Broken geodesics and inverse problems for radiative transfer equation

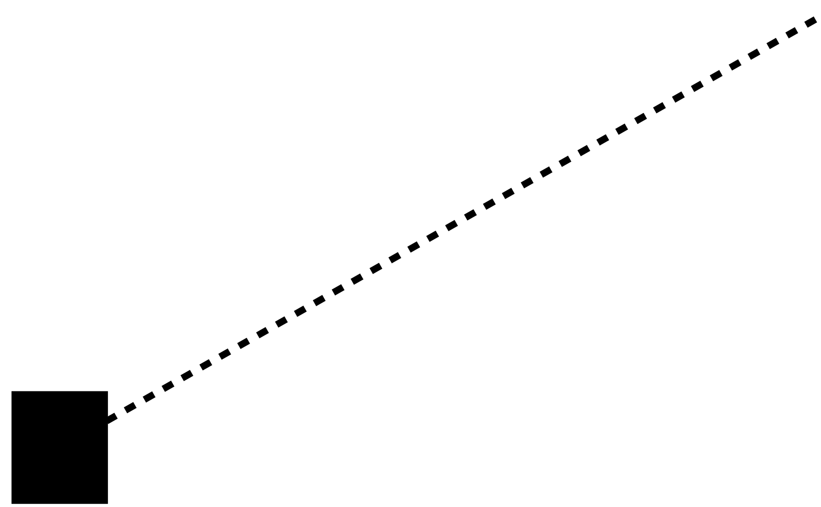
Matti Lassas

in collaboration with

Yaroslav Kurylev
Gunther Uhlmann

Finnish Centre of Excellence
in Inverse Problems Research

At Greenwich Royal Observatory, a green laser points exactly north along the prime meridian line.
Where the laser ray is exactly located?



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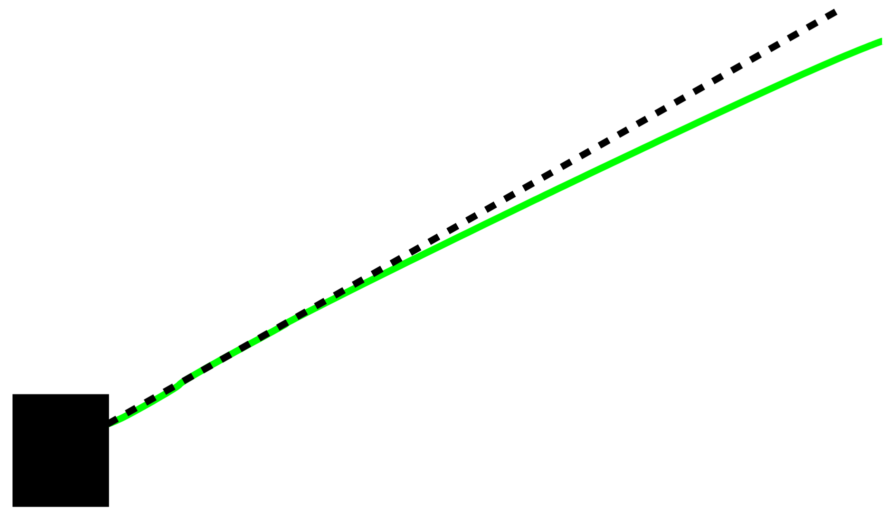

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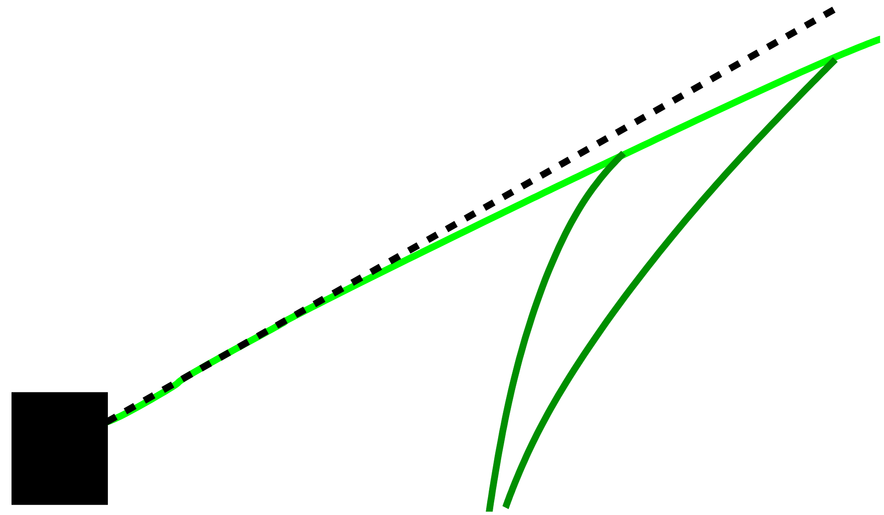

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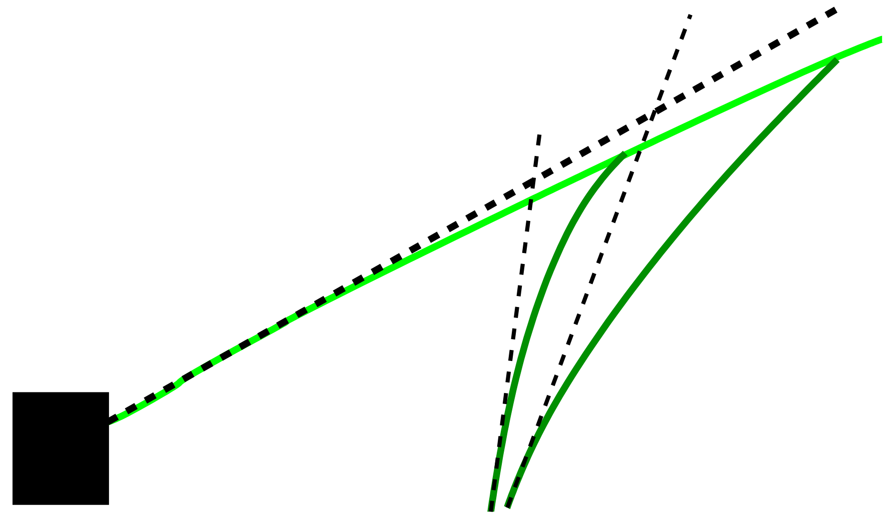
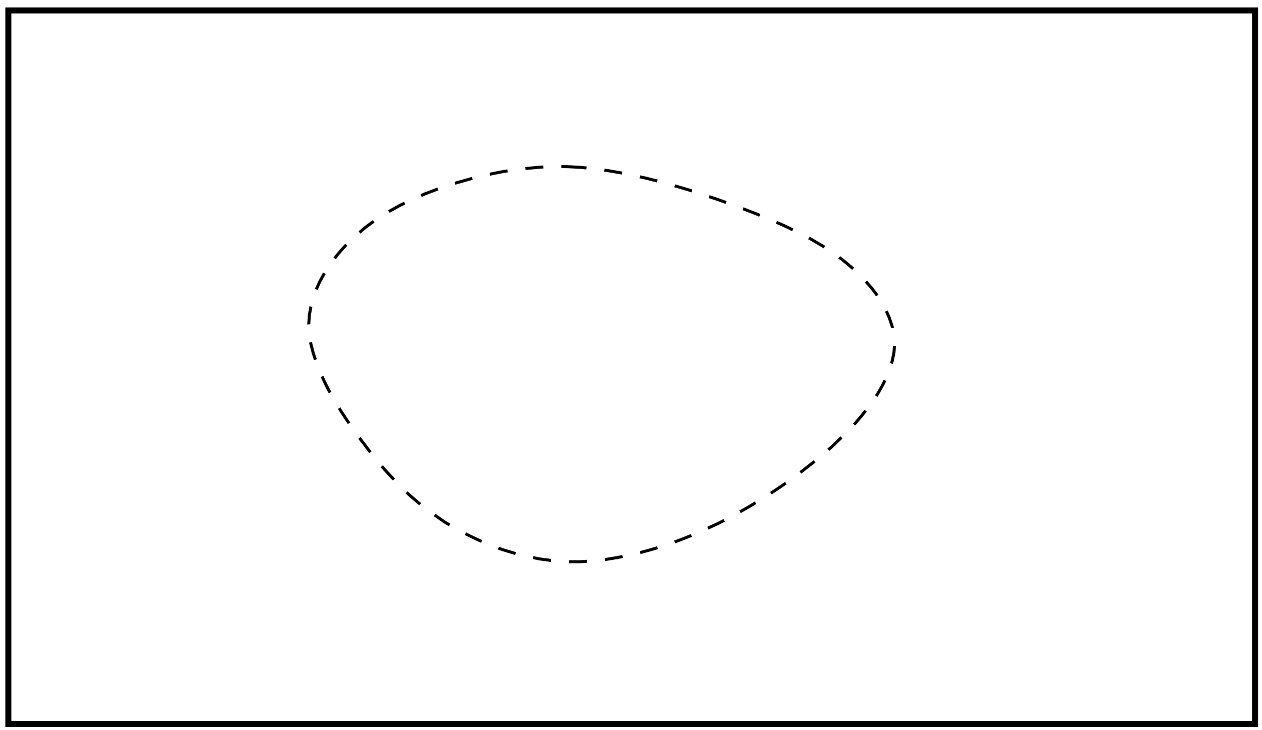


We consider radiative transfer equation with a varying light speed.
Let (N, g) be a complete Riemannian manifold and $M \subset N$ compact. Assume that g is known outside M
We send photons outside M that scatter inside M and measure the photons outside M.



