

Global controllability of the non linear Schrödinger equation on some compact manifolds of dimension 1 or 3

Camille LAURENT

CIRM february 2009

Conference "Control and Inverse Problems in PDE : Theoretical and Numerical Aspects"

$$\begin{cases} i\partial_t u + \Delta u &= \lambda |u|^2 u + g 1_\omega & \text{on } [0, T] \times M \\ u(0) &= u_0 \in H^s(M) \end{cases}$$

M is a compact Riemannian manifold, Δ the Laplace-Beltrami operator

$$\begin{cases} i\partial_t u + \Delta u &= \lambda |u|^2 u + g 1_\omega & \text{on } [0, T] \times M \\ u(0) &= u_0 \in H^s(M) \end{cases}$$

M is a compact Riemannian manifold, Δ the Laplace-Beltrami operator
If $g = 0$, two energies are formally conserved :

$$E(t) = \frac{1}{2} \int_M |\nabla u|^2 + \frac{\lambda}{4} \int_M |u|^4.$$

$$\begin{cases} i\partial_t u + \Delta u &= \lambda |u|^2 u + g \mathbf{1}_\omega & \text{on } [0, T] \times M \\ u(0) &= u_0 \in H^s(M) \end{cases}$$

M is a compact Riemannian manifold, Δ the Laplace-Beltrami operator
If $g = 0$, two energies are formally conserved :

$$E(t) = \frac{1}{2} \int_M |\nabla u|^2 + \frac{\lambda}{4} \int_M |u|^4.$$

- dim 1 : ω is arbitrary, $s \geq 0$, $\lambda \in \mathbb{R}$, $M = \mathbb{T}^1$ or $[0, 1]$ with Dirichlet or Neumann condition.
- dim 3 : ω fulfills geometric control and unique continuation, $s = 1$, $\lambda \geq 0$ (defocusing),
 $M = \mathbb{T}^3$, S^3 or $S^2 \times S^1$

2 strategies to get global controllability in large time :

- stabilization and then control near 0
- successive controls near trajectories and then control near to 0

Introduction

Dimension 1

Dimension 3

Idea of demonstration

Unique continuation

Conclusion

Bibliography

Non linear waves :

E. Zuazua, 1990 : fixed point

B. Dehman, G. Lebeau et E. Zuazua, 2003 : global in large time

B. Dehman and G. Lebeau, 2007 : with first harmoniques assumed small

Non linear Schrödinger :

B. Dehman, P. Gérard, et G. Lebeau, 2006 : global in large time, H^1 defocusing, dim 2 with geometric control assumption + unique continuation

L. Rosier et B.Y. Zhang, 2007 : local, L^2 , dim 1

Carleman estimates :

L. Baudouin et J.P. Puel ; A. Mercado, A. Osses et L. Rosier

Existence theorems :

J. Bourgain : \mathbb{T}^3 , irrational torus $\mathbb{R}^3 / (\theta_1 \mathbb{Z} \times \theta_2 \mathbb{Z} \times \theta_3 \mathbb{Z})$

N. Burq, P. Gérard and N. Tzvetkov : $S^3, S^2 \times S^1$

Definition of Bourgain spaces

The Bourgain spaces $X^{s,b}$ are equipped with the norm

$$\begin{aligned} \|u\|_{X^{s,b}}^2 &= \sum_k \int_{\mathbb{R}} \langle \lambda_k \rangle^s \langle \tau + \lambda_k \rangle^{2b} \left| \widehat{u}(\tau, k) \right|^2 d\tau \\ &= \left\| \langle \Delta \rangle^{s/2} \langle i\partial_t + \Delta \rangle^b u \right\|_{L^2(\mathbb{R} \times M)}^2 \\ &= \left\| u^\# \right\|_{H^b(\mathbb{R}, H^s(M))}^2 \end{aligned}$$

with λ_k the eigenvalues of Δ with multiplicity, $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$, $u = u(t, x)$, $t \in \mathbb{R}$, $x \in M$, and $u^\#(t) = e^{-it\Delta} u(t)$.

