Inverse Spectral Problem for ~~Orl~~ Riemannian Orbifolds

(with M. Lassas & T. Yamaguchi).

In this talk I'll be speaking about some inverse problems for orbifolds. What is an orbifold more precisely, Riemannian orbifold? The notion of an orbifold was introduced by Satake in late 50th (under the name of V-manifolds) and was given the second life by Thurston (see his book, GT3M) later in 90th. Having said so, it has never lost its appeal to theoretical physicists (I'll speak about this later).

Orbifold is the simplest extension of manifold allowing some singularities. The simplest picture occurs if we consider, in terminology of Thurston, "good orbifolds".

Def. A good (Riemannian) orbifold is a metric space X which may be represented as a quotient of a smooth R. manifold M by the action of a finite group of isometries of (M, g).

$$X = M / F, \quad F \text{ - finite (subgroup) of the group of isometries of } M.$$

This is the way to define the metric on X. Namely, ~~consider~~ for $x, y \in X$ consider all coimages $\tilde{x}_i, \tilde{y}_j \in M$ with respect to the projection π:

$\pi : M \to X$ ~~being~~

being the natural projectore from M onto $M/\mathbb{F}$.

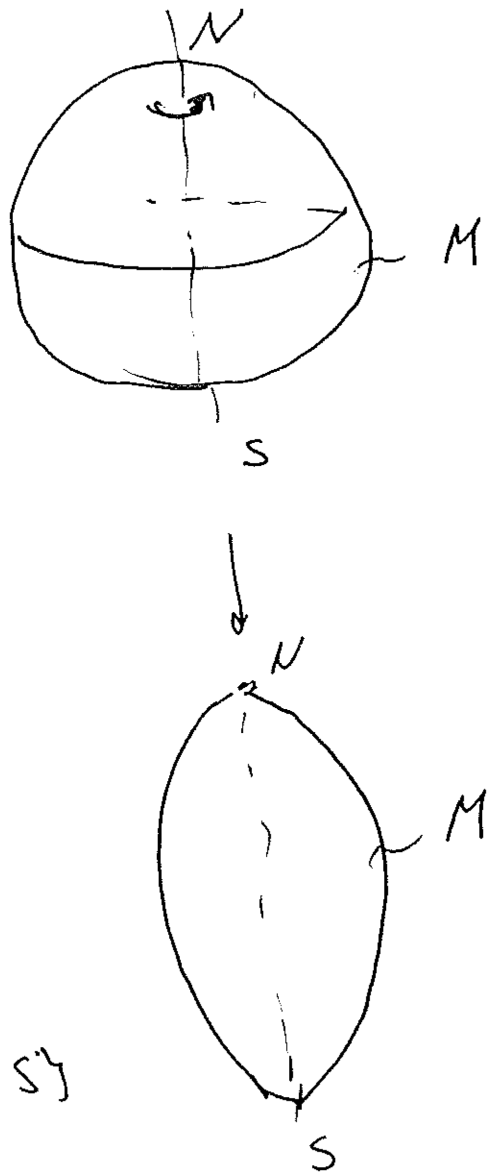
Then consider the distances (on M)

$$\tilde{d}(\tilde{x}_i, \tilde{y}_j) \text{ and } d(x,y) = \min_{i,j} \tilde{d}(\tilde{x}_i, \tilde{y}_j).$$

Let me provide a simple example:

Take $M = S_2$ - unit 2D sphere with the canonical metric and take $\mathbb{F}$ to be group of rotations by $2\pi/3$ around the vertical axis. Thus

$$\mathbb{F} \cong \mathbb{Z}_3.$$



Then X will look like a "rugby" ball with conic singularities at N and S and smooth elsewhere.

Thus, ~~x~~ if $x \in X \setminus \{N \cup S\}$ and, as intrinsic for the manifold and orbifold theory, we are working only in a small vicinity of x, the orbifold X, or better say a piece of X near x, looks just like a usual neighborhood for a manifold

This observation makes it possible to extend the notion of a (Riemannian) orbifold from ~~of~~ the good ones to general orbifolds.

Roughly speaking, ~~an~~ (Riemannian) orbifold is a metric space which locally looks like a quotient of an open set in $\mathbb{R}^n$ (with some metric tensor) under the action of a finite subgroup of the group of isometries. Thus, we have

$x \in X$, $\quad * \ U \subset X$ - open neighborhood of x,

$\tilde{B}_\rho = \tilde{U} \subset \mathbb{R}^n$, $\quad \tilde{g}$ - metric tensor on $\tilde{U}$,

a finite group Γ_x of isometries of $(\tilde{U}, \tilde{g})$, i.e.

$$\gamma^* \tilde{g} = \tilde{g}, \quad \gamma^* \in \Gamma_x$$

with

$$U \cong \tilde{U} / \Gamma_x,$$

thus defining a natural projection

$$\pi_x : \tilde{U} \to U$$

where $\pi_x^{-1}(x) = 0$.