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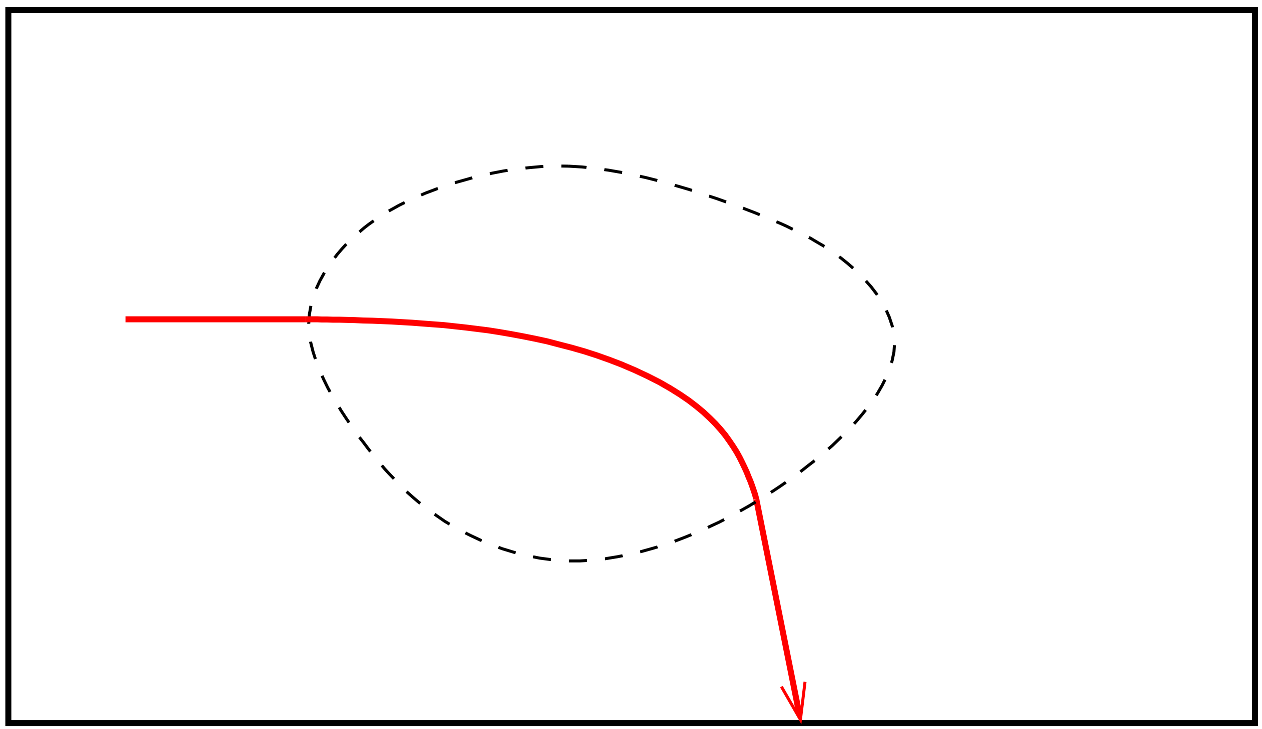

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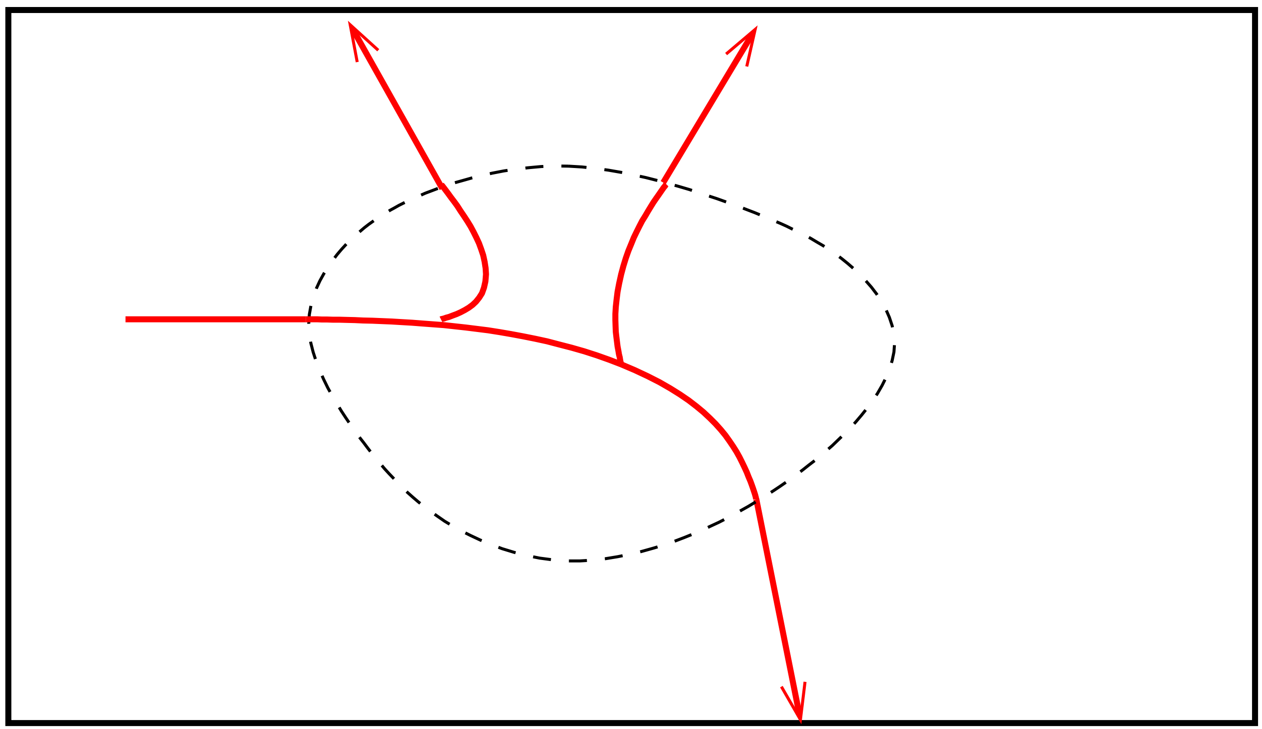

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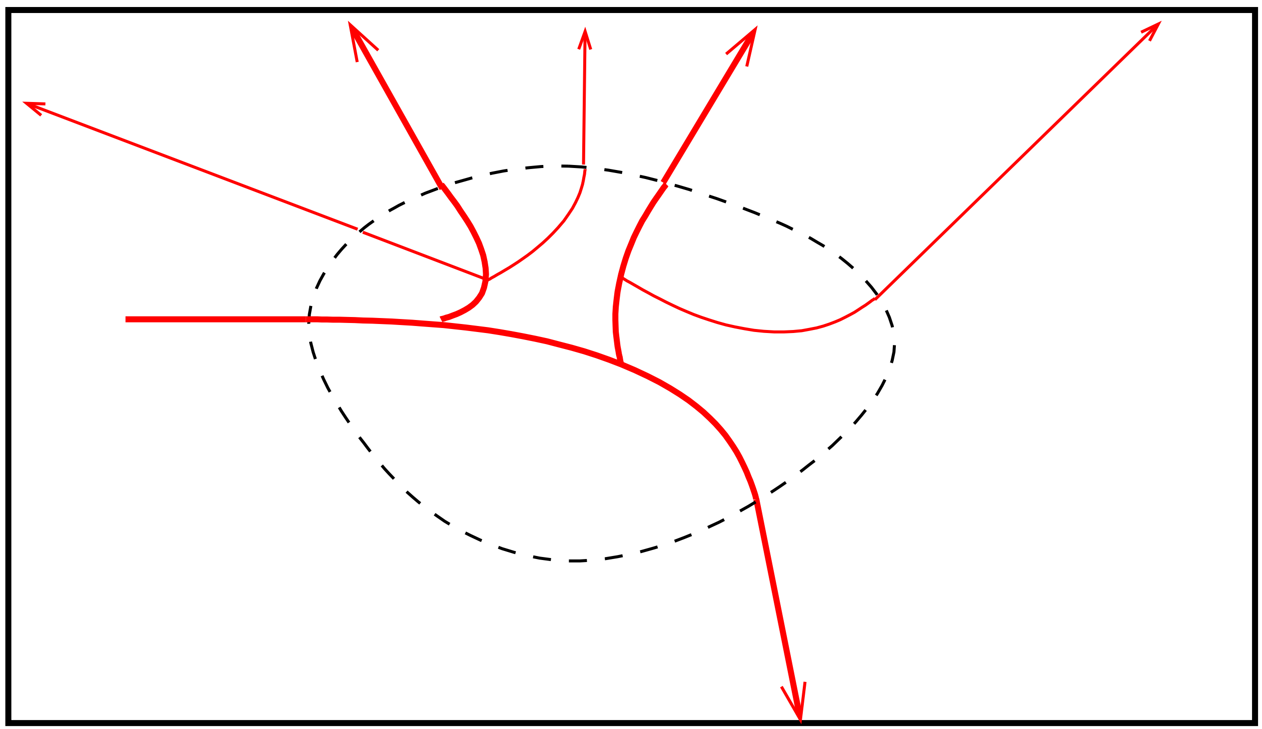


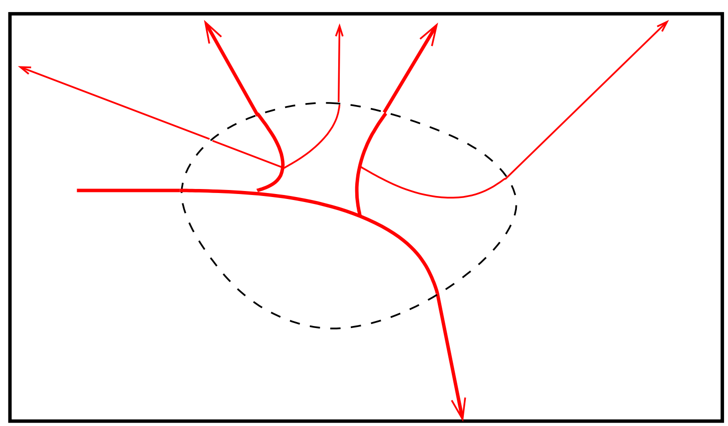
Radiative transfer equation. Consider

$$(Hu)(t,x,\xi) + \sigma(x,\xi)u(t,x,\xi) - (Ku)(t,x,\xi) = 0$$
$$u(t,x,\xi)|_{t=0} = w(x,\xi).$$

Here $t \in \mathbb{R}_+$ and $(x,\xi) \in SN = \{(x,\xi) \in TN : \ \|\xi\|_g = 1\}$.
H is the geodesic flow on the sphere bundle $SN \times \mathbb{R}$,

$$Hu(t,x,\xi) = \frac{\partial u}{\partial t} + \xi^i \frac{\partial u}{\partial x^i} - \xi^i \xi^j \Gamma^k_{ij}(x) \frac{\partial u}{\partial \xi^k},$$

$$Ku(t,x,\xi) = \int_{S_x N} K(x,\xi,\xi')u(t,x,\xi')\, dS_g(\xi').$$

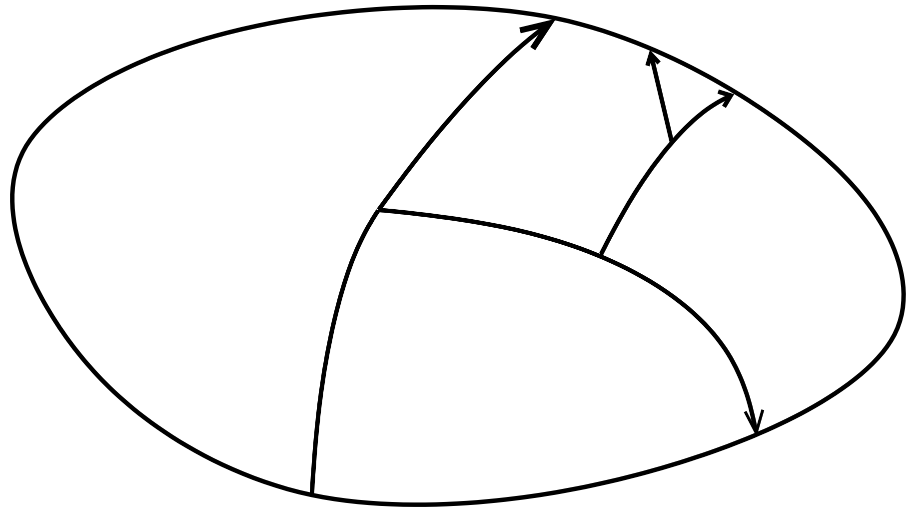


Previous results on radiative transfer the problem:

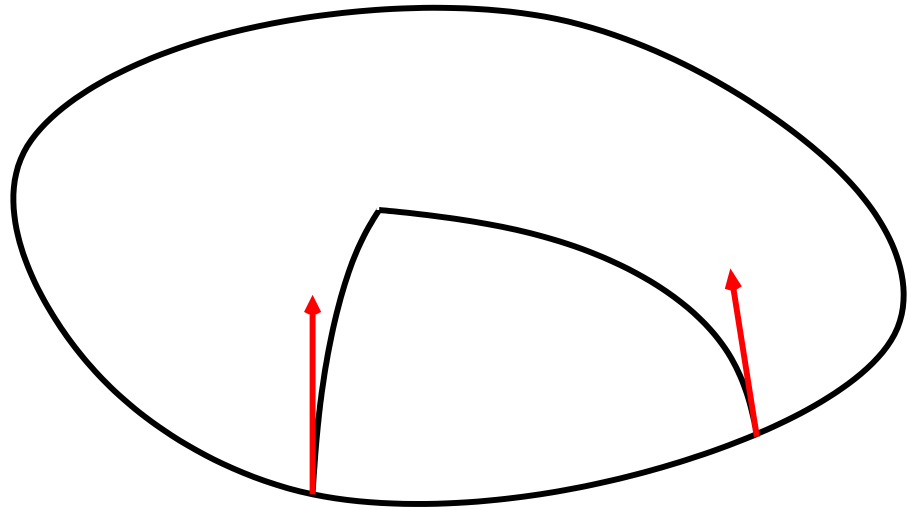
- Choulli-Stefanov, McDowall
- Arridge, Schotland
- Bal et al.

