

Remarks on the null controllability of elastic arch and shell

Arnaud MÜNCH

with Farid Ammar-Khodja and Giuseppe Geymonat

Laboratoire de Mathématiques, Université de Franche-Comté
Besançon, FRANCE

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Koiter type Shell Model :

$$\begin{cases} \rho \mathbf{y}_\varepsilon'' + \mathbf{A}_M \mathbf{y}_\varepsilon + \varepsilon^2 \mathbf{A}_F \mathbf{y}_\varepsilon = 0, & \omega \times (0, T) \\ + \text{Boundary and Initial Conditions} \end{cases} \quad (1)$$

- $\varepsilon > 0$: Thickness of the shell
- \mathbf{A}_M : membranal operator of mixed order
- \mathbf{A}_F : flexural operator of order 4
- $\omega \subset \mathbb{R}^N$ - mid surface of the shell ($N = 2$) / arch ($N = 1$);
- $\mathbf{y}_\varepsilon = (y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ (Shell) ; $\mathbf{y}_\varepsilon = (y_{1,\varepsilon}, y_{3,\varepsilon})$ (Arch)

- Boundary controllability of the system at time T large enough ?

$$y_\varepsilon(T) = y'_\varepsilon(T) = 0, \quad \omega \quad (2)$$

- Asymptotic behavior of the control as $\varepsilon \rightarrow 0$?

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$$\begin{cases} \rho(\varepsilon)\mathbf{y}_\varepsilon'' + \mathbf{A}_M\mathbf{y}_\varepsilon + \varepsilon^2\mathbf{A}_F\mathbf{y}_\varepsilon = 0, & \omega \times (0, T) \\ + \text{Boundary and Initial Conditions} \end{cases} \quad (3)$$

Asymptotic behavior as $\varepsilon \rightarrow 0$: " $\mathbf{y}_\varepsilon \rightarrow \mathbf{y}$ "

- $\rho(\varepsilon) = O(1)$, $\text{Ker}\mathbf{A}_M = \{\mathbf{0}\}$ (Inhibited shell)

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M\mathbf{y} = 0, & \omega \times (0, T) \\ + \text{Boundary and Initial Conditions} \end{cases} \quad (4)$$

- $\rho(\varepsilon) = O(\varepsilon^2)$, $\text{Ker}\mathbf{A}_M \neq \{\mathbf{0}\}$ (Non inhibited shell)

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_F\mathbf{y} = 0, & \omega \times (0, T) \\ \mathbf{A}_M\mathbf{y} = 0 \\ + \text{Boundary and Initial Conditions} \end{cases} \quad (5)$$

$$\begin{cases} \rho(\varepsilon) \mathbf{y}_\varepsilon'' + \mathbf{A}_M \mathbf{y}_\varepsilon + \varepsilon^2 \mathbf{A}_F \mathbf{y}_\varepsilon = 0, & \omega \times (0, T) \\ + \text{Boundary and Initial Conditions} \end{cases} \quad (3)$$

Asymptotic behavior as $\varepsilon \rightarrow 0$: " $\mathbf{y}_\varepsilon \rightarrow \mathbf{y}''$ "

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Membranal Operator A_M

$$A_M y = \begin{pmatrix} -(y_{1,1} + r^{-1} y_3)_{,1} \\ r^{-1} (y_{1,1} + r^{-1} y_3) \end{pmatrix} \quad (6)$$

Flexural Operator A_F

$$A_F y = \begin{pmatrix} 2r^{-1} (y_{3,111} - 2r^{-1} y_{1,11} - r^{-2} y_{3,1}) \\ y_{3,1111} - 2r^{-1} y_{1,111} - 2r^{-2} y_{3,11} + 2r^{-3} y_{1,1} + r^{-4} y_3 \end{pmatrix} \quad (7)$$

- r^{-1} - curvature ; r - radius of curvature
- $y_{1,1} + r^{-1} y_3$ - longitudinal deformation of the arch
- y_1 : tangential displacement; y_3 : normal displacement
- $y_{i,1} = \frac{\partial y_i}{\partial \xi}$
- $\xi \in \omega = (0, 1)$: curvilinear abscissae

Remark

If $r^{-1} = 0$, the system reduces to

$$y_1'' - y_{1,11} = 0, \quad y_3'' + \varepsilon^2 y_{3,1111} = 0, \quad \omega \times (0, T) \quad (8)$$

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$$\mathbf{A}_F \mathbf{y} = \begin{pmatrix} 2r^{-1}(y_{3,111} - 2r^{-1}y_{1,11} - r^{-2}y_{3,1}) \\ y_{3,1111} - 2r^{-1}y_{1,111} - 2r^{-2}y_{3,11} + 2r^{-3}y_{1,1} + r^{-4}y_3 \end{pmatrix} \quad (7)$$

- r^{-1} - curvature ; r - radius of curvature
- $y_{1,1} + r^{-1}y_3$ - longitudinal deformation of the arch
- y_1 : tangential displacement; y_3 : normal displacement
- $y_{i,1} = \frac{\partial y_i}{\partial \xi}$
- $\xi \in \omega = (0, 1)$: curvilinear abscissae

Remark

If $r^{-1} = 0$, the system reduces to

$$y_1'' - y_{1,11} = 0, \quad y_3'' + \varepsilon^2 y_{3,1111} = 0, \quad \omega \times (0, T) \quad (8)$$

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Step I - Controllability in the case $\varepsilon > 0$ fixed ?

$$\begin{cases} \mathbf{y}'' + \mathbf{A}^\varepsilon \mathbf{y} = \mathbf{f}, & \text{in } \omega \times (0, T), \\ \mathbf{y}_1 = \mathbf{v}_1, \quad y_3 = 0, \quad \partial_\nu \mathbf{y}_3 = \mathbf{v}_3, & \text{on } \partial\omega, \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (9)$$

$$\forall (\mathbf{y}^0, \mathbf{y}^1), \exists \mathbf{v}_1, \mathbf{v}_3 \in L^2(\partial\omega \times (0, T)) \text{ s.t. } \mathbf{y}(T) = \mathbf{y}'(T) = 0 \quad (10)$$

The answer is yes because the resolvent of $\mathbf{A}_M + \varepsilon^2 \mathbf{A}_F$ is compact ... but ...

$$\mathbf{V} = H_0^1(\omega) \times H_0^2(\omega), \quad \mathbf{H} = L^2(\omega) \times L^2(\omega) \quad (11)$$

$$\begin{cases} \phi'' + \mathbf{A}^\epsilon \phi = 0, & \text{in } \omega \times (0, T), \\ \phi = 0, & \text{on } \partial\omega, \\ (\phi(\cdot, 0), \phi'(\cdot, 0)) = (\phi^0, \phi^1) \in \mathbf{V} \times \mathbf{H}, & \text{in } \omega, \end{cases} \quad \begin{cases} \mathbf{y}'' + \mathbf{A}^\epsilon \mathbf{y} = 0, & \text{in } \omega \times (0, T), \\ \mathbf{y}_1 = \mathbf{v}_1, \quad \mathbf{y}_3 = 0, \quad \partial_\nu \mathbf{y}_3 = \mathbf{v}_3, & \text{on } \partial\omega, \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (12)$$