Then, Γ_x is isomorphic to a finite subgroup of the orthogonal group $O(n)$ (this is due to

the fact that radii ~~e~~emanating from $O \in U$ should go to radii ~~under~~ by the action of Γ_x.
This definition provides also a local metric structure on X near x. We can patch this local structures together (as we do for manifolds), if we introduce a proper notion of an analog of the "coordinate transformation".

Let we introduce some names.
The system $x, U, \tilde{U}, \Gamma_x, \pi: \tilde{U} \to U, \pi(O) = x,$
is called a local uniformising system for X near x.

Let we have $x' \in U$, $U' \subset X$ and the corresponding uniformizing system, $\tilde{U}', \Gamma_{x'}, \pi': \tilde{U}' \to U'$.

A homeomorphism

$$\lambda: \tilde{U}' \to \tilde{U} \quad \text{(into)}$$

is called an injection if

a) $\pi^*(\lambda \tilde{y}') = \pi'(\tilde{y}')$, $\tilde{y} \in \tilde{U}$

b) $\pi^*(\lambda \gamma' \tilde{y}') = \pi(\tilde{y}')$, $\gamma' \in \Gamma_{x'}$ $\quad \lambda\gamma' = \gamma\lambda$

Then there is ~~a~~ $\Gamma_{x'}$ is isomorphic to a subgroup in Γ_x;

$$\lambda^*: \Gamma_{x'} \xrightarrow{\sim} \Gamma_x$$

This homomorphism λ^* is uniquely defined by λ.

At last, when dealing with the Riemannian case, i.e. assuming that $\tilde{U}$, $\tilde{U}'$ are equipped with the metric tensors $\tilde{g}, \tilde{g}'$, we require

$$\lambda^*(\tilde{g}) = \tilde{g}'.$$

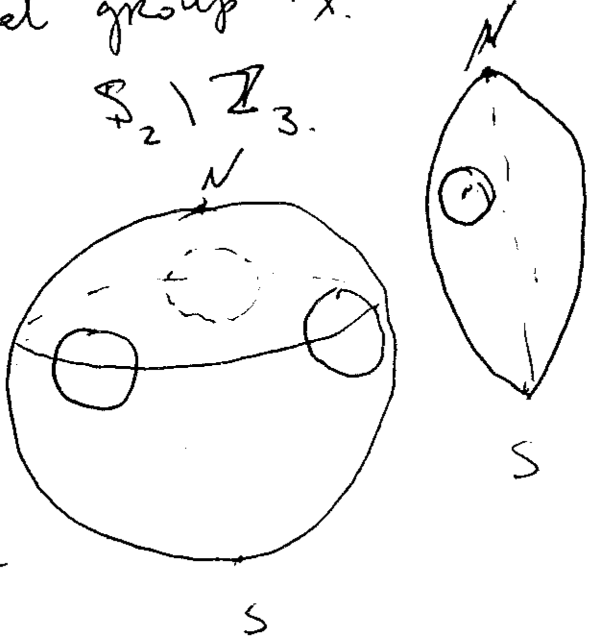
Returning to the notion of a (Riemannian) orbifold we require that, for any uniformizing systems $(x \in U, \tilde{U}, \Gamma_x, \tilde{g})$ and $(x' \in U', \tilde{U}', \Gamma_{x'}, \tilde{g}')$ with

$$U' \subset U,$$

there is a corresponding Riemannian injection

λ.

Roughly speaking, this definition reflects the fact that if we are on a smaller portion of U, the group of isometries of this portion may be smaller than the original group Γ_x. Let us look at our example $S_2 \setminus \mathbb{Z}_3$.



Let $x' \neq N, S$.

We can take a sufficiently small metric ball $B_\varepsilon(x') \subset S_2 \setminus \mathbb{Z}_3$

so that $\pi^{-1}(B_\varepsilon(x'))$ consists of 3 nonintersecting metric balls $B_\varepsilon(\tilde{x}'_1)$, $B_\varepsilon(\tilde{x}'_2)$, $B_\varepsilon(\tilde{x}'_3)$ on S_2.

Then a uniformizing system for x', $B_\varepsilon(x')$ ~~could~~ is be just, e.g. $\tilde{x}'_1$, $B_\varepsilon(\tilde{x}'_1)$, $\Gamma_{x'} = \{id\}$.

Def. A point $x \in X$ is called singular if $\Gamma_x \neq \{id\}$.

As for any x, $\Gamma_x \subset O(n)$, the set X^s of singular points is, at most, of dimension $(n-1)$. Moreover, if $\Gamma_x \subset SO(n)$, for any $x \in X$, ~~the~~ X^s is, at most, of dimension $(n-2)$.