Determination of a non-trapping metric using travel times

- Mukhomethov, Romanov
- Michel
- Gromov
- Croke, Otal
- Pestov-Uhlmann
- Burago-Ivanov
- Stefanov-Uhlmann



Let us consider single scattering in M.



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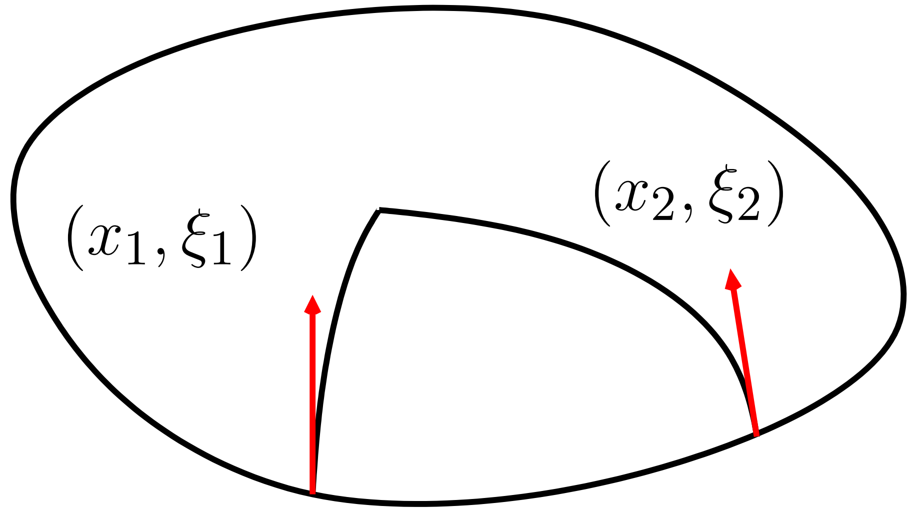


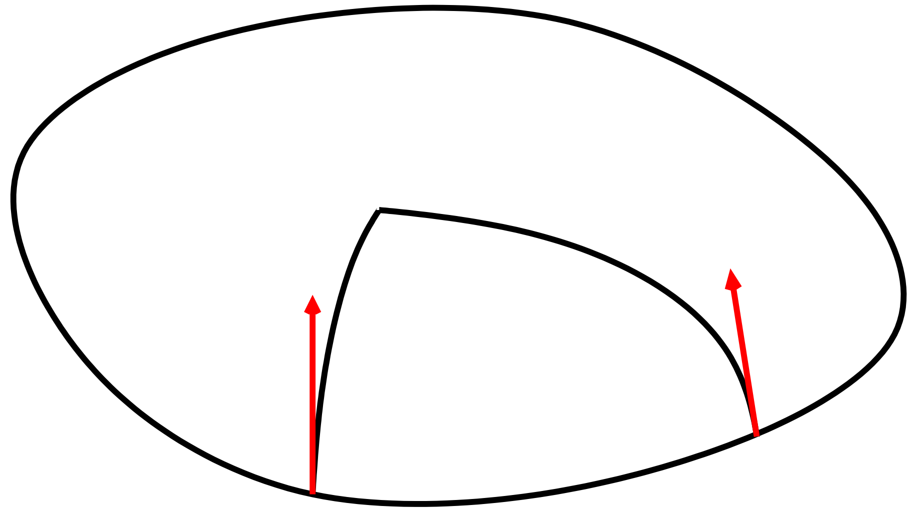
Denote by $\gamma_{x,\xi}$ be a geodesic with $\gamma_{x,\xi}(0)=x$, $\partial_t\gamma_{x,\xi}(0)=\xi$. Let $x_1,x_2\in\partial M$, $\xi_1\in S_{x_1}M$, $\xi_2\in S_{x_2}M$. We say that (x_1,ξ_1), (x_2,ξ_2) and time t are in broken scattering relation if

$$\gamma_{x_1,\xi_1}(s_1)=\gamma_{x_2,\xi_2}(s_2), \qquad \text{and } t=s_1+s_2,$$

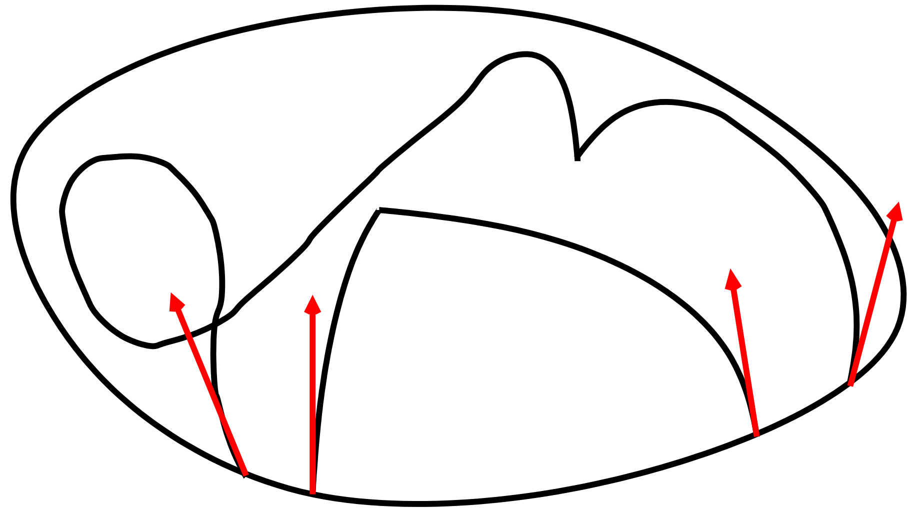
Then we denote

$$\big((x_1,\xi_1),(x_2,\xi_2),t\big)\in\mathcal{B}.$$

Then there is a broken geodesic from (x_1,ξ_1) to $(x_2,-\xi_2)$.

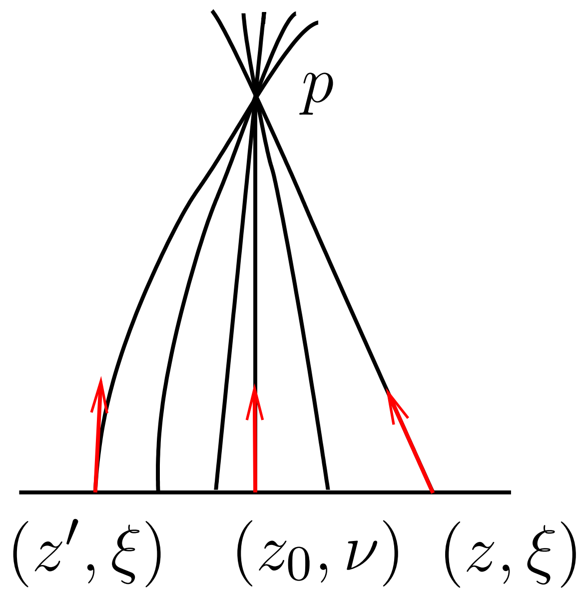


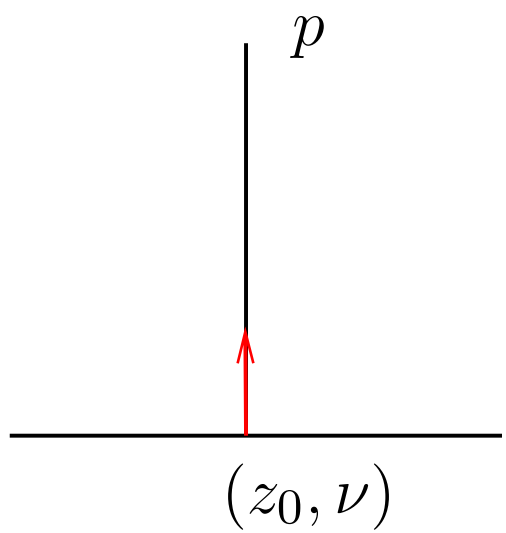
Theorem 1 *Let (M, g) be a compact Riemannian manifold with a non-empty boundary of dimension $n \geq 3$. Then ∂M and the broken scattering relation $\mathcal{B}$ determine the manifold (M, g) up to an isometry.*



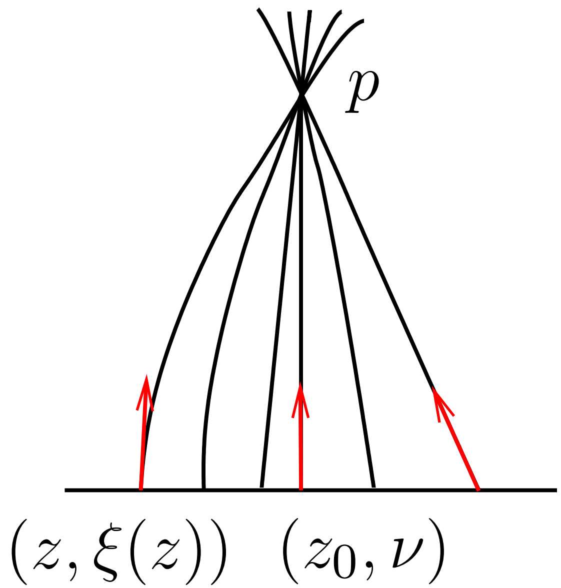
Theorem 2 *Let* (M, g) *be a compact Riemannian manifold with a non-empty boundary of dimension* $n \geq 3$*. Then* ∂M *and the broken scattering relation* $\mathcal{B}$ *determine the manifold* (M, g) *up to an isometry.*

Idea of the proof. Using boundary data we want to recognise when a family of geodesics intersect at the same point.





Let $z_0 \in \partial M$, $U \subset \partial M$ its neighborhood and $p = \gamma_{z_0,\nu}(t_0)$.