$\mathbf{y}(T) = \mathbf{y}'(T) = 0$ if and only $(\mathbf{v}_1, \mathbf{v}_3)$ satisfies

$$\int_0^T \int_{\partial\omega} \left(\phi_{1,1} \mathbf{v}_1 + \epsilon^2 (\phi_{3,11} - 2r^{-1} \phi_{1,1}) (\mathbf{v}_3 - 2r^{-1} \mathbf{v}_1) \right) \nu d\sigma dt = \langle \phi^0, \mathbf{y}^1 \rangle_{\mathbf{V}, \mathbf{V}'} - \langle \phi^1, \mathbf{y}^0 \rangle_{\mathbf{H}, \mathbf{H}} \quad (13)$$

Let $\mathcal{J} : \mathbf{V} \times \mathbf{H} \rightarrow \mathbb{R}$

$$\mathcal{J}(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \int_{\partial\omega} \left(\phi_{1,1}^2 + \epsilon^2 (\phi_{3,11} - 2r^{-1} \phi_{1,1})^2 \right) d\sigma dt - \langle \phi^0, \mathbf{y}^1 \rangle_{\mathbf{V}, \mathbf{V}'} + \langle \phi^1, \mathbf{y}^0 \rangle_{\mathbf{H}, \mathbf{H}} \quad (14)$$

\mathcal{J} is continuous and convex. If \mathcal{J} is coercive then the controllability holds with

$$\mathbf{v}_1 = \phi_{1,1} \nu, \quad \mathbf{v}_3 = \phi_{3,11} \nu \quad (15)$$

Coercivity of $\mathcal{J} \iff$ Observability inequality

$\implies \exists C_1 = C_1(r^{-1}, T, \epsilon) > 0$? T large enough, s.t.

$$C_1 \|(\phi^0, \phi^1)\|_{\mathbf{V} \times \mathbf{H}}^2 \leq \int_0^T \int_{\partial\omega} \left(\phi_{1,1}^2 + \epsilon^2 (\phi_{3,11} - 2r^{-1} \phi_{1,1})^2 \right) d\sigma dt \quad (16)$$

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Let

$$T^*(r^{-1}, \varepsilon) = \frac{2C}{1 - r^{-1}D(\varepsilon)}$$

and let $b^\varepsilon(\phi, \phi)$ such that $\int_\omega b^\varepsilon(\phi, \psi) d\xi = \int_\omega \mathbf{A}^\varepsilon \phi \cdot \psi d\xi$ for all $\phi, \psi \in \mathbf{V}$.

Theorem (Observability)

Let $\varepsilon > 0$ and $r^{-1} < 1/D(\varepsilon)$. For all $T > T^*(r^{-1}, \varepsilon)$, there exists a constant $C_2 = (1 - r^{-1}D(\varepsilon))$ such that the weak solution of the adjoint system satisfies the inequality

$$\int_0^T \int_{\partial\omega} b^\varepsilon(\phi, \phi) d\sigma dt \geq C_2(r^{-1}, \varepsilon)(T - T^*)E_0(\phi), \quad \forall (\phi^0, \phi^1) \in \mathbf{V} \times \mathbf{H}.$$

(From the Korn's inequality, $E_0(\phi)$ defines a norm over $\mathbf{V} \times \mathbf{H}$). ■

Theorem (Null controllability)

Let $\varepsilon > 0$, $r > D(\varepsilon)$ and $T > T^*$. For any $(\mathbf{y}^0, \mathbf{y}^1) \in \mathbf{H} \times \mathbf{V}'$, there exists a control $\mathbf{v} = (v_1, v_3)$ with $v_1, v_3 \in L^2(\partial\omega \times (0, T))$ such that $E(T, \mathbf{y}) = 0$. ■

2

²B. Miara, V. Valente, Exact controllability of a shallow Koiter Shell by a Boundary Action, J. Elasticity (1999).

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²B. Miara, V. Valente, Exact controllability of a shallow Koiter Shell by a Boundary Action, J. Elasticity (1999).

$$\mathbf{y}^0 = (\sin(\pi\xi), \sin^2(\pi\xi)), \quad \mathbf{y}^1 = (0, 0), \quad r^{-1} = C = \pi/4, \quad \varepsilon = 1/10, \quad T = 3 \quad (18)$$

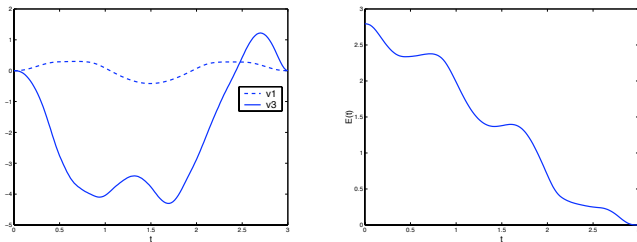


Figure: Control $(v_1, v_3) = (\partial_\nu u_1, \partial_{\nu\nu} u_3)$ at $\xi = 0$ (Left) - Energy $E_h(t, \mathbf{y})$ (Right).

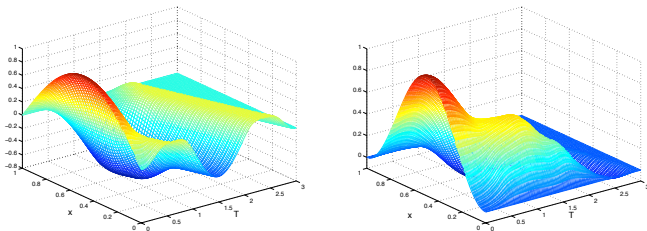


Figure: Controlled displacement y_1 (Left) and y_3 (Right) on $\omega \times (0, T)$.

$$\mathbf{y}^0 = (\sin(\pi\xi), \sin^2(\pi\xi)), \quad \mathbf{y}^1 = (0, 0), \quad C = \pi/4 \quad (19)$$

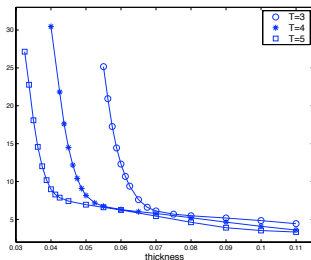
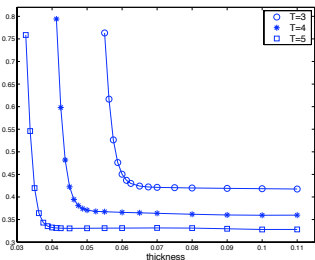


Figure: L^2 -norm $\|v_1\|_{L^2(0,T)}$ (Left) and $\|v_3\|_{L^2(0,T)}$ (Right) vs. the thickness ϵ .

$$T^* = 1 + O\left(\frac{r^{-1}}{\epsilon}\right) \quad (20)$$

4

⇒ Non uniform controllability with respect to ϵ !!!!!!!!!!!!!?!!