In our consideration we assume

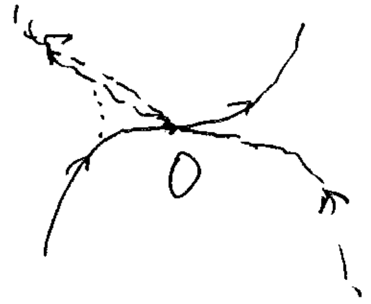
<u>Condition O</u>: For any $x \in X$, $\Gamma_x \subset SO(n)$.

Note that this condition was assumed by Satake and is related with orientability of the orbifold. Indeed, if we assume possibility of reflections in Γ_x then reflections change orientation contradicting orientability of X. I'll later underline the difficulties which may occur if we assume that Γ_x may contain reflections.

An important property of a Riemannian orbifold is that it is a <u>length space</u>. This means that for any $x, y \in X$ there is a shortest curve, $\mu_{xy}(t)$ with its length

$$|\mu_{xy}| = d(x,y).$$

Observe that, except for probably the end points, $\mu_{xy} \cap X^s = \emptyset$. Indeed, if $\Gamma_z \neq \{id\}$, the coimage $\tilde{\mu}_{xy}$ in the vicinity of O,



consists of several paths so we can short cut from one path to another to avoid O.

Also, μ_{xy} is a geodesic (for the same reasons as in the manifold case). Thus any two points $x, y \in X$ can be connected by a shortest geodesic avoiding X^s.

Until now we were speaking about orbifolds without boundary. The notion of the orbifold boundary (absent in the original paper of Satake) was introduced by Thurston (and thus is often called Thurston boundary). It is defined similar to the manifold case:

$x \in X$ is on the boundary $\partial_{Th} X$ of X if there exists a uniformizing system $x \in U$, $\tilde{U} \subset \mathbb{R}^n_+$, Γ_x

~~such that~~ and a map $\pi : \tilde{U} \to U$, with π being a natural projection onto $\tilde{U}/\Gamma_x$ with $\pi(x) = 0$.
(Typically $\tilde{U} = B_\rho(0)$ if we are inside X or $B_\rho^+(0)$ if we are on the boundary).

Observe that $\Gamma_x : \partial_{Th} X \cap U \to \partial_{Th} X \cap U$, so that $\Gamma_x \in O(n-1)$ (or $SO(n-1)$ if we avoid reflections).

Orbifolds with boundary are still length spaces and any shortest curve is a C^1-curve avoiding X^s with its components in X^{int} or $\partial_{Th} X$ being pieces of geodesics.

$\partial_{Th} X$ is itself an $(n-1)$-dimensional R. orbifold and, from the above, if $x \in \partial_{Th} X$, then
$$\Gamma_x = \Gamma_x^\partial,$$
where Γ_x^∂ is the group for x corresponding to the ~~orb~~ boundary orbifold $\partial_{Th} X$.

Remark. We can now look at the difficulties due to an assumption that Γ_x, for some x, contain reflections. Let $M = S^2_{\pm}$, D - equator.

Consider now to objects:

a) Upper half-sphere S^2_+ (closed, with D being its boundary. S^2_+ is a nice manifold (orientable) with boundary.

b) Orbifold $X = S^2_+ \setminus \{id, r\}$, where r is the reflection wrt to D. X is (non-orientable) orbifold without boundary. But this is, in some sense the same geometric object - same geodesics, same distance

So we see that we can hardly distinguish X and S^2_+ geometrically. Certainly, not all manifolds with boundary can be orbifolds without boundary.

A necessary condition is that the boundary should be totally geodesic. However, this clearly is a difficult

~~Later we look at analysis, ...~~

Analysis on ~~mani~~ orbifolds. Using uniformizing systems we can easily introduce various classes of functions and differential operators on orbifolds. Say $u \in C(X)$, if in any uniformising coordinate system $\tilde{U} \xrightarrow{\pi} U = \tilde{U}/\Gamma_X$, $\pi^* u = u^\circ$ is continuous (and, clearly, Γ_X-invariant).

A local differential operator (with Γ_X-invariant coefficients) is defined as

$A(x,D)u \overset{def}{=} A^*(\tilde{x},D)u^*/|_x$,

where $A^*(\tilde{x},D)$ is an operator with Γ_x- ~~per~~ invariant coefficients so that $A^*(\tilde{x},D)u^*$ is Γ_x-invariant and we can define its push forward by π, to V.

In particular, we can use this way to determine the Laplace operator, Δ, on X as also the corresponding heat and wave equations

$$\partial_t + \Delta \text{ and } \partial_t^2 - \Delta.$$

Traditional results regarding uniqueness, solvability and smoothness of solutions remain (with the corresponding change of understanding) valid for these objects. In particular, if X is compact and we require Dirichlet b.c. on $\partial_{Th} X$, the spectrum of $-\Delta$ is discreet, with the corresponding λ_j satisfying ~~the~~ Weyl's law, eigenfunctions φ_j are C^∞ and form an orthonormal basis in $L^2(X)$. (These and similar results are now well-known in analysis due to mainly works by E. Dreyden, C. Gordon and Elizabeth Stanhope).