$$\|u\|_{X_T^{s,b}}^2 = \inf \left\{ \|\tilde{u}\|_{X^{s,b}}^2 \mid u = \tilde{u} \text{ sur } [0, T] \right\}$$

Global controllability in dimension 1

Theorem

For every open set $\omega \subset \mathbb{T}^1$ and $R_0 > 0$, there exist $T > 0$ and $C > 0$ such that for any u_0 and u_1 with $L^2(\mathbb{T}^1)$ with

$$\|u_0\|_{L^2} \leq R_0 \quad \text{and} \quad \|u_1\|_{L^2} \leq R_0$$

there exists a control $g \in C([0, T], L^2)$ supported in $[0, T] \times \omega$, such that the unique solution u in $X_T^{0,b}$ of the Cauchy problem

$$\begin{cases} i\partial_t u + \partial_x^2 u &= \lambda|u|^2 u + g & \text{on } [0, T] \times \mathbb{T}^1 \\ u(0) &= u_0 \in L^2(\mathbb{T}^1) \end{cases}$$

satisfies $u(T) = u_1$.

Moreover, if u_0 and $u_1 \in H^s$, with $s \geq 0$, one can impose $g \in C([0, T], H^s)$.

Stabilization theorem

Theorem

If $a \in L^\infty(\mathbb{T}^1)$ and $a(x)^2 > \eta > 0$ on a non empty open set. Then, for any $R_0 > 0$, there exist $C > 0$ and $\gamma > 0$ such that we have

$$\|u(t)\|_{L^2} \leq C e^{-\gamma t} \|u_0\|_{L^2} \quad t > 0$$

for any solution u of the system

$$\begin{cases} i\partial_t u + \partial_x^2 u + ia^2 u &= \lambda |u|^2 u \quad \text{on } [0, T] \times \mathbb{T}^1 \\ u(0) &= u_0 \in L^2(\mathbb{T}^1). \end{cases}$$

with Cauchy data u_0 with $\|u_0\|_{L^2} \leq R_0$.

Control near 0

Theorem

Let $\omega \neq \emptyset$ an open set of \mathbb{T}^1 and $T > 0$. Then, there exists $\varepsilon > 0$ such that for any $u_0 \in L^2$ and $\|u_0\|_{L^2} < \varepsilon$, there exists $g \in C([0, T], L^2)$ supported in $]0, T[\times \omega$ such that the unique solution u in $X_T^{0,b}$ of

$$\begin{cases} i\partial_t u + \partial_x^2 u &= \lambda |u|^2 u + g \\ u(x, 0) &= u_0(x) \end{cases}$$

satisfies $u(T) = 0$.

Moreover, if $u_0 \in H^s$, with $s \geq 0$, eventually with a large H^s norm, one can impose $g \in C([0, T], H^s)$.

Assumptions in dimension 3

We only treat the case of $M = \mathbb{T}^3$ (with rational or irrational sides), S^3 or $S^2 \times S^1$ because otherwise the flow map is not known to be locally lipschitz.

The assumptions for ω are

- geometric control
- unique continuation.

Examples :

- \mathbb{T}^3 with

$$\omega = \{x \in \mathbb{R}^3 / (\theta_1 \mathbb{Z} \times \theta_2 \mathbb{Z} \times \theta_3 \mathbb{Z}) \mid \exists i \in \{1, 2, 3\}, x_i \in]0, \varepsilon[+ \theta_i \mathbb{Z}\}.$$

- S^3 with ω a neighborhood of $\{x_4 = 0\}$ in $S^3 \subset \mathbb{R}^4$.
- $S^2 \times S^1$ with $\omega = (\omega_1 \times S^1) \cup (S^2 \times]0, \varepsilon[)$ where ω_1 is a neighborhood of the equator of S^2 .