⁴ Geymonat G., Loreti P., Valente V., *Exact controllability of thin elastic hemispherical shell via harmonic analysis*, (1993)

Controllability of an elastic arch ($\epsilon > 0$)

	$\epsilon = 1/10$	$\epsilon = 1/20$	$\epsilon = 1/30$	$\epsilon = 1/40$
Nb. iterations	8	9	16	n.c.
$\ v_{1h}\ _{L^2(\Sigma_T)}$	2.942×10^{-1}	2.952×10^{-1}	3.095×10^{-1}	n.c.
$\ \epsilon v_{3h}\ _{L^2(\Sigma_T)}$	3.38×10^{-1}	3.56×10^{-1}	8.52×10^{-1}	n.c.
$\ b^\epsilon(\phi, \phi)\ _{L^1(\Sigma_T)}$	1.24×10^{-1}	1.295×10^{-1}	3.37×10^{-1}	n.c.
$E_h(T)/E_h(0)$	7.29×10^{-6}	3.36×10^{-5}	1.90×10^{-3}	n.c.

Table: Approximation of the control vs. ϵ - $\mathbf{T} = \mathbf{3}$ - $C = \pi/32$ - $\Sigma_T = \partial\omega \times (0, T)$. n.c. stands for non controllability.

	$\epsilon = 1/10$	$\epsilon = 1/20$	$\epsilon = 1/30$	$\epsilon = 1/40$
Nb. iterations	8	8	8	8
$\ v_{1h}\ _{L^2(\Sigma_T)}$	2.942×10^{-1}	2.551×10^{-1}	2.302×10^{-1}	2.151×10^{-1}
$\ \epsilon v_{3h}\ _{L^2(\Sigma_T)}$	3.38×10^{-1}	2.49×10^{-1}	2.41×10^{-1}	2.49×10^{-1}
$\ b^\epsilon(\mathbf{u}, \mathbf{u})\ _{L^1(\Sigma_T)}$	1.24×10^{-1}	8.57×10^{-2}	7.20×10^{-2}	6.70×10^{-2}
$E_h(T)/E_h(0)$	7.29×10^{-6}	1.04×10^{-5}	2.14×10^{-4}	1.38×10^{-4}

Table: Approximation of the control vs. ϵ - $\mathbf{T}(\epsilon) = \mathbf{3}/(10\epsilon)$ - $C = \pi/32$ - $\Sigma_T = \partial\omega \times (0, T)$.

Step 1: Non uniform controllability in $\mathbf{H} \times \mathbf{V}'$ w.r.t. ε

Step 2: What is about the case $\varepsilon = 0$?

$$\mathbf{V} = H_0^1(\omega) \times L^2(\omega), \mathbf{H} = L^2(\omega) \times L^2(\omega) \quad (21)$$

$$\omega = (0, 1), T > 0, r > 0, (\mathbf{y}^0, \mathbf{y}^1) \in \mathbf{H}' \times \mathbf{V}'$$

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M \mathbf{y} = \mathbf{0}, & \text{in } \omega \times (0, T), \\ y_1(0, t) = 0, \quad \mathbf{y}_1(1, t) = \mathbf{v}(t), & t \in (0, T), \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (22)$$

$$\mathbf{v} \in L^2(0, T)? \quad \text{s.t.} \quad \mathbf{y}(\xi, T) = \mathbf{y}'(\xi, T) = \mathbf{0}, \quad \forall \xi \in \omega. \quad (23)$$

\implies Exact controllability of y_1 and y_3 by only one control !

5

⁵F. Ammar-Khodja, Geymonat G., AM, *On the exact controllability of a system of mixed order with essential spectrum*, C.R.Acad. Sci. Paris Série I, (2008).

Adjoint system -

$$\begin{cases} \phi'' + \mathbf{A}_M \phi = \mathbf{0}, & \text{in } \omega \times (0, T), \\ \phi_1(0, \cdot) = \phi_1(1, \cdot) = 0, & t \in (0, T) \\ (\phi(\cdot, 0), \phi'(\cdot, 0)) = (\phi^0, \phi^1), & \text{in } \omega, \end{cases} \quad (24)$$

Weak formulation - For all $(\phi^0, \phi^1) \in \mathbf{V} \times \mathbf{H}$, there exists a unique weak solution $\phi \in C(0, T; \mathbf{V}) \cap C^1(0, T; \mathbf{H})$ that satisfies the variational problem

$$\int_{\omega} \phi'' \cdot \mathbf{v} \, d\xi + \int_{\omega} b_M(\phi, \mathbf{v}) \, d\xi = 0, \quad \forall \mathbf{v} \in \mathbf{V}. \quad (25)$$

$$b_M(\phi, \mathbf{v}) = (\phi_{1,1} + r^{-1} \phi_3)(v_{1,1} + r^{-1} v_3)$$

Energy -

$$E(t, \phi) = \frac{1}{2} \int_{\omega} (|\phi'|^2 + b_M(\phi, \phi)) \, d\xi = \frac{1}{2} \int_{\omega} (|\phi'_1|^2 + |\phi'_3|^2 + (\phi_{1,1} + r^{-1} \phi_3)^2) \, d\xi = E(0, \phi) \quad \forall t \in (0, T) \quad (26)$$

$$A_M \psi = \lambda \psi, \quad \xi \in \omega, \quad \psi_1 = 0, \quad \xi \in \partial\omega \quad (27)$$

$$\sigma(A_M) = \{0, \lambda_0 = r^{-2}, \lambda_k = r^{-2} + (k\pi)^2, k \geq 1\}, \quad \sigma_{\text{ess}}(A_M) = \{0\}. \quad (28)$$

Since

$$\text{Ker } A_M = \{v_\zeta = (-r^{-1}\zeta, \zeta_1) \in H, \zeta \in H_0^1(\omega)\} \quad (29)$$

and the eigenfunctions associated with λ_0 and λ_k are respectively :

$$v_0 = (0, 1), \quad v_k = \left(\sin(k\pi\zeta), \frac{r^{-1}}{k\pi} \cos(k\pi\zeta) \right), \quad (30)$$

An orthogonal basis in H of $\text{Ker } A_M$ is

$$w_k = \left(-\frac{r^{-1}}{k\pi} \sin(k\pi\zeta), \cos(k\pi\zeta) \right), \quad k = 1, 2, \dots \quad (31)$$

and $\{w_k, v_0, v_k\}$ is an orthogonal basis in H . This permits to expanded the weak solution of (24) in term of a Fourier serie (setting $\mu_0 = r^{-1}$ and $\mu_k = \sqrt{\lambda_k}$):

$$\phi(\cdot, t) = \sum_{k=1}^{\infty} (a_k + b_k t) w_k + (A_0 \cos(\mu_0 t) + B_0 \sin(\mu_0 t)) v_0 + \sum_{k=1}^{\infty} (A_k \cos(\mu_k t) + B_k \sin(\mu_k t)) v_k \quad (32)$$