Thus, we are, in principle, able to formulate and try to s~~olve~~ answer various inverse problems which are familiar from the manifold case.

However, before going into IP per se let me make two remarks.

a) let us return to the example of S^2_+ and

$X = \mathbb{S}^2/\{id, r\}$. Clearly, the eigenfunctions of the Laplace on X are just those eigenfunctions of the Laplace on $\mathbb{S}^2$ (up to a normalizing factor) which are even wrt to the reflection r. Thus, they satisfy the Neumann "boundary c." on D and coincide with the eigenfunctions of the Neumann Laplacian on $\mathbb{S}^2_+$. This means that adding "analytic" information would still not allow us to distinguish between X and $\mathbb{S}^2_+$. In the case of Dirichlet, we would be able to make this distinction, however in a more general IP, when we do not know the whole $\partial_{Th} X$ and the type of bc on this Thurston boundary we would again come to an ambiguity.

b). A natural question is why should we be interested in the Laplace and other type of equation on orbifolds, in particular, IP? There are two major reasons for that:

1. R. orbifolds is currently a common place in the modern theoretical physics, especially different types of the field theory. Again, there are, I believe, two principal reasons, why physicists like orbifolds. Firstly, they want to deal with the models which allow for singularities.

There ~~should~~ singularities should not be nasty, should occupy only a small fraction of the space and allow for some symmetries in the considered models (physicists adore symmetries!) Orbifolds perfectly satisfy all those requirements. Secondly, the unified fields theory starts from the assumption that we live on a higher dimensional manifold where all fields (gravitational, em, strong and weak quantum mech., etc) is actually one and the same field. This higher dimensional manifold is metrically very small in some directions/dimensions (imagine a very thin but a finite length donut - an ideal for the capitalist "profit margins") Then, ~~tending this~~ collapsing in this dimension we arrive at a smaller dimensional object where the laws for gravitational, em and other fields decouple. (And in the case of a donut we obtain a circle) So, in some sense, we are living on a higher dimensional space-time manifold but in some directions it is very thin so that we feel it to be 4D.

Among physicists, it is quite popular to think of this manifold as 5=4+1 dimensional (Kaluza-Klein theory and its modern analogs). It is, however known that if, under some mild restrictions on the curvature, we allow a smooth manifold to collapse in one-dimension, we arrive to an orbifold. Wonderful! That's brings physicists back to the object which they like for various different reasons. (At this stage I speak about orbifolds without boundary, whether a one-dimensional collapse of manifolds with boundary gives rise to an orbifold - with or without boundary is still under investigation

b). Analysis of stability of IP on manifold / anisotropic spaces. As we see collapse in one-dimensions gives rise to an orbifold of a 1-smaller dimensions. On the other hand, in real life collapse is the question of multiscales: if we have an object extremely thinner than the wave lengths we use to look at it, it would appear to be of a smaller dimensions. Therefore, in our study of stability ~~for~~ of IP which are extremely thin in some directions we should consider simultaneously IP on these manifolds _and_ the spaces which appear as the result of their collapse. The simplest object which occur, different from just a manifold, is the orbifold. Moreover, in the case of a 1D collapse (and it is possible to formulate some simple invariant conditions to guarantee that collapse is no worse than 1D) the resulting object is an orbifold (hopefully) so that we are obliged (~~even~~ to study the stability of IP even for the higher dimensional thin manifolds) to study IP on orbifolds. (By the way ~~the~~ conditions which would guarantee collapse of manifolds to a manifold, except for very restrictive and akward to formulate, are yet unknown).

I am now in the position to formulate the IP we are interested in.

Def. a). Let $\Omega \subset X^{int} \setminus X^s$ be a non-empty open set. The local spectral data (LSD) for X is the set

$$(\Omega, \{\lambda_k, \varphi_k|_\Omega\}|_{k=1}^{\infty}),$$

where, let me remind you, λ_k and φ_k are the eigenvalues and orthonormal eigenfunction of ~~the~~ a Laplace operator on X with some classical boundary conditions.

b). Let $\Sigma \subset \partial_{Th} X \setminus X^s$ be a non-empty open set on $\partial_{Th} X$. The local boundary spectral data (LBSD) for X is the set

$$(\Sigma, \{\lambda_k, \partial_\nu \varphi_k|_\Sigma\}_{k=1}^{\infty}),$$

Here ∂_ν is the normal derivative to $\partial_{Th} X$, and we assume that, on Σ, we have Dirichlet b.c.

Thm a) Let $(\Omega, \{\lambda_k, \varphi_k|_\Omega\}_{k=1}^{\infty})$ and $(\Omega', \{\lambda_k', \varphi_k'|_{\Omega'}\}_{k=1}^{\infty})$ be LSD of R. orbifolds X, X' which satisfy Cond O (orientability, absense of reflections).