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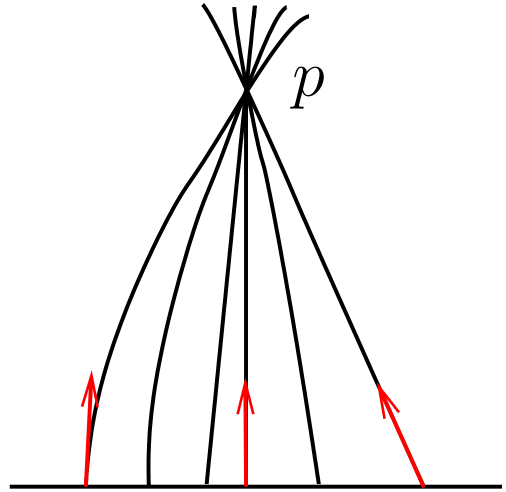
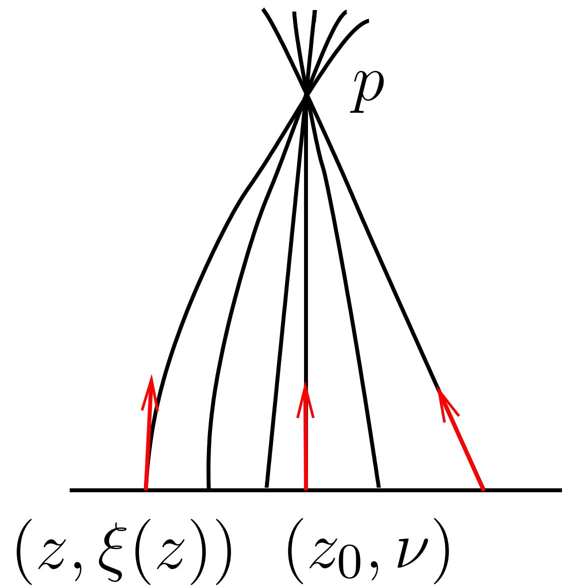


If all geodesics $\gamma_{z,\xi(z)}$ intersect at p and $t(z) = \text{dist}\,(z,p)$, then

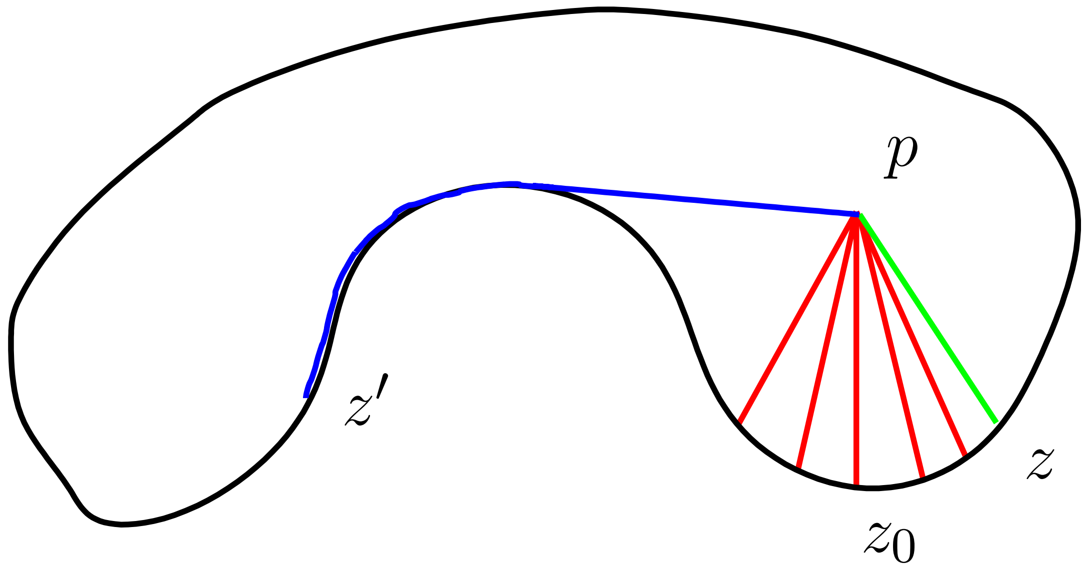
$$\big((z,\xi(z)),(z',\xi(z')),\, t(z)+t(z')\big) \in \mathcal{B}, \quad z,z' \in U \subset \partial M, \tag{1}$$

$$\xi(z_0) = \nu, \quad t(z_0) = t_0, \quad dt(z)|_{z_0} = 0. \tag{2}$$

Definition 1 $(U,\xi(\,\cdot\,),t(\,\cdot\,)\,)$ *is a family of focusing directions for z_0 and t_0 if (1) and (2) are valid.*



Lemma 1 *Let* $(U, \xi(\,\cdot\,), t(\,\cdot\,))$ *be a family of focusing directions for* $z_0 \in \partial M$ *and* $t_0 < \tau(z_0)$ *where* $\tau(z_0)$ *is a critical distance determined by the boundary data. Then all geodesics* $\gamma_{z,\xi(z)},\ z \in U$ *intersect at the point* p *and* $t(z) =$ *dist*(z, p)*.*



Lemma 2 *Let* $(U, \xi(\,\cdot\,), t(\,\cdot\,))$ *be a family of focusing directions for* $z_0 \in \partial M$ *and* $t_0 < \tau(z_0)$*. The broken scattering relation* $\mathcal{B}$ *determines function*

$$z \mapsto \textit{dist}\,(z, p), \quad z \in \partial M,\ p = \gamma_{z_0,\nu}(t_0).$$

Boundary distance functions.

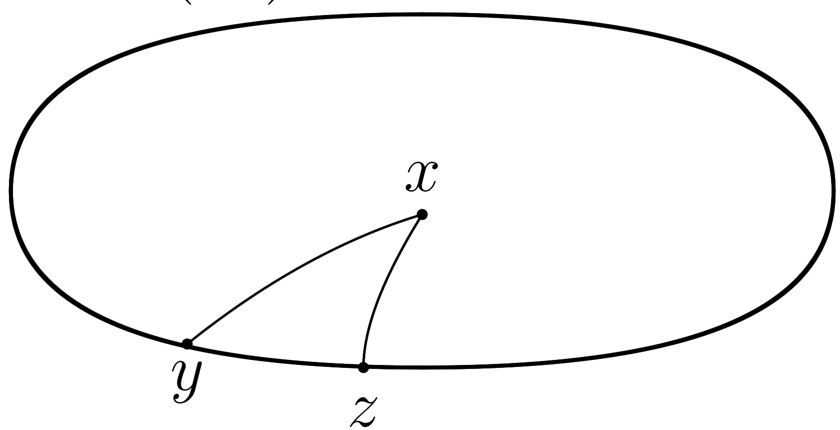
For $x \in M$ define

$$r_x(z) = \text{dist}\,(x, z), \;\; z \in \partial M.$$

Let

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.



In the Boundary Control method the boundary distance functions are used to solve hyperbolic inverse problems.

Lemma 3 (Kurylev) *The set $R(M)$ has a Riemannian manifold structure which is isometric to (M, g).*

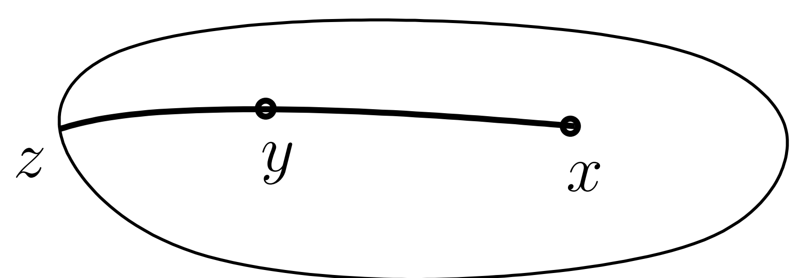
Example: Assume that (M, g) is compact and all points are connected by unique geodesics that are globally shortest paths.
By triangular inequality we have

$$\|r_x - r_y\|_{C(\partial M)} \leq \text{dist}\,(x, y), \quad x, y \in M.$$

For any $x, y \in M$ the geodesic from x to y hits later to $z \in \partial M$ and

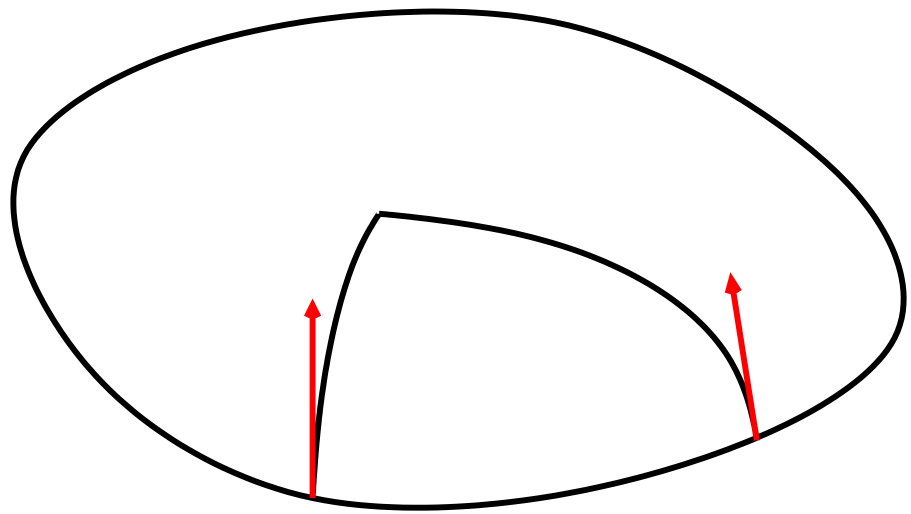
$$\|r_x - r_y\|_{C(\partial M)} \geq |r_x(z) - r_y(z)| = \text{dist}\,(x, y)$$

Then (M, d) is isometric to $(R(M), \|\cdot\|_\infty)$.



Lemma 4 (Kurylev) *The set $R(M)$ has a Riemannian manifold structure which is isometric to (M, g).*

The broken scattering relation $\mathcal{B}$ determines the boundary distance functions $R(M)$ and thus (M, g) upto an isometry.

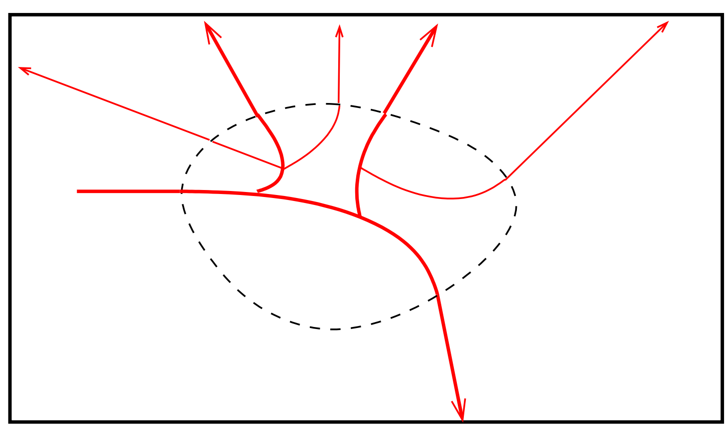


Inverse problem for radiative transfer equation.
Consider the equation

$$(Hu)(t,x,\xi)+\sigma(x,\xi)u(t,x,\xi)-(Ku)(t,x,\xi)=0,$$
$$u(t,x,\xi)|_{t=0}=w(x,\xi).$$

on a complete and simple Riemannian manifold (N,g).