The coefficients $a_k, b_k, A_0, B_0, A_k, B_k$ are determined from the expansion of the initial data :

$$\phi^0 = \sum_{k=1}^{\infty} a_k w_k + A_0 v_0 + \sum_{k=1}^{\infty} A_k v_k, \quad \phi^1 = \sum_{k=1}^{\infty} b_k w_k + \mu_0 B_0 v_0 + \sum_{k=1}^{\infty} \mu_k B_k v_k. \quad (33)$$

The assumption $(\phi^0, \phi^1) \in V \times H$ implies that

$$\sum_{k=1}^{\infty} (a_k^2 + A_k^2) < \infty, \quad \sum_{k=1}^{\infty} (r^{-1} a_k + k\pi A_k)^2 < \infty, \quad \sum_{k=1}^{\infty} (b_k^2 + \lambda B_k^2) < \infty. \quad (34)$$

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$$C_1 \|(\phi^0, \phi^1)\|_{\mathbf{V} \times \mathbf{H}}^2 \leq \int_0^T (\phi_{1,1} + r^{-1} \phi_3)^2(1, t) dt \leq C_2 \|(\phi^0, \phi^1)\|_{\mathbf{V} \times \mathbf{H}}^2 \quad \forall (\phi^0, \phi^1) \in \mathbf{V} \times \mathbf{H}. \quad (35)$$

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Remark

From the Korn's inequality, the energy $E(t, \phi)$ defines a norm over $\mathbf{V}^\perp \times \mathbf{H}^\perp$.

Proposition

Let $r > 0$. For all time T such that

$$T > T^*(r) \equiv \frac{2\pi}{\gamma}, \quad \gamma = \min\left(2r^{-1}, \sqrt{r^{-2} + \pi^2} - r^{-1}\right) \quad (38)$$

there exist two strictly positive constants $C_1(r)$ and $C_2(r)$ such that

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We obtain

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using

Theorem (Ingham, 1936)

Let $K \in \mathbb{Z}$ and $(w_k)_{k \in K}$ be a family of real numbers satisfying the uniform gap condition

$$\gamma = \inf_{k \neq n} |w_k - w_n| > 0. \quad (41)$$

If I is a bounded interval of length $|I| > 2\pi/\gamma$, then there exist two positives constants C_1 and C_2 such that

$$C_1 \sum_{k \in K} |x_k|^2 \leq \int_I |x(t)|^2 dt \leq C_2 \sum_{k \in K} |x_k|^2 \quad (42)$$

for all functions given by the sum

$$x(t) = \sum_{k \in K} x_k e^{iw_k t} \quad (43)$$

with square-summable complex coefficients x_k . ■

with $x(t) = (\phi_{1,1} + r^{-1}\phi_3)(1, t)$ and $I = (0, T)$.

6

⁶Ingham A.E., *Some trigonometrical inequalities with applications to the theory of series*, Math. Z., (1936). ≡ 🔍 ↻

We compute that

$$E(0, \phi) = \frac{\mu_0^2}{2} (A_0^2 + B_0^2) + \frac{1}{4} \sum_{k=1}^{\infty} \frac{\mu_k^4}{k^2 \pi^2} (A_k^2 + B_k^2). \quad (44)$$

On the other hand, we have

$$\begin{aligned} \phi_{1,1}(1, t) + r^{-1} \phi_3(1, t) &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k \mu_k^2}{k\pi} (A_k + iB_k) e^{-i\mu_k t} + \frac{r^{-1}}{2} (A_0 + iB_0) e^{-i\mu_0 t} \\ &\quad + \frac{r^{-1}}{2} (A_0 - iB_0) e^{i\mu_0 t} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k \mu_k^2}{k\pi} (A_k - iB_k) e^{i\mu_k t}. \end{aligned} \quad (45)$$

We then apply Ingham's theorem with $I = (0, T)$ and the sequence

$$w = (\dots, -\mu_2, -\mu_1, -\mu_0, \mu_0, \mu_1, \mu_2, \dots) \quad (46)$$

to obtain that there exists a positive constant C_1 such that

$$C_1 \left(\frac{\mu_0^2}{2} (A_0^2 + B_0^2) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\mu_k^4}{(k\pi)^2} (A_k^2 + B_k^2) \right) \leq \int_0^T (\phi_{1,1}(1, t) + r^{-1} \phi_3(1, t))^2 dt \quad (47)$$

under the condition $T > 2\pi/\gamma$ with $\gamma = \min(\mu_0 - (-\mu_0), \inf_{k \in \mathbb{N}} |\mu_k - \mu_{k-1}|)$ leading to $T > T^*(r)$.

We compute that

$$E(0, \phi) = \frac{\mu_0^2}{2} (A_0^2 + B_0^2) + \frac{1}{4} \sum_{k=1}^{\infty} \frac{\mu_k^4}{k^2 \pi^2} (A_k^2 + B_k^2). \quad (44)$$

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We compute that

$$E(0, \phi) = \frac{\mu_0^2}{2} (A_0^2 + B_0^2) + \frac{1}{4} \sum_{k=1}^{\infty} \frac{\mu_k^4}{k^2 \pi^2} (A_k^2 + B_k^2). \quad (44)$$

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We then apply Ingham's theorem with $I = (0, T)$ and the sequence

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under the condition $T > 2\pi/\gamma$ with $\gamma = \min(\mu_0 - (-\mu_0), \inf_{k \in \mathbb{N}} |\mu_k - \mu_{k-1}|)$ leading to $T > T^*(r)$.

$$T^*(r) = \frac{\pi}{r^{-1}} \chi_{(r^{-1} \leq \pi^2/8)} + \frac{2\pi}{\sqrt{r^{-2} + \pi^2} - r^{-1}} \chi_{(r^{-1} > \pi^2/8)} \quad (48)$$

$$\sqrt{8} = T^*\left(\frac{\pi^2}{8}\right) \leq T^*(r), \quad \forall r > 0 \quad (49)$$

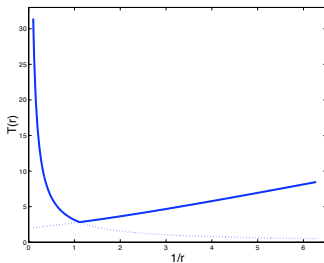


Figure: Lower bound T^* of the time of controllability T with respect to the curvature r^{-1} .

$$\forall (\phi^0, \phi^1) \in \mathbf{v}_0$$

$$2 \min\left(T, \frac{T^3}{3} r^{-2}\right) E_0(\phi) \leq \int_0^T (\phi_{1,1} + r^{-1} \phi_3)^2(1, t) dt \leq 2 \max\left(T, \frac{T^3}{3} r^{-2}\right) E_0(\phi). \quad (50)$$

We denote by H_K , V_K the closed subspace of H and V generated by v_k , for all $k \geq 1$.