Global controllability in dimension 3

Theorem

Let M , ω as before. For any $R_0 > 0$, there exist $T > 0$ and $C > 0$ such that for every u_0 and u_1 in $H^1(M)$ with

$$\|u_0\|_{H^1(M)} \leq R_0 \quad \text{and} \quad \|u_1\|_{H^1(M)} \leq R_0$$

there exists a control $g \in C([0, T], H^1)$ with $\|g\|_{L^\infty([0, T], H^1)} \leq C$ supported in $[0, T] \times \bar{\omega}$, such that the unique solution u in $X_T^{1,b}$ of the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u &= |u|^2 u + g \quad \text{on} \quad [0, T] \times M \\ u(0) &= u_0 \in H^1(M) \end{cases} \quad (1)$$

satisfies $u(T) = u_1$.

Stabilization

Theorem

Let M, ω as before. Let $a \in C^\infty(M)$, with $a(x) \neq 0$ on ω . There exists $\gamma > 0$ such that for every $R_0 > 0$, there is a constant $C > 0$ such that inequality

$$\|u(t)\|_{H^1} \leq Ce^{-\gamma t} \|u_0\|_{H^1} \quad t > 0$$

holds for every solution u of system

$$\begin{cases} i\partial_t u + \Delta u - a(x)(1 - \Delta)^{-1} a(x)\partial_t u &= (1 + |u|^2)u \quad \text{on } [0, T] \times M \\ u(0) &= u_0 \in H^1(M). \end{cases}$$

with initial data u_0 such that $\|u_0\|_{H^1} \leq R_0$.

Necessity of geometric control on S^3

Let Γ be a geodesic of S^3 (or S^2). If $Supp(a) \cap \Gamma = \emptyset$, the former theorem (exponential stabilization) is false.

Dem : Take $\Phi_n = c_n(x_1 + ix_2)^n$, eigenfunctions that concentrate on a geodesic $\Gamma = \{x_3 = x_4 = 0\}$.

Control near trajectories

Theorem

Let $T > 0$ and $M, \bar{\omega}$ as before. Let $1 \geq s > s_0$ and $w \in X_T^{1,b}$ be a solution of

$$\begin{cases} i\partial_t w + \Delta w \pm |w|^2 w & = g \\ w(x, 0) & = w_0(x) \end{cases} \quad (3)$$

with $g \in C([0, T], H^1)$ supported in $[0, T] \times \bar{\omega}$.

Then, there exists $\varepsilon(\|w\|_{X_T^{1,b}}) > 0$, such that for every $u_0 \in H^s$ with

$\|u_0 - w(0)\|_{H^s} < \varepsilon$, there exists $g_1 \in C([0, T], H^s)$ supported in $[0, T] \times \bar{\omega}$ such that the unique solution u in $X_T^{s,b}$ of (3) with $u(0) = u_0$ and g replaced by g_1 fulfills $u(T) = w(T)$.

Moreover, for any $u_0 \in H^1$ with $\|u_0 - w(0)\|_{H^s} < \varepsilon$, the same conclusion holds with $g \in C([0, T], H^1)$.

Corollary

For any time T (eventually small), the set of reachable states is open in H^1 .

If we fix a bound for the H^1 norm, it gives control in H^1 near a trajectory with some smallness only of a fixed number of harmonics.

It gives global controllability in large time with successive controls near trajectories.

Control near 0 with smaller ω

Theorem

If $w = 0$ (control near 0) and (M, ω) is either :

- $(\mathbb{T}^3, \text{any open set})$ (see also the recent work of L. Rosier and B.

Y.Zhang)

- $(S^2 \times S^1, \omega_1 \times S^1)$ where ω_1 is a neighborhood of the equator of S^2

- $(S^2 \times S^1, S^2 \times]0, \epsilon[)$

Then, the same conclusion as the former Theorem is true : control near 0.

Idea of demonstration

Observability estimates

For stabilization and control of the linearized equation, we need to prove an observability estimates for solutions of a pertubated (non linear or with rough potential) Schrödinger equation of the kind :

$$\|u_0\|_{H^s}^2 \leq C \int_0^T \|au(t)\|_{H^s}^2 dt$$

By contradiction : let u_n a sequence of solutions contradicting the observability :

$$\int_0^T \|au_n(t)\|_{H^s}^2 dt \leq C \|u_{n,0}\|_{H^s}^2$$

- By propagation of compactness u_n converges strongly to one u : we use microlocal defect measure (or an elementary argument in dimension 1).
- By propagation of regularity, the weak limit u is regular.
- By unique continuation, $u = 0$, which makes a contradiction.

