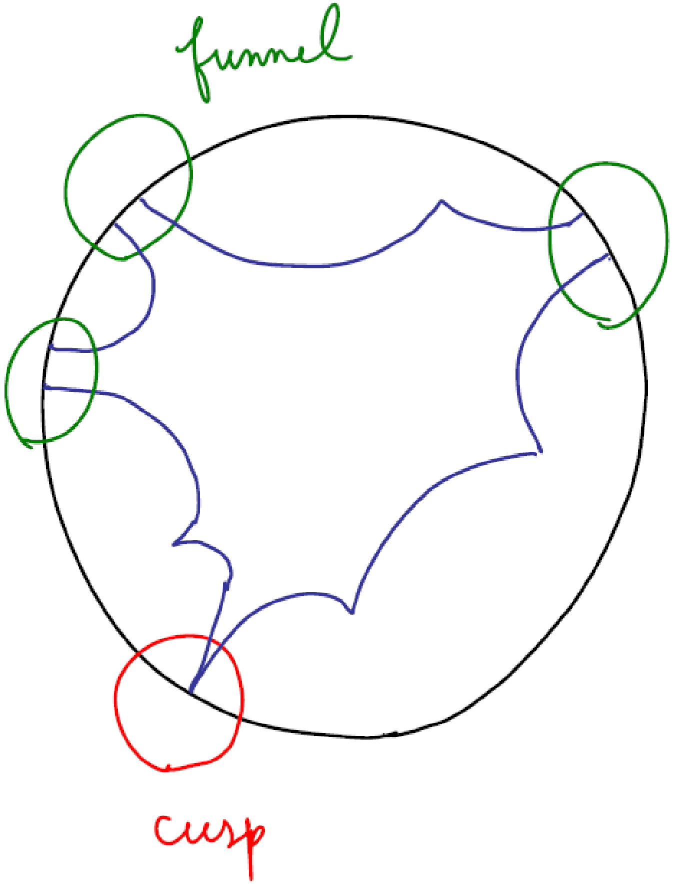
$$\Gamma \text{ is elementary} \iff {}^\sharp\Lambda(\Gamma) < \infty$$

Classification theorem

Theorem $M = \Gamma \backslash \mathbf{H}^2$ is non-elementary, geometrically finite

$\Longrightarrow$ $\exists K$ compact set, s.t. $M \setminus K$ is s finite union of cusps and funnels.

Almost general example



Asymptotically hyperbolic manifold

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N,$$

$$\mathcal{K} = \text{compact set}$$

For $1 \leq i \leq m$, $\mathcal{M}_i$ is regular in the sense that

$$\mathcal{M}_i \simeq M_i \times (0,1), \quad M_i = \text{compact manifold},$$

$$ds^2 \simeq \frac{(dy)^2 + h_i(x, dx)}{y^2},$$

$$\left|(y\partial_y)^k(\partial_x)^l\left(y^2(ds)^2 - (dy)^2 - h_i(x,dx)\right)\right|$$
$$\leq C_{kl}(1 + |\log y|)^{-1-\epsilon_0},$$

For $m+1 \le i \le N$, $\mathcal{M}_i$ is a cusp, i.e.

$$\mathcal{M}_i \simeq M_i \times (1, \infty),$$

$$ds^2 \simeq \frac{(dy)^2 + h_i(x, dx)}{y^2},$$

$$\left|(y\partial_y)^k (y\partial_x)^l \left(y^2(ds)^2 - (dy)^2 - h_i(x, dx)\right)\right|$$
$$\le C_{kl}(1 + |\log y|)^{-1-\epsilon_0},$$

The Besov type space

M = compact manifold

$$\mathcal{H} = L^2(\mathbf{R}; L^2(M))$$

Rigged Hilbert space

$$\mathcal{B} \subset \mathcal{H} \subset \mathcal{B}^*$$

$u \in \mathcal{B}^* \iff u$ is an $L^2(M)$-valued function of $y \in \mathbf{R}$ such that

$$\|u\|_{\mathcal{B}*} = \left(\sup_{R>e} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|u(y)\|^2 \frac{dy}{y^n} \right)^{1/2} < \infty$$

$n \geq 2$ a fixed integer

$\mathcal{B}$ is the space of $L^2(M)$-valued function $f(y)$ such that

$$\|f\|_{\mathcal{B}} = \sum_{k\in\mathbf{Z}} e^{|k|/2} \left(\int_{I_k} \|f(y)\|^2 \frac{gy}{y^n} \right)^{1/2} < \infty$$

$$I_k = \begin{cases} (\exp(e^{k-1}), \exp(e^k)], & k \geq 1, \\ (e^{-1}, e], & k = 0, \\ (\exp(-e^{|k|}, \exp(-e^{|k|-1})], & k \leq -1. \end{cases}$$