Proposition

Let $r > 0$. For all $T > 0$ such that

$$T > T^{**}(r) \equiv \frac{2\pi}{\gamma}, \quad \gamma = \sqrt{r^{-2} + 4\pi^2} - \sqrt{r^{-2} + \pi^2} \quad (51)$$

there exist two positive constants C_1 and C_2 independent of r such that

$$C_1 \|(\phi^0, \phi^1)\|_{V \times H}^2 \leq \int_0^T (\phi_{1,1} + r^{-1} \phi_3)^2(1, t) dt \leq C_2 \|(\phi^0, \phi^1)\|_{V \times H}^2 \quad (52)$$

$\forall (\phi^0, \phi^1) \in V_K \times H_K$. ■

$$T^{**}(r) < T^*(r), \quad \forall r > 0; \quad \lim_{r \rightarrow 0} T^{**}(r) = 2. \quad (53)$$

We note $V^{\perp'}$ the closure of V^{\perp} in V' . The control v is characterized by

$$\int_0^T (\phi_{1,1} + r^{-1} \phi_3)(1, t) v(t) dt = \langle \phi^0, y^1 \rangle_{V, V'} - \langle \phi^1, y^0 \rangle_{H, H} \quad (54)$$

We introduce the continuous and convex functional $\mathcal{J} : V \times H \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(\phi^0, \phi^1) = \frac{1}{2} \int_0^T (\phi_{1,1} + r^{-1} \phi_3)^2(1, t) dt - \langle (\phi^0, \phi^1), (y^1, -y^0) \rangle_{V \times H, V' \times H} \quad (55)$$

Moreover, We then have to enforce that y^0 be in the dual of H^{\perp} and that y^1 be in the dual of V^{\perp} in order that

$$\langle \phi^0, y^1 \rangle_{V, V'} - \langle \phi^1, y^0 \rangle_{H, H} \neq 0, \quad \forall (\phi^0, \phi^1) \in V^{\perp} \times H^{\perp}. \quad (56)$$

Let $r > 0$. For any $T > T^*(r)$ and any initial data $(y^0, y^1) \in H^{\perp} \times V^{\perp'}$, there exists a control function $v \in L^2(0, T)$ which drives to rest at time T the solution y of (22) associated with (y^0, y^1) . Moreover, the control of minimal L^2 -norm is given by $v = (\phi_{1,1} + r^{-1} \phi_3)(1, \cdot)$ where ϕ is solution of (24) and associated with (ϕ^0, ϕ^1) minimum of \mathcal{J} defined by (55) over $V^{\perp} \times H^{\perp}$. ■

The non controllable modes $w_k \in \text{Ker } A_M$, $k \leq 1$, do not correspond to modes of arbitrarily small energy. For $(y^0, y^1) = \sum_{k \geq 1} (a_k, b_k) w_k$, the norm of the solution at time $t = T$ is such that (since the control v has no effect on w_k)

$$\|y(T)\|_V^2 = \sum_{k \geq 1} (a_k + b_k T)^2 \left(r^{-2} + \frac{\lambda_k}{(k\pi)^2} \right), \quad \|y'(T)\|_H^2 = \frac{1}{2} \sum_{k \geq 1} b_k^2 \frac{\lambda_k}{(k\pi)^2} \quad (57)$$

Consequently, approximate controllability for the system (22) does not hold anymore. ■

We note $\mathbf{V}^{\perp'}$ the closure of \mathbf{V}^{\perp} in \mathbf{V}' . The control v is characterized by

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Moreover, We then have to enforce that \mathbf{y}^0 be in the dual of \mathbf{H}^{\perp} and that \mathbf{y}^1 be in the dual of \mathbf{V}^{\perp} in order that

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Theorem

Let $r > 0$. For any $T > T^*(r)$ and any initial data $(\mathbf{y}^0, \mathbf{y}^1) \in \mathbf{H}^{\perp} \times \mathbf{V}^{\perp'}$, there exists a control function $v \in L^2(0, T)$ which drives to rest at time T the solution \mathbf{y} of (22) associated with $(\mathbf{y}^0, \mathbf{y}^1)$. Moreover, the control of minimal L^2 -norm is given by $v = (\phi_{1,1} + r^{-1} \phi_3)(1, \cdot)$ where ϕ is solution of (24) and associated with (ϕ^0, ϕ^1) minimum of \mathcal{J} defined by (55) over $\mathbf{V}^{\perp} \times \mathbf{H}^{\perp}$. ■

Remark

The non controllable modes $\mathbf{w}_k \in \text{Ker } \mathbf{A}_M$, $k \leq 1$, do not correspond to modes of arbitrarily small energy. For $(\mathbf{y}^0, \mathbf{y}^1) = \sum_{k \geq 1} (a_k, b_k) \mathbf{w}_k$, the norm of the solution at time $t = T$ is such that (since the control v has no effect on \mathbf{w}_k)

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$$\mathcal{J}(\phi^0, \phi^1) = \frac{1}{2} \int_0^T (\phi_{1,1} + r^{-1} \phi_3)^2(1, t) dt - \langle (\phi^0, \phi^1), (\mathbf{y}^1, -\mathbf{y}^0) \rangle_{\mathbf{V} \times \mathbf{H}, \mathbf{V}' \times \mathbf{H}} \quad (55)$$

Moreover, We then have to enforce that \mathbf{y}^0 be in the dual of \mathbf{H}^{\perp} and that \mathbf{y}^1 be in the dual of \mathbf{V}^{\perp} in order that

$$\langle \phi^0, \mathbf{y}^1 \rangle_{\mathbf{V}, \mathbf{V}'} - \langle \phi^1, \mathbf{y}^0 \rangle_{\mathbf{H}, \mathbf{H}} \neq 0, \quad \forall (\phi^0, \phi^1) \in \mathbf{V}^{\perp} \times \mathbf{H}^{\perp}. \quad (56)$$

Theorem

Let $r > 0$. For any $T > T^*(r)$ and any initial data $(\mathbf{y}^0, \mathbf{y}^1) \in \mathbf{H}^{\perp} \times \mathbf{V}^{\perp'}$, there exists a control function $v \in L^2(0, T)$ which drives to rest at time T the solution \mathbf{y} of (22) associated with $(\mathbf{y}^0, \mathbf{y}^1)$. Moreover, the control of minimal L^2 -norm is given by $v = (\phi_{1,1} + r^{-1} \phi_3)(1, \cdot)$ where ϕ is solution of (24) and associated with (ϕ^0, ϕ^1) minimum of \mathcal{J} defined by (55) over $\mathbf{V}^{\perp} \times \mathbf{H}^{\perp}$. ■