Propagation of compactness

Proposition

Let $r \in \mathbb{R}$. Suppose that ω satisfies geometric control condition, $a \in C^\infty(M)$ and $a(x) \neq 0$ for $x \in \omega$.

Let u_n be a sequence bounded in $X_T^{r,b}$ with $0 \leq b < 1$, weakly convergent to 0 and satisfying

$$\begin{cases} i\partial_t u_n + \Delta u_n \rightarrow 0 \text{ in } X_T^{r-1+b, -b} \\ a(x)u_n \rightarrow 0 \text{ in } L^2([0, T], H^r) \end{cases} \quad (4)$$

Then, we have $u_n \rightarrow 0$ in $L_{loc}^2([0, T], H^r)$.

Propagation of regularity

Proposition

Let $T > 0$, $0 \leq b < 1$ and $u \in X_T^{r,b}$, $r \in \mathbb{R}$ solution of

$$i\partial_t u + \Delta u = f \in X_T^{r,-b}$$

Suppose that ω satisfies geometric control condition, $a \in C^\infty(M)$ and $a(x) \neq 0$ for $x \in \omega$.

If $a(x)u \in L_{loc}^2([0, T[, H^{r+\rho})$ for some $\rho \leq \frac{1-b}{2}$ then $u \in L_{loc}^2([0, T[, H^{r+\rho}(M))$.

Idea of demonstration

Regularity of the control

First step : case of the linearized equation : Let S be the HUM operator, we have proved that it is an homeomorphism from H^{-s} into H^s . We want to prove that it is an homeomorphism from $H^{-s+\varepsilon}$ into $H^{s+\varepsilon}$. See Dehman-Lebeau + commutator estimates involving rough potentials :

$u_3 \mapsto [\Delta^{\varepsilon/2}, u_1 u_2] u_3$ maps any $X^{s,b}$ into $X^{s,-b}$ for $u_1, u_2 \in X^{s+\varepsilon,b}$.

Second step : extend this to the nonlinear control near a trajectory using tame estimates

$$\| |u|^2 u \|_{X^{s+\varepsilon,-b}} \leq C \|u\|_{X^{s,b}}^2 \|u\|_{X^{s+\varepsilon,b}}$$

Once the norm H^s is bounded, the behavior is "almost linear" in $H^{s+\varepsilon}$.

Unique continuation

It is obtained thanks to global Carleman estimates for compactly supported functions. For a Riemannian manifolds, the main assumptions on the weight becomes :

If $\nabla\Psi(x) \neq 0$ and $\text{Hess}\Psi(x)(\xi, \xi) + |\nabla\Psi(x) \cdot \xi|^2 > 0$ for $x \notin \omega$, we get strong Carleman estimates.

If $\nabla\Psi(x) \neq 0$ and $\text{Hess}\Psi(x)(\xi, \xi) + |\nabla\Psi(x) \cdot \xi|^2 \geq 0$ for $x \notin \omega$, we get weak Carleman estimates.

Hess is the Hessian of a function with respect to the metric.

Further problems

Adaptation of the method for KdV (in collaboration with L. Rosier and B. Y. Zhang)

Open problems : open set with boundary, optimal condition for ω (even for the linear equation), large time necessary ?

THANKS FOR YOU ATTENTION

For more precision :

- for dimension 1 : ESAIM-COCV on the web
- for dimension 3 : soon available on arxiv and at <http://www.math.u-psud.fr/~laurent>

or contact me at camille.laurent@math.u-psud.fr

Strong Carleman estimates

$$\begin{aligned} & \iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi} \\ & \leq C \iint |L(q)|^2 e^{-2s\varphi} + C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi} \end{aligned} \quad (5)$$

Weak Carleman estimates

$$\begin{aligned} & \iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda^2 \theta |\nabla \Psi \cdot \nabla q|^2 \right] e^{-2s\varphi} \\ & \leq C \iint |L(q)|^2 e^{-2s\varphi} + C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi} \end{aligned} \quad (6)$$