Assume that $\lambda_k = \lambda_k'$, $k=1,2,\ldots$, and there exists a homeomorphism $\Phi: \Omega \to \Omega'$ such that

$$\varphi_k = \Phi^* \varphi_k', \quad k=1,2,\ldots$$

Then the orbifolds X, X' are Riemannian isomorphic.

b) Let $(\Sigma, \{\lambda_k, \partial_\nu \varphi_k|_\Sigma\})$ and $(\Sigma', \{\lambda_k', \partial_\nu \varphi_k'|_{\Sigma'}\})$ be LBSD of R. orbifolds X, X' which satisfy Cond. C

Assume that $\lambda_k = \lambda_k'$ and there exists a homeomorphism $\Phi: \Sigma \to \Sigma'$ such that

$$\partial_\nu \varphi_k = \Phi^*(\partial_\nu \varphi_k'), \quad k=1,2\ldots$$

Then the orbifolds X, X' are ~~iso~~ Riemann isomorphi

Let me introduce the notion of isomorphism of R. orbifolds. It is a natural extension of the notion of isometry of R. manifolds.

We deal with orbifolds X, X'. ~~and et~~ Assume that there are coverings of $X = \bigcup_\alpha U_\alpha$ and $X' = \bigcup_\alpha U_\alpha'$, $\alpha \in A$, such that, for any α

there is ~~$\tilde{F}_\alpha: \tilde{U}_\alpha \to \tilde{U}_\alpha'$ - a homeomorphism~~, a homeomorphism

$$F_\alpha : U_\alpha \to U_\alpha',$$

its lift

$$\tilde{F}_\alpha : \tilde{U}_\alpha \to \tilde{U}_\alpha'$$

and associated group isomorphism

$$F_\alpha^* : \Gamma_{x_\alpha} \to \Gamma_{x_\alpha'}$$

s.t.

$$\pi'(\tilde{F}_\alpha \tilde{x}) = F_\alpha(\pi \tilde{x}); \quad \tilde{x} \in \tilde{U}_\alpha$$

$$\tilde{g}_\alpha = \tilde{F}_\alpha^* \tilde{g}_\alpha' ;$$

$$\tilde{F}_\alpha(\gamma \tilde{x}) = (F_\alpha^* \gamma)(\tilde{F}_\alpha \tilde{x}), \quad \gamma \in \Gamma_{x_\alpha}$$

Clearly, there should be some natural conditions satisfied if we have two injections

$$V \xrightarrow{\lambda} U, \quad V' \xrightarrow{\lambda'} U',$$

where V and V', U and U' are associted pairs of "coordinate charts" but I am not going to go into details (they are straightforward but lengthy to accurately define)

Extra: Let $\lambda: V \to U$, $\lambda': V' \to U'$,

$F, \tilde{F}, F^{*}$ and $G, \tilde{G}, G^{*}$

are the introduced functions for V, V' and U, U'.

~~Then~~ Let also $\lambda^{+}, \lambda'^{+}$ be the homomorphism (injective of Γ_y into Γ_x, $y \in V$, and $\Gamma_{y'}$ into $\Gamma_{x'}$, $y' \in V'$.

Then

$$\cancel{\tilde{F}(\lambda z) = \lambda \tilde{F}(z), \quad z \in V},$$

$$\tilde{F}(\lambda z) = \lambda' \tilde{F}(z), \quad z \in V; \qquad (1)$$

$$F^{*} \lambda^{*} = (\lambda')^{*} G^{*}$$

where the left and right side are injective homomorphism from Γ_y to $\Gamma'_{x'}$. Similar identities should be valid when dealing with F^{-1} and G^{-1} instead of F and G.

Note. ~~Strictly speaking condition (1) may be weaken. Observe that $\Gamma_y = \Gamma_x / St_y$, where St_y is the stabilizer subgroup~~ Strictly speaking (1) may be weaken. Indeed, Γ_y is the stabilizer subgroup of $\{y\}$ in Γ_x. In principle, stabilizer subgroups of other points $y_2, .., y_p$, in the orbit of Γ_x through y, may be different (although adjoint))

$$\Gamma_{y_j} = \gamma_j \circ \Gamma_y \circ \gamma_j^{-1},$$

where $\gamma_j(y) = y_j$.

This non-uniqueness may be reflected in the notion of $(\lambda')^*$ but we would not pay much attention to these nuances.