Remark

The non controllable modes $\mathbf{w}_k \in \text{Ker } \mathbf{A}_M$, $k \leq 1$, do not correspond to modes of arbitrarily small energy. For $(\mathbf{y}^0, \mathbf{y}^1) = \sum_{k \geq 1} (a_k, b_k) \mathbf{w}_k$, the norm of the solution at time $t = T$ is such that (since the control v has no effect on \mathbf{w}_k)

$$\|\mathbf{y}(T)\|_{\mathbf{V}}^2 = \sum_{k \geq 1} (a_k + b_k T)^2 \left(r^{-2} + \frac{\lambda_k}{(k\pi)^2} \right), \quad \|\mathbf{y}'(T)\|_{\mathbf{H}}^2 = \frac{1}{2} \sum_{k \geq 1} b_k^2 \frac{\lambda_k}{(k\pi)^2} \quad (57)$$

Consequently, approximate controllability for the system (22) does not hold anymore. ■

We assume that the initial condition $(\mathbf{y}^0, \mathbf{y}^1)$ are generated by $\{\mathbf{v}_k\}$ and that $(\mathbf{y}^0, \mathbf{y}^1) \rightarrow (\widetilde{\mathbf{y}}^0, \widetilde{\mathbf{y}}^1)$ in $\mathbf{H} \times \mathbf{V}'$ as r^{-1} goes to zero.

Let y_1 solution of

$$\begin{cases} y_1'' - y_{1,11} = 0, & \omega \times (0, T), \\ y_1(0, t) = 0, \quad y_1(1, t) = \widetilde{v}, & (0, T), \\ (y_1(\xi, 0), y_1'(\xi, 0)) = (\widetilde{y}_1^0, \widetilde{y}_1^1). \end{cases} \quad (58)$$

Theorem

Let $r > 0$. For any $T > T^{**}(r)$ and any initial data $(\mathbf{y}^0, \mathbf{y}^1) \in (\mathbf{H}'_K \times \mathbf{V}'_K)$, there exists a control function $v \in L^2(0, T)$ which drives to rest at time T the solution \mathbf{y} of (22) associated with $(\mathbf{y}^0, \mathbf{y}^1)$. Moreover, the control of minimal L^2 -norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1, \cdot)$ where ϕ is solution of (24) and associated with (ϕ^0, ϕ^1) minimum of \mathcal{J} defined by (55) over $\mathbf{V}_K \times \mathbf{H}_K$. Finally, this control converges weakly in $L^2(0, T)$ as r^{-1} goes to zero toward the control of minimal L^2 -norm which drives to rest the solution y_1 of (58). ■

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M \mathbf{y} = \mathbf{0}, & \text{in } \omega \times (0, T), \\ r^{-1} y_1 + y_{3,1} = 0, & \text{in } \omega \times (0, T) \\ y_1(0, t) = 0, y_1(1, t) = v(t), & t \in (0, T) \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (59)$$

If $v \in L^2(0, T)$ is the HUM control, i.e. of minimal L^2 -norm for (59), then, using formally the relation $r^{-1} y_1 + y_{3,1} = 0$, v is also a control (but *a priori* not the control of minimal L^2 -norm, except at the limit as r^{-1} goes to zero) for y_1 solution of

$$\begin{cases} y_1'' - y_{1,11} + r^{-2} y_1 = 0, & \text{in } \omega \times (0, T), \\ y_1(0, t) = 0, y_1(1, t) = v(t), & t \in (0, T) \\ (y_1(\cdot, 0), y_1'(\cdot, 0)) = (y_1^0, y_1^1) \in L^2(\omega) \times H^{-1}(\omega) \end{cases} \quad (60)$$

and also a control for y_3 solution of

$$\begin{cases} y_3'' - y_{3,11} + r^{-2} y_3 = 0, & \text{in } \omega \times (0, T), \\ y_{3,1}(0, t) = 0, y_{3,1}(1, t) = -r^{-1} v(t), & t \in (0, T) \\ (y_3(\cdot, 0), y_3'(\cdot, 0)) = (y_3^0, y_3^1) \in L^2(\omega) \times L^2(\omega). \end{cases} \quad (61)$$

assuming a compatibility condition at time $t = 0$ such that:

$$y_1(\xi, 0) = -r y_{3,1}(\xi, 0), \quad y_1'(\xi, 0) = -r y_{3,1}'(\xi, 0), \quad \text{in } \omega. \quad (62)$$

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M \mathbf{y} = \mathbf{0}, & \text{in } \omega \times (0, T), \\ r^{-1} y_1 + y_{3,1} = 0, & \text{in } \omega \times (0, T) \\ y_1(0, t) = 0, y_1(1, t) = v(t), & t \in (0, T) \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (59)$$

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and also a control for y_3 solution of

$$\begin{cases} y_3'' - y_{3,11} + r^{-2} y_3 = 0, & \text{in } \omega \times (0, T), \\ y_{3,1}(0, t) = 0, y_{3,1}(1, t) = -r^{-1} v(t), & t \in (0, T) \\ (y_3(\cdot, 0), y_3'(\cdot, 0)) = (y_3^0, y_3^1) \in L^2(\omega) \times L^2(\omega). \end{cases} \quad (61)$$

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$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M \mathbf{y} = \mathbf{0}, & \text{in } \omega \times (0, T), \\ (y_{1,1} + r^{-1}y_3)(0, t) = 0, \quad (y_{1,1} + r^{-1}y_3)(1, t) = v(t), & t \in (0, T) \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (63)$$

assuming $\mathbf{y}^0 \in H^1(\omega) \times L^2(\omega)$, and $\mathbf{y}^1 \in L^2(\omega) \times H^{-1}(\omega)$. If we introduce the variable $z = y_{1,1} + r^{-1}y_3$, we first observe that z is solution of

$$\begin{cases} z'' - z_{,11} + r^{-2}z = 0, & \text{in } \omega \times (0, T), \\ z(0, t) = 0, \quad z(1, t) = v(t), & t \in (0, T), \\ (z(\cdot, 0), z'(\cdot, 0)) = (y_{1,1}^0 + r^{-1}y_3^0, y_{1,1}^1 + r^{-1}y_3^1), & \text{in } \omega; \end{cases} \quad (64)$$

There exists $v \in L^2(0, T)$ such that $z(\xi, T) = (y_{1,1} + r^{-1}y_3)(\xi, T) = 0$ and $z'(\xi, T) = 0$ for T large enough.