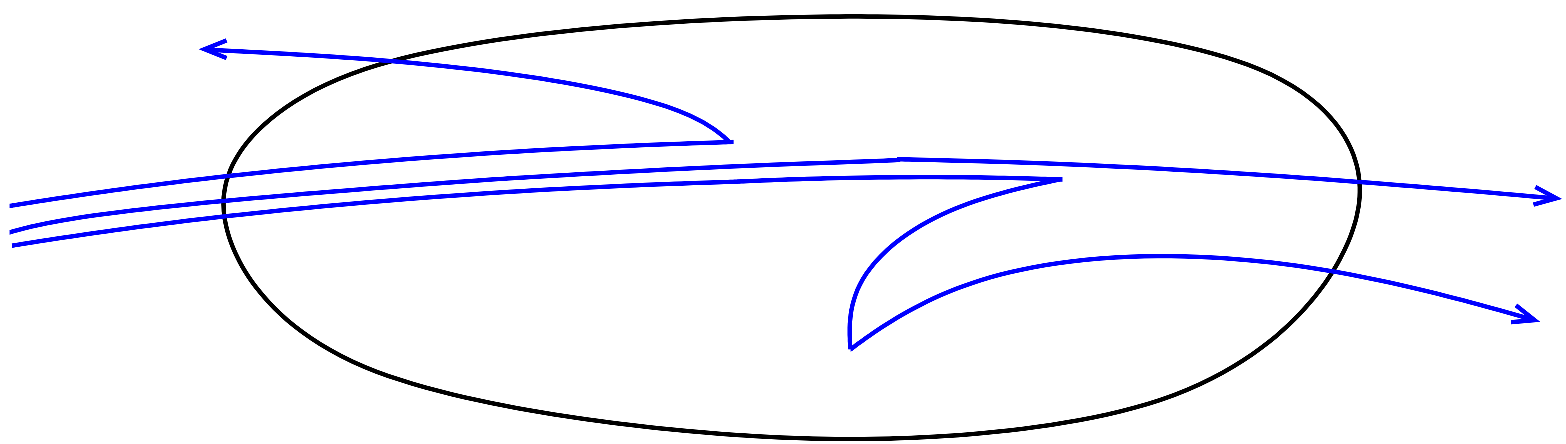
$$Hu(t,x,\xi)=\frac{\partial u}{\partial t}+\xi^i\frac{\partial u}{\partial x^i}-\xi^i\xi^j\Gamma^k_{ij}(x)\frac{\partial u}{\partial \xi^k},$$
$$Ku(t,x,\xi)=\int_{S_xN}K(x,\xi,\xi')u(t,x,\xi')\,dS_g(\xi').$$



Let $M \subset N$ be compact, $U = N \setminus M$. Assume that we are given the measurement map

$$A : C_0^\infty(SU) \to C^\infty(\mathbb{R}_+ \times SU), \quad A(u|_{t=0}) = u|_{\mathbb{R}_+ \times SU}.$$

Theorem 3 *Let N be a complete simple manifold of dimension $n \geq 3$, $M \subset N$ be compact and strictly convex. Assume that $K \in C_0^\infty(SM \dot{\times} SM)$ and $K(x, \xi, \xi') > 0$ for all $x \in M^{\rm int}$.*
Then $U = N \setminus M$ and the measurement map A determine uniquely (M, g).



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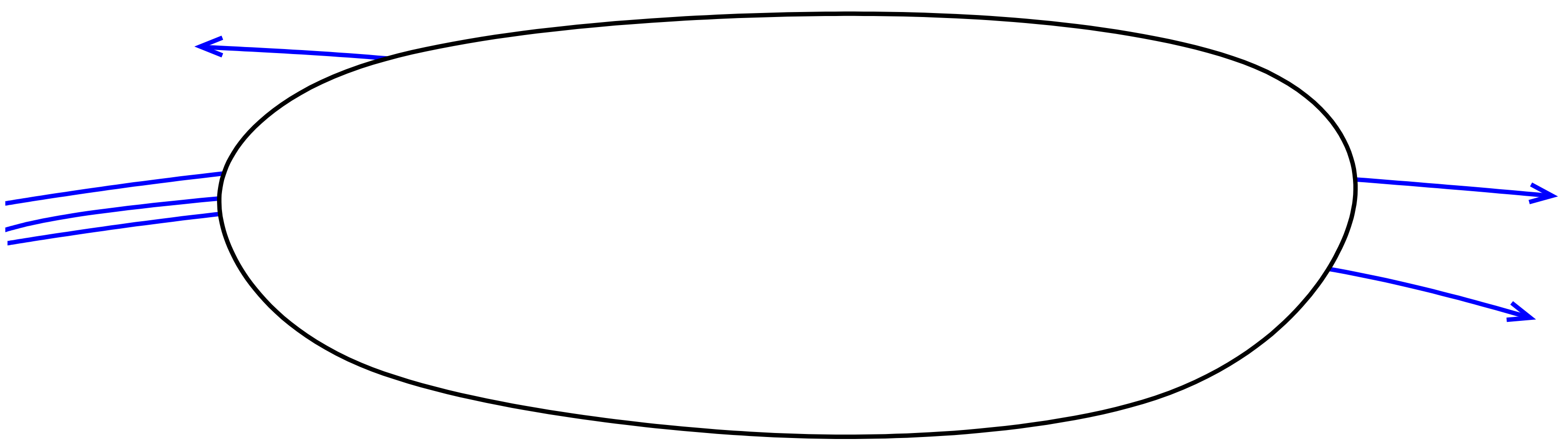


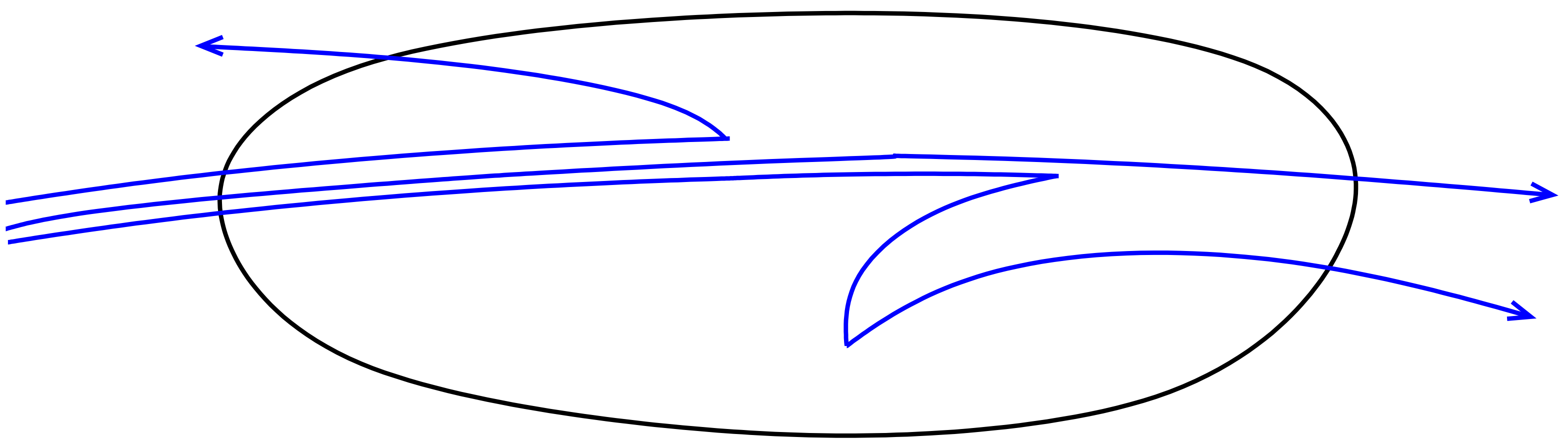
Idea of solution: Consider the Born series

$$u = u_0 + u_1 + u_2 + u_3 + \dots, \quad u_{j+1} = (H+\sigma)^{-1} K u_j.$$

Singularities of u_j can be analyzed using

- Asymptotic methods used by Choulli-Stefanov and McDowall, or
- Guillemin-Melrose-Uhlmann calculus of conormal distributions.

The ballistic photons u_0 and the single scattering photons u_1 dominate in the Born series.



Consider $X = \mathbb{R}^n$ with coordinates $x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^{n-d}$. Denote

$$S = \{x' = 0\}, \quad \Lambda = N^*S.$$

We say that $u \in \mathcal{D}'(X)$ is a Lagrangian distribution associated with Λ and denote $u \in I^m(X; \Lambda)$, if u can locally in written in form

$$u(x) = \int_{\mathbb{R}^d} e^{ix' \cdot \theta} a(x, \theta)\, d\theta, \quad a(x, \theta) \in S^{m+n/4-d/2}(X \times \mathbb{R}^d).$$

Note that $WF(u) \subset \Lambda$.
In the following we use $X = SN$.

Let $X = \mathbb{R}^n$, $x = (x', x'', x''') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n-d_1-d_2}$,

$$S_1 = \{x' = 0\}, \quad \Lambda_1 = N^* S_1, \quad S_2 = \{x' = x'' = 0\}, \quad \Lambda_2 = N^* S_2.$$

Then

$$u \in I^{p,l}(X; \Lambda_1, \Lambda_2), \quad WF(u) \subset \Lambda_1 \cup \Lambda_2,$$

if locally

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x' \cdot \theta' + x'' \cdot \theta'')} a(x, \theta', \theta'')\, d\theta' d\theta'',$$

where $a \in S^{\mu,\nu}(\mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, μ, ν depending on p, l,

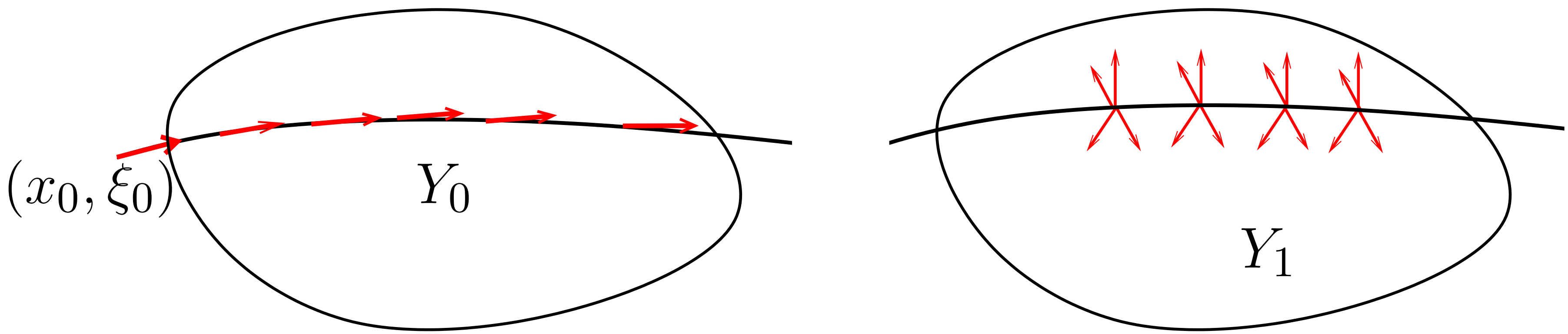
$$|\partial_{\theta'}^{\alpha} \partial_{\theta''}^{\beta} \partial_x^{\gamma} a(x, \theta', \theta'')| \leq C_{\alpha\beta\gamma}(\langle \theta' \rangle + \langle \theta'' \rangle)^{\mu - |\alpha|} \langle \theta'' \rangle^{\nu - |\beta|}.$$

Generelly, $I^{p,l}(X; \Lambda_1, \Lambda_2)$ can be defined for two cleanly intersecting Lagrangian manifolds $\Lambda_1, \Lambda_2 \subset T^* X \setminus 0$.