The proof of this Th. consists of two non-related steps:

Step A. To show that if the conditions of Th. are satisfied then X and X' are isometric as <u>metric spaces</u>, namely, that there is a homeomorphism $\mathbb{F}: X \to X'$ s.t.

$$d_X(x, y) = d_{X'}(x', y'), \quad x' = \mathbb{F}(x), \; y' = \mathbb{F}(y).$$

This is proven by a proper modification of the BC-method. I am not going into details of this construction as, first, it will take too much time, and, second, is well described, for the case of manifolds, in KKL "Inverse Boundary Spectral Problems". Let me, however, briefly touch upon some features due to the existence of singular points. Basically, we try to avoid them and here the fact that a shortest between $x, y \in X \setminus X^s$ lies completely in $X \setminus X^s$ is of critical importance. Let me recall that this property is valid for any orbifold, not necessarily orientable. In this connection, the fact that if X, X' satisfy conditions of Th, then they are isometric, is valid for general orbifolds.

Having shown that $X \setminus X^s$ and $X' \setminus (X')^s$ are isometric we extend this isometry to the whole X and X' just by closure.

Let me also note that, in addition to the metric isometry, we can show that, ~~if~~

$$\varphi_k(x) = \varphi_k(Fx), \quad k=1,2,\dots$$

Note: strictly speaking this is valid only for the case of simple eigenvalues, but if we assume this on Ω or Σ, the above equality remains valid throughout X.

Also, we can show that geodesic goes into geodesic and, indeed, recover geodesics on X and X'.

Step B consists of the proof of the fact that if X, X' are isometric orbifolds, then they are isomorphic.

Although the use of the above facts that the eigenfunctions on X and X' coincide in the corresponding points does slightly helps in the proof, we prove the above result using only the isometry of orbifolds. At this stage we heavily use the fact, that the stabilizer, F_x, of $x \in X$, is from $SO(n)$.

Recall the famous Meyers-Steenrod theorem for R manifolds:

Th (Meyers-Steenrod). Let (M,g), (M',g') be two R. manifolds. Assume that M and M' are isometric as metric spaces with the distance functions defined by the R. metrics g, g'. Then (M,g) and (M',g') are isometric as R. manifolds, i.e. the ~~homeomorphism~~ distance preserving homeomorphism $F: M \to M'$ is actually an isometry

Thus I am going to prove the following extension of the M.-S. Th for orbifolds:

Th (M-S. for orbifolds): Let X, X' be two R. orbifolds which are ~~compact and~~ orientable. Let X, X' are isometric ~~wrt to the metric~~ Then X, X' are isomorphic as R. orbifolds.

The proof of this result is rather long. I'll provide only the main ideas of the proof. Moreover, to avoid technicalities, I will consider only orbifolds without boundary.

The crucial step is the proof of this result for simply-connected good orbifolds.

L. Let $X = \tilde{X}/\Gamma$, $X' = \tilde{X}'/\Gamma'$ be two good orbifolds. Let $F: X \to X'$ be a distance preserving homeomorphism. Then we can lift F to $\tilde{X}$, $\tilde{F}: \tilde{X} \to \tilde{X}'$ so that the following diagram commutes:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{F}"] \arrow[d, "\pi"'] & \tilde{X}' \arrow[d, "\pi'"] \\
X \arrow[r, "F"] & X'
\end{tikzcd}$$

~~Moreover~~ Thus, $\tilde{F}$ turns out to be an isometry between R. manifolds $\tilde{X}$ and $\tilde{X}'$ and, moreover, Γ and Γ' are isomorphic

The proof of this statement is by induction. We start with $n=2$. Take balls $B_1(x)$, $B_1'(x')$, $x' = Fx$, on X and X', correspondingly. Then $\pi^{-1} B_1(x) = \tilde{B}_1(\tilde{x})$, $(\pi')^{-1} B_1'(x') = \tilde{B}_1'(\tilde{x}')$ and wlg we can assume that the injectivity radii of $\tilde{X}$ and $\tilde{X}$ are ≥ 1 (otherwise, just take B_ρ with ρ less than injectivity radii).

Observe, that ∂B_ρ, $\partial B_\rho'$ are $(n-1)$-dimensional good orbifolds:

$$\partial B_\rho = \partial \tilde{B}_\rho \approx S^{n-1} / \Gamma_x$$

with similar representation for $\partial B_\rho'$.

For $n=2$, $\Gamma_x < SO(2)$ is a finite subgroup of rotations, $\Gamma_x \approx \mathbb{Z}_p$, $\Gamma_{x'}' \approx \mathbb{Z}_{p'}$. It follows from isometry of B_ρ, B'_ρ that $p = p'$! ~~Indeed~~ Indeed, for small ε,

$$\frac{\operatorname{diam} \partial B_\varepsilon}{\pi \varepsilon} \approx p .$$

However, the distance function of B_ρ ~~agrees~~ determines uniquely the distance function on ∂B_ε, $\varepsilon \le \rho$.