$$\begin{aligned} \phi(\xi, t) = & \sum_{k \geq 0} \begin{pmatrix} r^{-1}(a_k(\xi) + b_k(\xi)t) \\ -(a_k(\xi) + b_k(\xi)t)_{,1} \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \sum_{k=1}^{\infty} (A_k \cos(\mu_k t) + B_k \sin(\mu_k t)) \begin{pmatrix} \cos(k\pi\xi) \\ -\frac{r^{-1}}{k\pi} \sin(k\pi\xi) \end{pmatrix} \end{aligned} \quad (65)$$

$$C_1 \sum_{k=0}^{\infty} (A_k^2 + B_k^2) \leq \int_0^T \phi_1(1, t)^2 dt \leq C_2 \sum_{k=0}^{\infty} (A_k^2 + B_k^2) \quad (66)$$

provided

$$T > \frac{2\pi}{\sqrt{\pi^2 + r^{-2}}} \quad (67)$$

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M \mathbf{y} = \mathbf{0}, & \text{in } \omega \times (0, T), \\ (y_{1,1} + r^{-1}y_3)(0, t) = 0, \quad (y_{1,1} + r^{-1}y_3)(1, t) = v(t), & t \in (0, T) \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (63)$$

assuming $\mathbf{y}^0 \in H^1(\omega) \times L^2(\omega)$, and $\mathbf{y}^1 \in L^2(\omega) \times H^{-1}(\omega)$. If we introduce the variable $z = y_{1,1} + r^{-1}y_3$, we first observe that z is solution of

$$\begin{cases} z'' - z_{,11} + r^{-2}z = 0, & \text{in } \omega \times (0, T), \\ z(0, t) = 0, \quad z(1, t) = v(t), & t \in (0, T), \\ (z(\cdot, 0), z'(\cdot, 0)) = (y_{1,1}^0 + r^{-1}y_3^0, y_{1,1}^1 + r^{-1}y_3^1), & \text{in } \omega; \end{cases} \quad (64)$$

There exists $v \in L^2(0, T)$ such that $z(\xi, T) = (y_{1,1} + r^{-1}y_3)(\xi, T) = 0$ and $z'(\xi, T) = 0$ for T large enough.

$$\begin{aligned} \phi(\xi, t) = & \sum_{k \geq 0} \begin{pmatrix} r^{-1}(a_k(\xi) + b_k(\xi)t) \\ -(a_k(\xi) + b_k(\xi)t)_{,1} \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \sum_{k=1}^{\infty} (A_k \cos(\mu_k t) + B_k \sin(\mu_k t)) \begin{pmatrix} \cos(k\pi\xi) \\ -\frac{r^{-1}}{k\pi} \sin(k\pi\xi) \end{pmatrix} \end{aligned} \quad (65)$$

$$C_1 \sum_{k=0}^{\infty} (A_k^2 + B_k^2) \leq \int_0^T \phi_1(1, t)^2 dt \leq C_2 \sum_{k=0}^{\infty} (A_k^2 + B_k^2) \quad (66)$$

provided

$$T > \frac{2\pi}{\sqrt{\pi^2 + r^{-2}}} \quad (67)$$

$$\begin{cases} \mathbf{y}'' + \mathbf{A}_M \mathbf{y} = \mathbf{0}, & \text{in } \omega \times (0, T), \\ (y_{1,1} + r^{-1}y_3)(0, t) = 0, \quad (y_{1,1} + r^{-1}y_3)(1, t) = v(t), & t \in (0, T) \\ (\mathbf{y}(\cdot, 0), \mathbf{y}'(\cdot, 0)) = (\mathbf{y}^0, \mathbf{y}^1), & \text{in } \omega, \end{cases} \quad (63)$$

assuming $\mathbf{y}^0 \in H^1(\omega) \times L^2(\omega)$, and $\mathbf{y}^1 \in L^2(\omega) \times H^{-1}(\omega)$. If we introduce the variable $z = y_{1,1} + r^{-1}y_3$, we first observe that z is solution of

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For the circular arch, the map is $\phi(\xi) = \left(r \sin(r^{-1}\xi), r \cos(r^{-1}\xi) \right)$.

In the general case $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, we have

$$A_M \mathbf{y} = \begin{pmatrix} -\gamma_{11,1}(y) - 2\gamma_{11}(y)\Gamma_{11}^1 \\ -\gamma_{11}(y)b_{11} \end{pmatrix}. \quad (68)$$

with

$$\begin{cases} t = \phi_{1,1}^2 + \phi_{2,1}^2; \Gamma_{11}^1 = t^{-1}(\phi_{1,1}\phi_{1,11} + \phi_{2,1}\phi_{2,11}); \\ \gamma_{11}(y) = y_{1,1} - \Gamma_{11}^1 y_1 - b_{11} y_3; \\ b_{11} = t^{-3/2}(-\phi_{2,1}\phi_{1,11} + \phi_{1,1}\phi_{2,11}) \end{cases} \quad (69)$$

$$\begin{aligned} \text{Ker } A_M &= \left\{ (y_1, y_3) \in H_0^1(\omega) \times L^2(\omega), \gamma_{11}(y) = 0 \right\} \neq \emptyset \\ &= \{ v = (\zeta, b_{11}^{-1}(\zeta, 1 - \Gamma_{11}^1 \zeta)) \in H, \zeta \in H_0^1(\omega) \} \end{aligned} \quad (70)$$

Once again, $\sigma_{\text{ess}}(A_M) = \{0\}$. But, in that case, the discrete spectrum is not explicit.

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$(y^0, y^1) \in H \times V'$, $(y_0^T, y_1^T) \in L^2(\omega) \times H^{-1}(\omega)$. Is there exists a control $v \in L^2(0, T)$, such that

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$$\begin{cases} y_1''(\xi, t) - y_{1,11}(\xi, t) + r^{-1} \int_0^t \sin(r^{-1}(t-u)) y_{1,11}(\xi, u) du = 0 & \text{in } \omega \times (0, T) \\ y_1(0, t) = 0; y_1(1, t) = v(t) & (0, T) \end{cases} \quad (72)$$

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$$r^{-1} = \pi/5, T = 3.5 > T^*(r) \approx 3.2552, \mathbf{y}^0 = \mathbf{v}_0 + \mathbf{v}_1 \quad \mathbf{y}^1 = \mu_0 \mathbf{v}_0 + \mu_2 \mathbf{v}_2 \quad (76)$$

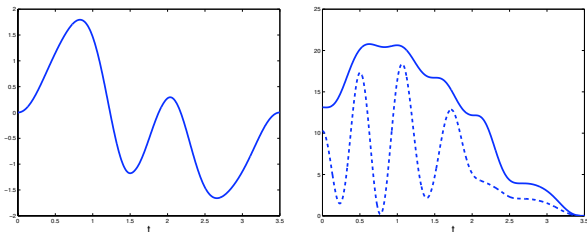


Figure: Left: HUM control v ; Right: Energy (solid line) and kinetic energy (dashed line)

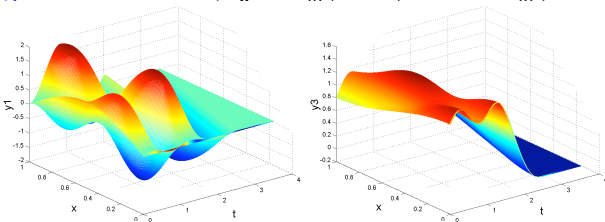


Figure: Controlled solution $\mathbf{y} = (y_1, y_3)$ in $(0, T) \times \omega$.