Asymptotic relation

We write $f \simeq g$ if

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|f(y) - g(y)\| \frac{dy}{y^n} = 0$$

The generalized Fourier transform

Theorem Let $f \in \mathcal{B}$, $k^2 \in \sigma_e(H) \setminus \sigma_p(H)$, and χ_j the partition of unity. Then we have

$$R(k^2 \pm i0)f$$

$$\simeq \ \omega_\pm(k) \sum_{j=1}^{M} \chi_j y^{(n-1)/2 \mp ik} \mathcal{F}_j^{(\pm)}(k) f$$

$$+\omega_\pm^{(c)}(k) \sum_{j=M+1}^{N} \chi_j y^{(n-1)/2 \pm ik} \mathcal{F}_j^{(\pm)}(k) f.$$

Theorem We define $(\mathcal{F}^{(\pm)}f)(k) = \mathcal{F}^{(\pm)}(k)f$ for $f \in \mathcal{B}$. Then $\mathcal{F}^{(\pm)}$ is uniquely extended to a bounded operator from $L^2(\mathcal{M})$ to $\widehat{\mathcal{H}}$ with the following properties.

(1) Ran $\mathcal{F}^{(\pm)} = \widehat{\mathcal{H}}$.

(2) $\|f\| = \|\mathcal{F}^{(\pm)}f\|$ for $f \in \mathcal{H}_{ac}(H)$.

(3) $\mathcal{F}^{(\pm)}f = 0$ for $f \in \mathcal{H}_p(H)$.

(4) $\left(\mathcal{F}^{(\pm)}Hf\right)(k) = k^2\left(\mathcal{F}^{(\pm)}f\right)(k)$ for $f \in \mathrm{Dom}\, H$.

(5) $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$ and $(H - k^2)\mathcal{F}^{(\pm)}(k)^* = 0$ for $k^2 \in (0,\infty) \setminus \sigma_p(H)$.

(6) For $f \in \mathcal{H}_{ac}(H)$, the inversion formula holds:

$$f = \left(\mathcal{F}^{(\pm)}\right)^* \mathcal{F}^{(\pm)} f = \sum_{j=1}^{N} \int_0^\infty \mathcal{F}_j^{(\pm)}(k)^* \left(\mathcal{F}_j^{(\pm)} f\right)(k)dk.$$

The Helmholtz equation

Theorem (1) For any $u \in \mathcal{B}^*$ satisfying $(H - k^2)u = 0$, there exists a unique $\psi^{(\pm)} \in \mathbf{h}_\infty$ such that

$$u \simeq \omega_-(k) \sum_{p=1}^{M} \chi_p y^{(n-1)/2+ik} \psi_p^{(-)}$$

$$+ \omega_-(k) \sum_{p=M+1}^{N} \chi_p y^{(n-1)/2-ik} \widehat{\psi}_{p0}^{(-)}$$

$$- \omega_+(k) \sum_{p=1}^{M} \chi_p y^{(n-1)/2-ik} \psi_p^{(+)}$$

$$- \omega_+^{(c)}(k) \sum_{p=M+1}^{N} \chi_p y^{(n-1)/2+ik} \psi_p^{(+)}.$$

The S-matrix

(2) For any $\psi^{(-)} \in \mathbf{h}_\infty$, there exists a unique $\psi^{(+)} \in \mathbf{h}_\infty$ and $u \in \mathcal{B}^*$ satisfying $(H - k^2)u = 0$, for which the expansion (1) holds. Moreover

$$\psi^{(+)} = \widehat{S}(k)\psi^{(-)}.$$

Result (Y.Kurylev-H.I)

Theorem Suppose we are given two such manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ equipped with metrics $G^{(1)}$, $G^{(2)}$. Suppose $m \geq 1$ (which means that the associated end is not the cusp) and

$$S_{11}^{(1)}(\lambda) = S_{11}^{(2)}(\lambda), \quad \forall \lambda.$$

Moreover $G^{(1)} = G^{(2)}$ on $\mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)}$.

$\Longrightarrow \mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric.

Inverse scattering from cusp (work in progress)

- Kurylev-Lassas-I are trying to extend this result to the case of inverse scattering from cusp.
- The arithmetic surface $C_+ / SL(2, Z)$ is a basic example of hyperbolic manifold with cusp, which is also a classical example of orbifold.