This makes it possible to ~~ex~~ lift

$$F\big|_{\partial B_\rho} : \partial B_\rho(x) \longrightarrow \partial B'_\rho(x')$$

to

$$\tilde F\big|_{S^1 = \partial \tilde B_\rho} : S^1 \longrightarrow S^1 = \partial \tilde B'_\rho$$

with the following diagram being commutative

$$\begin{tikzcd}
\partial B_\rho \arrow[r, "F"] \arrow[d, "\pi \cong \mathbb{Z}_p"'] & \partial B'_\rho \arrow[d, "\pi' \cong \mathbb{Z}_p"] \\
S^1_\rho \arrow[r, "\tilde F"] & S^1_\rho
\end{tikzcd}$$

~~in a way com~~.

We can now extend $\tilde F$ from S^1 onto B_ρ. To this end observe that if $\tilde y$ lies on the radius through $\tilde z$, ~~then~~ $\in S^1_\rho$ which we denote by $R_{\tilde z}$, then

$\pi(R_{\tilde z}) = R_z$,

where R_z is the geodesic on X from x through $z = \pi \tilde z$. Moreover if $\tilde y$ satisfies

$$d_{\tilde X}(O, \tilde y) + d_{\tilde X}(\tilde y, \tilde z) = d_{\tilde X}(O, \tilde z), \qquad (1)$$

then this equation determines uniquely the point $\tilde y$. Similar on X, if

$$d_X(x, y) + d_X(y, z) = d_X(x, z),$$

then y is uniquely determined and

$$y = \pi(\tilde y).$$

Similar relations are valid on X', $\tilde X'$.

Continue now $\tilde F$ onto $\tilde B_\rho(O)$ in the following way: If $\tilde y$ satisfies (1) and $\tilde z' = (\tilde F|_{\partial \tilde B_\rho})(\tilde z)$, we define

$$\tilde F(\tilde y) = \tilde y',$$

where $\tilde y'$ is uniquely defined by

$$d_{\tilde X'}(O, \tilde y') + d_{\tilde X'}(\tilde y', \tilde z') = d_{\tilde X'}(O, \tilde z'). \qquad (2)$$

Let us show that $\tilde{F}$ is the desired isomorphism.

a) $F \circ \pi = \pi' \circ \tilde{F}$

As F is an iso~~morphism~~metry, $F(y)$ satisfies

$d_{X'}(x', z') = d_{X'}(x', F(y)) + d_{X'}(F(y), z)$, $\quad y = \pi \tilde{y}$.

But, by construction, $\tilde{y}' = \tilde{F}(\tilde{y})$ satisfies (2), so that $\pi'(\tilde{y}') = y'$ satisfies ~~the~~

$$d_{X'}(x', y') + d_{X'}(y', z'),$$

ie.
$$\pi'(\tilde{y}') = F \circ \pi(\tilde{y}).$$

b) Moreover, as for $x \in B_\rho \setminus X^S$, $\tilde{F}$ covers the isometry $F: B_\rho \to B'_\rho$, by M.-S. Th,

$$(\tilde{F}_*)^* \, \tilde{g}_{\tilde{X}'} = \tilde{g}_{\tilde{X}}$$

there. As $\tilde{g}_{\tilde{X}}$ and $\tilde{g}_{\tilde{X}'}$ are smooth, this equality is valid on the whole $\tilde{B}_\rho$.

Note that this construction implies that if $\tilde{F}|_{\partial \tilde{B}_\rho}$ is known it determines uniquely

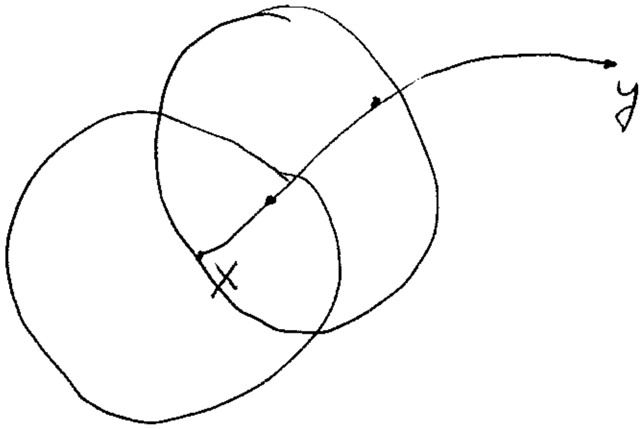
$\tilde{F} : \tilde{B}_1 \longrightarrow \tilde{B}'_1$

Note also that this construction is not specifically 2D. It is valid each time we can construct $\tilde{F}|_{\partial \tilde{B}_\rho}$.

Next we extend $\tilde{F}$ to the whole $\tilde{X}$.

Let $\tilde{x}_1 \in B_1(\tilde{x})$ together with a small ball $\tilde{B}_\varepsilon(\tilde{x}_1)$. Take $\tilde{F}|_{\partial B_\varepsilon(x_1)}$ and (uniquely) extend it to $\tilde{B}_1(\tilde{x}_1)$.