Let $\gamma_0 = \gamma_{x_0,\xi_0}$ be geodesic starting from (x_0, ξ_0) and

$$Y_0 = \{(\gamma_0(t), \partial_t \gamma_0(t)) \in SN :\ t \in \mathbb{R}\},$$
$$Y_1 = \{(x, \xi) \in SN :\ x \in \gamma_0(\mathbb{R})\}$$

and define $\Lambda_0 = N^* Y_0$ and $\Lambda_1 = N^* Y_1$.



Let u be solution with the initial data $u|_{t=0} = \delta_{x_0,\xi_0}$.
Let $\widehat{u}(k,x,\xi) = (\mathcal{L}u(\cdot\,,x,\xi))(k)$ be the Laplace transform of u in time t. Then

$$(P + \sigma + k)\widehat{u} - K\widehat{u} = w_0 \quad \text{in } (x,\xi) \in SN,$$

where $w_0(x,\xi) = \delta_{(x_0,\xi_0)}(x,\xi)$ and

$$Pv(x,\xi) = \xi^j \frac{\partial v}{\partial x^j}(x,\xi) - \xi^l \xi^j \Gamma^m_{lj}(x) \frac{\partial v}{\partial \xi^m}(x,\xi).$$

The operator $P + \sigma + k$ has a right inverse

$$\widehat{Q}_k : C_0^\infty(SN) \to C^\infty(SN).$$

Mapping properties of $\widehat{Q}_k$ and the parametrix Q of H are known by results of Melrose, Uhlmann, and Greenleaf.

We can write $K = K_1 K_2$,

$$K_j f(x,\xi) = \int_{S_x N} K_j(x,\xi,\xi') f(x,\xi')\, dS_g(\xi'), \quad j = 1,2,$$

where $K_j(x,\xi,\xi') \in C_0^\infty(SN \dot{\times} SN)$.
The terms in the Born series can be written as

$$\begin{aligned} \widehat{u}_j(k) &= \widehat{Q}_k (K\widehat{Q}_k)^{j-1} K \widehat{u}_0(k) \\ &= \widehat{Q}_k K_1 G^{j-1} K_2 \widehat{u}_0(k), \quad j \geq 1, \end{aligned}$$

where

$$G = K_2 \widehat{Q}_k K_1.$$

Consider k as a parameter. The operators

$$K_j f(x,\xi) = \int_{S_xN} K_j(x,\xi,\xi')f(x,\xi')\,dS_g(\xi'), \quad j = 1,2$$

have C^∞ smooh kernels. Thus the operator

$$G = K_2 \widehat{Q}_k K_1$$

has a kernel $G(x,\xi,x',\xi') \in I(SN \times SN; N^*\{x = x'\})$.
The kernel can be considered as a pseudodifferential operator of order (-1) in variables x, x', depending smoothly on paremeters ξ, ξ'.
Thus in the Born iteration

$$\widehat{u}_j(k) \;\; = \;\; \widehat{Q}_k K_1 G^{j-1} K_2 \widehat{u}_0(k), \quad j \geq 1,$$

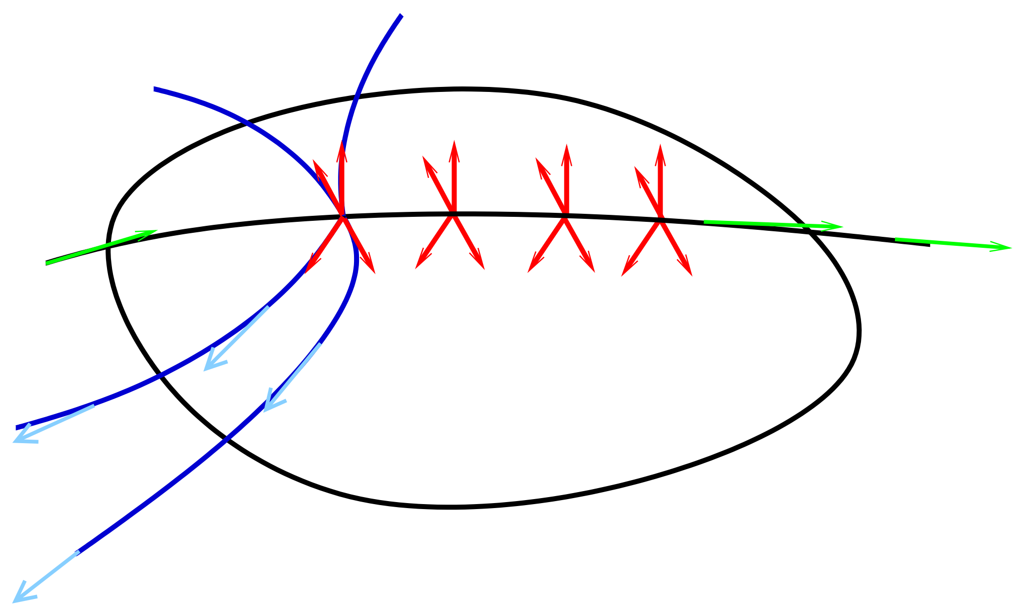
the term $\widehat{u}_{j+1}(k)$ is smoother than $\widehat{u}_j(k)$ is some variables.

Lemma 5 *We have*

$$\widehat{u}_0(k,x,\xi) = c_0(x,k)\delta_{Y_0}(x,\xi) \in I^{r_0}(SN;\Lambda_0), \quad r_0 = (2n-3)/4,$$
$$\widehat{u}_j(k) \in I^{r_j,-\frac{1}{2}}(SN;\Lambda_1,\Lambda_2), \;\; r_j = -j + \frac{1}{4} + \epsilon, \;\; \epsilon > 0, j \geq 1$$

where Λ_2 *is the flow-out of* Λ_1 *in char* (P^{-1}).

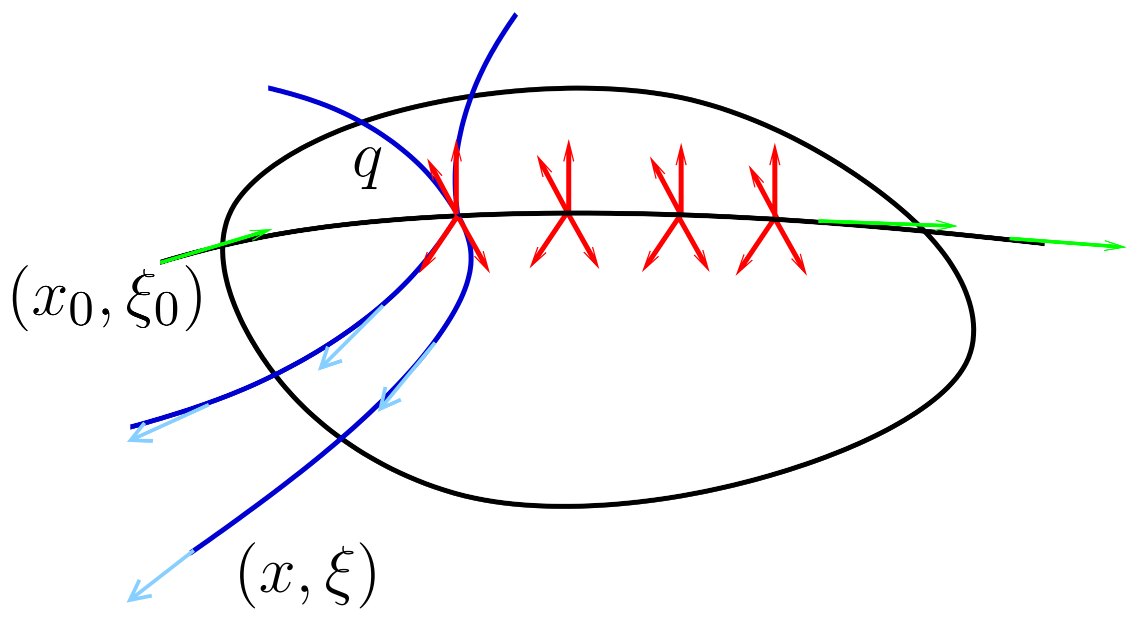
Thus with a fixed k the solution $\widehat{u}(x,\xi,k)$ determines the singularities of $\widehat{u}_1(x,\xi,k)$.



Singularities of $\widehat{u}(x,\xi,k)$ determine all points (x,ξ), $x\notin M$, such that there is a broken geodesic from (x_0,ξ_0) to (x,ξ) with a breaking point in $M^{\rm int}$. Let $q=\gamma_0(s)$ be the breaking point.

As $k\to\infty$, the principal symbol of $\widehat{u}(k)$ near $\Lambda_2\setminus\Lambda_1$ has the asymptotics

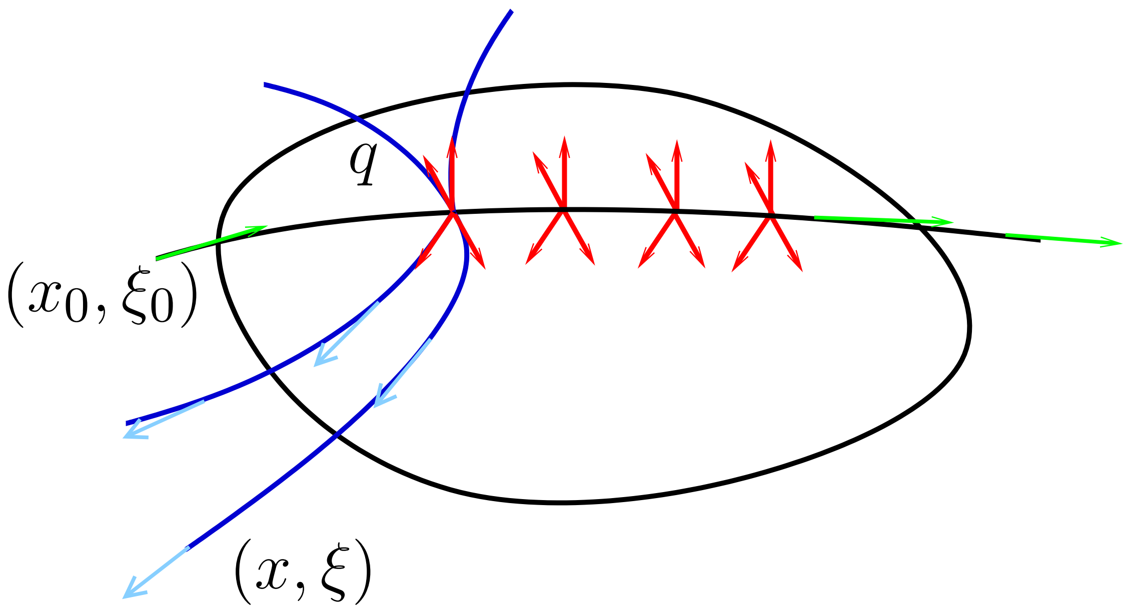
$$a^p(x,\theta;k)=e^{k(dist\,(x_0,q)+dist\,(q,x))}(c_1(x,\theta)+\mathcal{O}(k^{-1}))$$