As $\tilde{X}$ is path connected, for any $\tilde{y} \in \tilde{X}$ take a path μ



from x to y and extend $\tilde{F}$ onto $B_1(\tilde{y})$ along this path. Let us show that the result, $\tilde{F}_\mu(\tilde{y})$, does not depend on μ.

Indeed, as $\tilde{X}$ is simply connected for any μ_0, μ_1 from $\tilde{x}$ to $\tilde{y}_*$ there is a continuous deformation $\mu(s)$, $\mu(0)=\mu_0$, $\mu(1)=\mu_1$, so that

$$\tilde{F}_{\mu(s)}(\tilde{y}) = \tilde{y}'(s) \in \tilde{X}'.$$

However,

$$\pi'(\tilde{y}'(s)) = (\pi' \circ \tilde{F})(\tilde{y}) = (F \circ \pi)(\tilde{y}) = F(y) = \text{const}.$$

Since ~~$\tilde{y}'$~~ Γ is finite and, therefore, discreet this implies that $\tilde{y}'(s) = \text{const}$.

Let us show that $\tilde{F}$ is an isometric covering. The same reasoning as above shows that

$$(\tilde{F})^* \cancel{k}\, \tilde{g}_{\cancel{*},} = \tilde{g}_*.$$

~~Since $\tilde{F}$ is open by construction and, as is easy to see, closed~~

Consider $\tilde{F}(\tilde{X})$. By construction $\tilde{F}(\tilde{X})$ is open in $\tilde{X}'$. Let $\tilde{y}' \in cl\, \tilde{F}(\tilde{X})$. Let $\tilde{x}' \in \tilde{F}(\tilde{X})$, $\tilde{x}' = \tilde{F}(\tilde{x})$ and $d_{\tilde{X}'}(\tilde{x}', \tilde{y}') < \varepsilon/2$

Here ε is so small that

$$\Gamma' \tilde{y}' \cap \tilde{B}'_\varepsilon(\tilde{y}') = \tilde{y}'$$

Denote $x' = \pi' \tilde{x}'$, $x = \pi \tilde{x}$, $y' = \pi' \tilde{y}'$, $y = F^{-1} y'$.

As $x = F^{-1} x'$, $d_X(x, y) < \frac{\varepsilon}{2}$.

Consider now $\tilde{F}(\tilde{y})$ where $\tilde{y}$ be a point in $\tilde{B}_\varepsilon(\tilde{x})$ with $\pi \tilde{y} = y$. As $\tilde{F}$ is an isometry

$$d_{\tilde{X}'}(\tilde{F}(\tilde{y}), \tilde{y}') \leq d_{\tilde{X}'}(\tilde{F}(\tilde{y}), \tilde{F}(\tilde{x})) + d_{\tilde{X}'}(\tilde{F}(\tilde{x}), \tilde{y}') \leq \frac{\varepsilon}{2} + d_{X'}(\tilde{x}', \tilde{y}') < \varepsilon.$$

as $\tilde{F}(\tilde{x}) = \tilde{x}'$

On the other hand,

$$\pi' \tilde{F}(\tilde{y}) = F \pi(\tilde{y}) = F y = y'.$$

Thus, $\tilde{F}(\tilde{y}) = \tilde{y}'$.

Summarizing, $\tilde{F}$ is an isometric covering of $\tilde{X}'$. As $\tilde{X}'$ is simply connected, $\tilde{F} : \tilde{X} \to \tilde{X}'$ is an isometry.

c) At last, let us show that $\Gamma \approx \Gamma'$.

For $\gamma \in \Gamma$, consider

$$\gamma' = \tilde{F} \circ \gamma \circ \tilde{F}^{-1}$$

Clearly, γ' is an isometry of $\tilde{X}'$. To show that $\gamma' \in \Gamma'$ we should show that

$$\pi' \circ \gamma' = \pi'.$$

Indeed,

$$\pi' \circ \gamma' = \pi' \circ \tilde{F} \circ \gamma \circ \tilde{F}^{-1} =$$

$$= F \circ \pi \circ \gamma \circ \tilde{F}^{-1} = F \circ \pi \circ \tilde{F}^{-1} = \pi' \circ \tilde{F} \circ \tilde{F}^{-1} = \pi'.$$

The map $\gamma \to \tilde{F} \circ \gamma \circ \tilde{F}^{-1}$ is obviously a homomorphism from Γ into Γ'. However it has an inverse $\gamma' \longrightarrow \tilde{F}^{-1} \circ \gamma' \circ \tilde{F}$.

Thus, it is an isomorphism.

This proves Lemma.

To obtain M.-S. Th for general, not only good orbifolds, observe that if U, U', $U' = FU$ are charts for uniformising systems

$$U = \tilde{B}_\rho(\tilde{x}) / \Gamma_x \, ; \quad U' = \tilde{B}'_\rho(\tilde{x}') / \Gamma_{x'},$$

we can apply our considerations to the good orbifolds U and U' to show that $\Gamma_x \simeq \Gamma_{x'}$.