Thus the singularities of $\widehat{u}(x, \xi, k)$ determine all points (x, ξ), $x \notin M$, such that there is a broken geodesic from (x_0, ξ_0) to (x, ξ) with a breaking point in M^{int} and the function

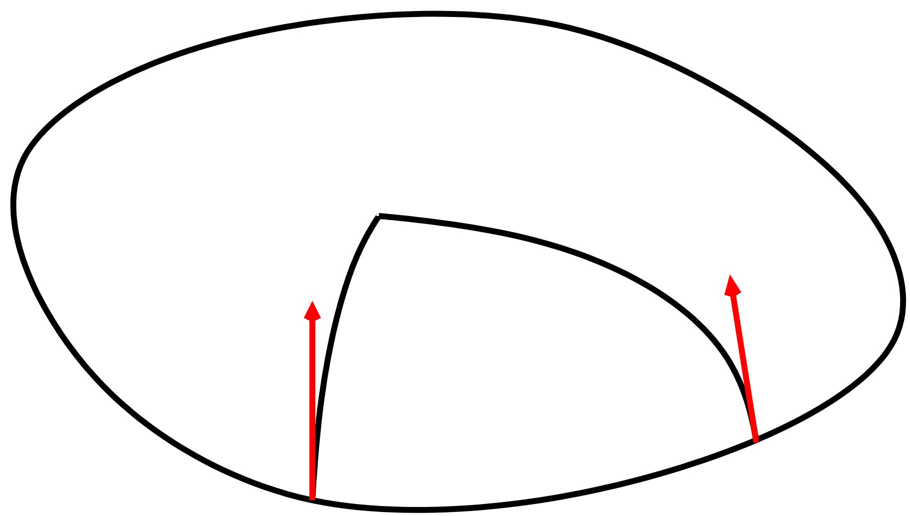
$$\text{dist}\,(x_0, q) + \text{dist}\,(q, x), \quad q = \gamma_0(s_1).$$

Thus the singularities of $\widehat{u}(x, \xi, k)$ determine the broken scattering relation $\mathcal{B}$.



Summarizing:

- The observations of the solutions of the radiative transfer equation determine the broken scattering relation $\mathcal{B}$.
- The broken scattering relation determines the boundary distance functions $R(M)$.
- The boundary distance functions determines (M, g) up to an isometry.



Boundary distance functions and wave equation

Next we apply control theory for wave equation on compact manifold (M, g).

We modify the classical time reversal algorithms (M. Fink et al). Our aim is to focus waves at the time T to a single point $x \in M$ and construct the boundary distance functions.

Results are made in collaboration with K. Bingham, Y. Kurylev and S. Siltanen.

Wave equation. Let $u = u^f(x,t)$ be the solution of

$$\begin{aligned} &u_{tt} - \Delta_g u = 0 \quad \text{on } M \times \mathbb{R}_+, \\ &\partial_\nu u|_{\partial M \times \mathbb{R}_+} = f, \\ &u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \end{aligned}$$

Here M is a compact manifold, ν is the unit normal of ∂M,

$$\Delta_g u = \sum_{j,k=1}^{n} |g|^{-1/2} \frac{\partial}{\partial x^j}(|g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u),$$

where $|g| = \det(g_{ij})$ and $[g_{ij}] = [g^{jk}]^{-1}$. Define

$$\Lambda_T f = u^f|_{\partial M \times (0,T)}, \qquad \Lambda = \Lambda_\infty$$

Assume that we are given the **boundary data** $(\partial M, \Lambda)$.

Seminal results on the problem:

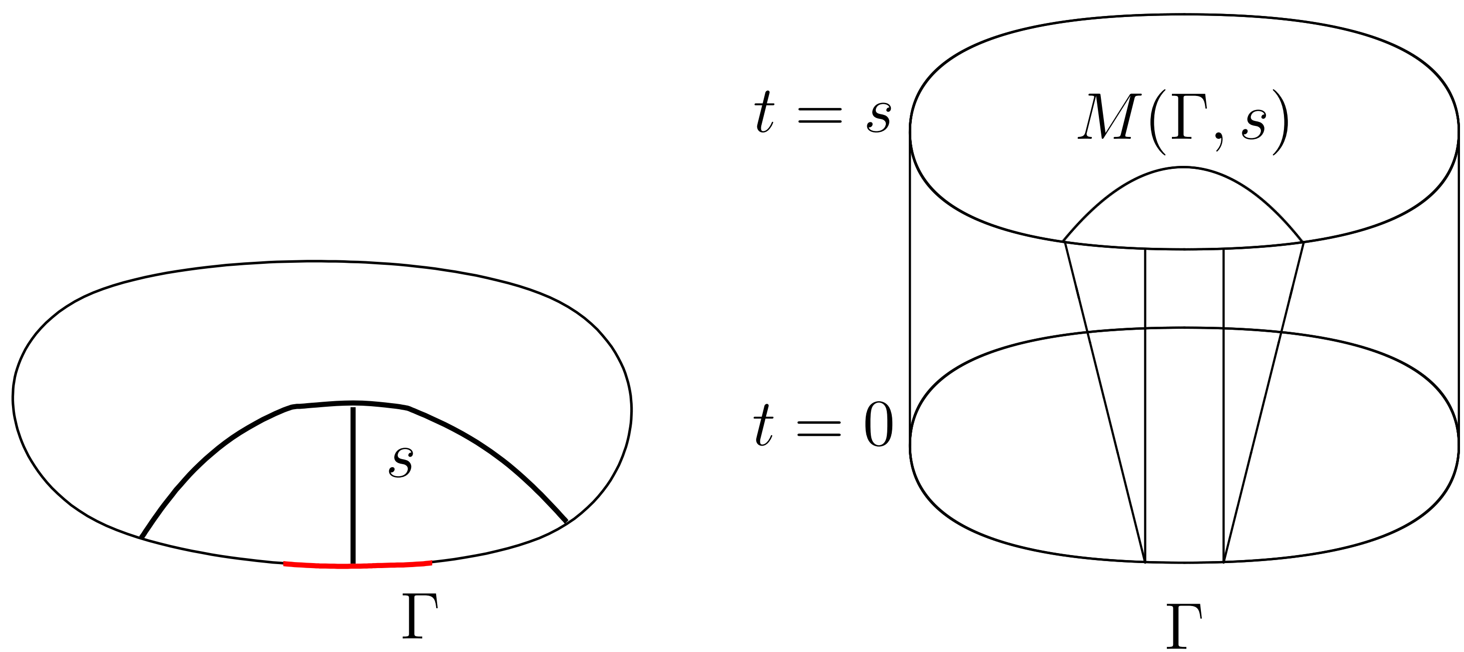
- Belishev 1987: $c(x)^2\Delta$ in $\mathbb{R}^m$ by boundary control method.
- Belishev-Kurylev 1992: Reconstruction of a Riemannian manifold from boundary spectral data.
- Tataru 1995: Local controllability.

Domains of influence

Definition 2 *Let $s > 0$ and $\Gamma \subset \partial M$. The set*

$$M(\Gamma, s) = \{x \in M : \text{ dist}_g(x, \Gamma) \le s\}$$

is the domain of influence of Γ at time s.



Using the the Blagovestchenskii identity one can obtain

$$\int_M u^f(x,T)u^h(x,T)\,dV_g(x) = \int_{\partial M\times[0,2T]} (Kf)(x,t)h(x,t)\,dS_g(x)dt$$

where

$$K = J\Lambda_{2T} - R\Lambda_{2T}RJ,$$

$$Rf(x,t) = f(x,2T-t)$$ "time reversal operator",

$$Jf(x,t) = \frac{1}{2}\int_t^{2T-t} f(x,s)ds$$ "time filter".

Note that

$$\Lambda_{2T}^* = R\Lambda_{2T}R.$$

Iterated measurements and boundary control.

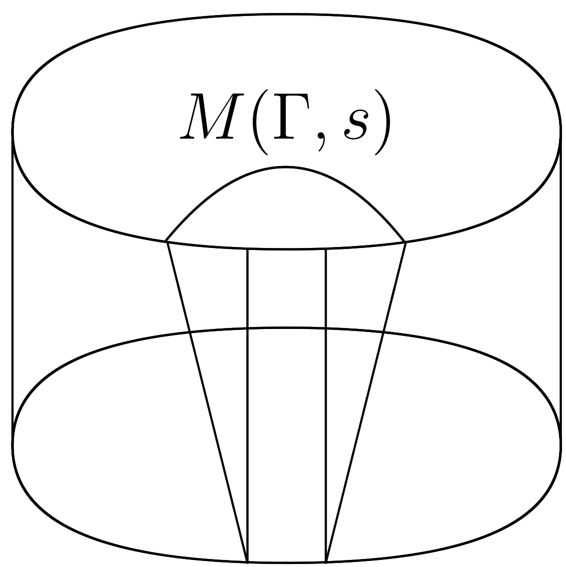
First goal: Let f be given. Can we find h such that

$$u^h(x,T) = \chi_{M(\Gamma,s)}(x) u^f(x,T)?$$

This is equivalent of the minimization of

$$\|u^f(T) - u^h(T)\|^2_{L^2(M)} = \langle K(h-f), h-f \rangle_{L^2(\partial M \times [0,2T])}$$

over $h \in C_0^\infty(\Gamma \times [T-s,T])$.

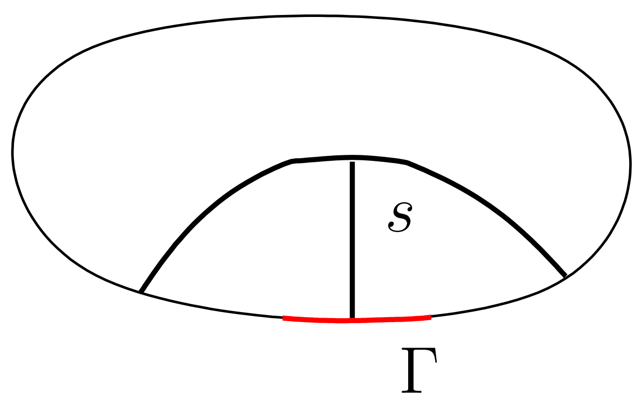


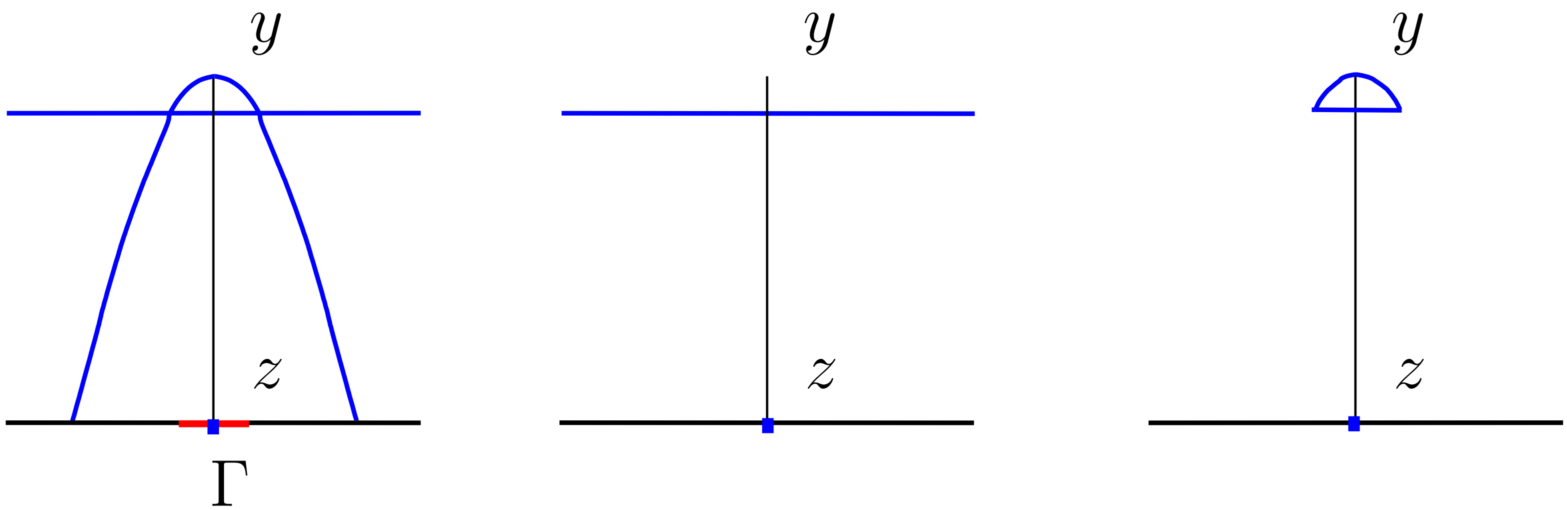
The minimization problem can be solved using a modified time reversal iteration:

$$
\begin{aligned}
F &:= \alpha P(J\Lambda_{2T} - R\Lambda_{2T}RJ)f, \\
a_n &:= \Lambda_{2T}(h_n), \\
b_n &:= \Lambda_{2T}(RJh_n), \\
h_{n+1} &:= (1-\alpha^2)h_n + \alpha(PRb_n - PJa_n) + F,
\end{aligned}
$$

where $f \in L^2(\partial M \times [0, 2T])$, $\alpha > 0$, and $Pf = \chi_{\Gamma \times [T-s,T]} f$. Iteration starts at $h_0 = 0$. Then $h_n = h_n(\alpha)$ satisfy

$$\lim_{\alpha \to 0} \lim_{n \to \infty} u^{h_n(\alpha)}(x,T) = \chi_{M(\Gamma,s)}(x) u^f(x,T), \qquad \text{in } L^2(M).$$



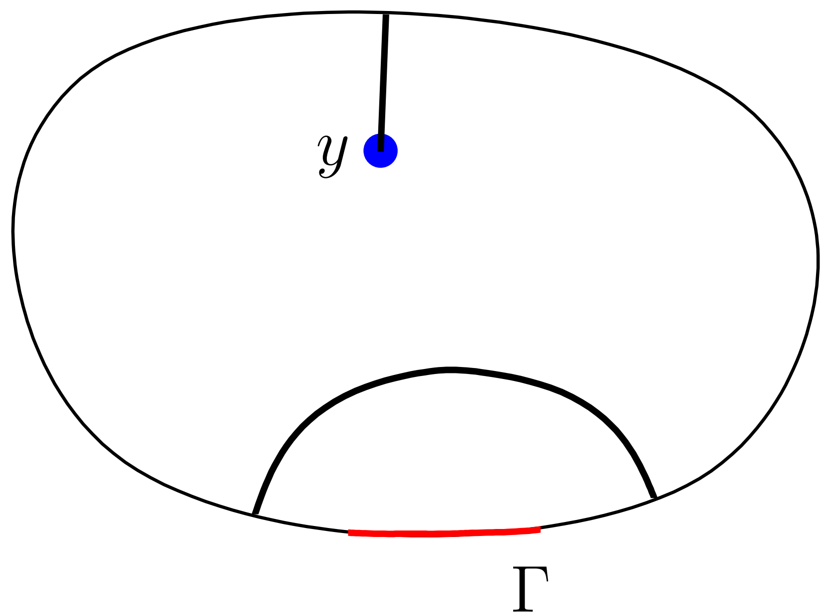


Theorem 5 (Bingham-Kurylev-L.-Siltanen) *Let* $z \in \partial M$*,* $s > 0$ *small enough, and* $y = \gamma_{z,\nu}(s)$*.*

Let h_n *and* $\widetilde{h}_n$ *be results of the modified time reversal iteration corresponding to sets* $M(\Gamma, s) \cup M(\partial M, s - \epsilon)$ *and* $M(\partial M, s - \epsilon)$*, respectively. Then*

$$\lim_{\epsilon \to 0} \lim_{\Gamma \to z} \lim_{\alpha \to 0} \lim_{n \to \infty} \frac{1}{\epsilon^{(m+1)/2}} \, u^{h_n - \widetilde{h}_n}(T) = C \delta_y(x)$$

in $\mathcal{D}'(M)$*. Moreover,* $C = c(y) u^f(y, T)$*.*

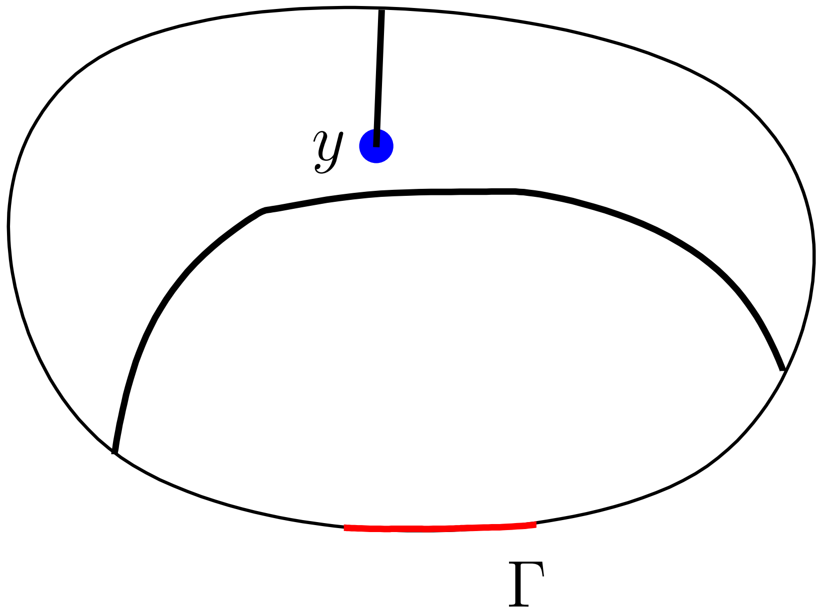


Let $z_0 \in \partial M$ and $y = \gamma_{z_0,\nu}(s)$. Boundary data determines

$$I(r) = \sup \left\{ \int_M \delta_y(x)\, u^f(x,T)\, dV_g(x) : \; f \in C_0^\infty(\Gamma \times [T-r,T]) \right\},$$

dist $(y,\Gamma) = \sup\{r \geq 0 : \; I(r) = 0\}$.

Thus we can find the boundary distance functions $r_y(z) =$ dist (y,z), $z \in \partial M$ for all $y \in M$.



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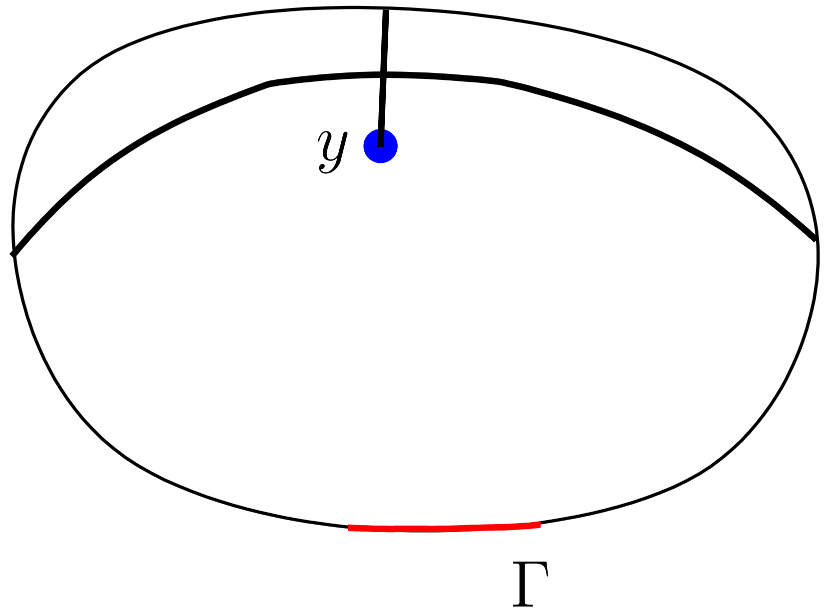

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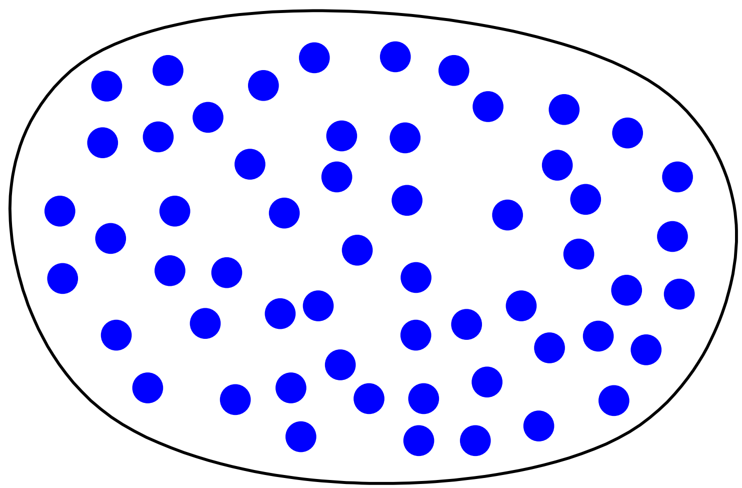
Theorem 6 (Bingham-Kurylev-L.-Siltanen) *Assume we are given ∂M and Λ. Using the modified time reversal iteration one can find the boundary distance functions $R(M)$.*

Thus using the modified time reversal iteration one can reconstruct the manifold (M, g) up to an isometry.



Theorem 7 (Bingham-Kurylev-L.-Siltanen) *Assume we are given ∂M and Λ. Using the modified time reversal iteration one can find the boundary distance functions $R(M)$.*

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