$$y(x_1, x_3) = y_1(\xi)\tau(\xi) + y_3(\xi)\nu(\xi), \quad \xi \in \omega \quad (x_1, x_3) = \phi(\xi) \quad (77)$$

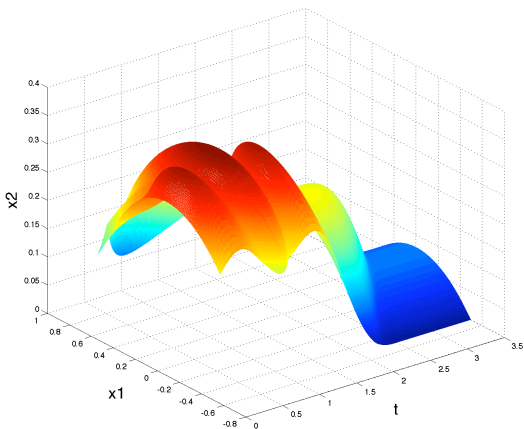


Figure: Evolution in the cartesian plane (O, x_1, x_3) of the controlled arch vs. $t \in (0, T)$.

$$y^0(\xi) = \alpha v_0 + v_1, \quad y^1(\xi) = \alpha \mu_0 v_0 + \mu_2 v_2 \quad (78)$$

	$r^{-1} = \pi$	$r^{-1} = \pi/4$	$r^{-1} = \pi/16$	$r^{-1} = \pi/64$	$r^{-1} = \pi/256$
Nb. Iterations	8	8	9	7	8
$\ v\ _{L^2(0,T)}$	1.415	1.601	4.823	15.791	59.220
$E(0)/E(T)$	2.51×10^{-5}	4.47×10^{-7}	3.12×10^{-6}	3.26×10^{-5}	4.49×10^{-4}
$\frac{\ v\ _{L^2(0,T)}^2}{\ (\phi^0, \phi^1)\ _{V \times H}^2}$	5.33×10^{-1}	7.95×10^{-2}	1.67×10^{-3}	9.86×10^{-5}	6.1804×10^{-6}

Table: $\alpha = 1$ - Evolution of the L^2 -norm of the control vs. the curvature r^{-1} .

	$r^{-1} = \pi$	$r^{-1} = \pi/4$	$r^{-1} = \pi/16$	$r^{-1} = \pi/64$	$r^{-1} = \pi/256$
Nb. Iterations	5	4	4	4	4
$\ v\ _{L^2(0,T)}$	0.823	0.692	0.679	0.678	0.678
$E(0)/E(T)$	1.12×10^{-6}	8.47×10^{-8}	4.28×10^{-6}	2.83×10^{-7}	1.05×10^{-7}
$\frac{\ v\ _{L^2(0,T)}^2}{\ (\phi^0, \phi^1)\ _{V \times H}^2}$	2.019	1.565	1.515	1.5126	1.5124
$\frac{\ v - v_{r=\infty}\ _{L^2(0,T)}}{\ v_{r=\infty}\ _{L^2(0,T)}}$	1.679	1.24×10^{-1}	7.87×10^{-3}	5.77×10^{-4}	4.28×10^{-5}

Table: $\alpha = 0$ - Evolution of the L^2 -norm of the control vs. the curvature r^{-1} .

$$r^{-1} = \pi/5, T = 3.5 > T^*(r) \approx 3.2552, \quad (\mathbf{y}^0, \mathbf{y}^1) = (\mathbf{0}, \mathbf{0}), \quad (y_T^0(\xi), y_T^1(\xi)) = (\sin(\pi\xi), \mu_2 \sin(2\pi\xi)) \quad (79)$$

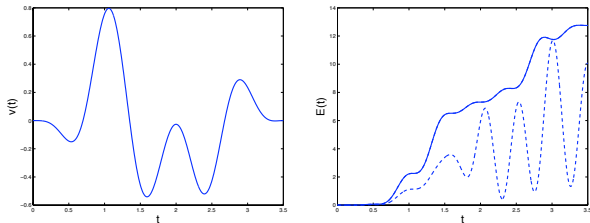


Figure: **Left:** RHUM control v ; **Right:** Energy (solid line) and kinetic energy (dashed line)

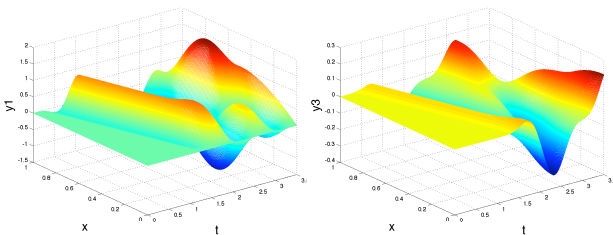


Figure: Evolution of the controlled component y_1 vs. t (**Left**) and corresponding evolution of the component y_3 (**Right**), starting from $(\mathbf{0}, \mathbf{0})$.

Step 1: $\varepsilon > 0$ - Non uniform controllability in $\mathbf{H} \times \mathbf{V}'$ w.r.t. ε

Step 2: $\varepsilon = 0$ - Controllability only in $(\text{Ker} \mathbf{A}_M)^\perp$

Step 3: $\varepsilon \rightarrow 0$ in $(\text{Ker} \mathbf{A}_M)^\perp$

Let us assume that $(\mathbf{y}_\varepsilon^0, \mathbf{y}_\varepsilon^1) \rightarrow (\mathbf{y}^0, \mathbf{y}^1) \in \text{Ker} \mathbf{A}_M^\perp$.

Question: Is the observability uniform w.r.t. ε in that case ?

The answer is no !!!!!!!?!!

(From now, to simplify,

$$\mathbf{A}_F \mathbf{y} = \begin{pmatrix} 2r^{-1}(y_{3,111} - 2r^{-1}y_{1,11} - r^{-2}y_{3,1}) \\ y_{3,1111} - 2r^{-1}y_{1,111} - 2r^{-2}y_{3,11} + 2r^{-3}y_{1,1} + r^{-4}y_3 \end{pmatrix} \rightarrow \mathbf{A}_F^* \mathbf{y} = \begin{pmatrix} 0 \\ y_{3,1111} \end{pmatrix} \quad (80)$$

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$$\begin{cases} -\psi_{1,11} - r^{-1}\psi_{3,1} = \lambda^\varepsilon \psi_1 \\ r^{-1}\psi_1 + r^{-2}\psi_3 + \varepsilon^2\psi_{3,1111} = \lambda^\varepsilon \psi_3 \\ \psi_1 = \psi_3 = \psi_{3,1} = 0 \end{cases} \quad (81)$$

$$\varepsilon^2\psi_i^{(6)} + \lambda\varepsilon^2\psi_i^{(4)} - \lambda\psi_i^{(2)} - \lambda(\lambda - r^{-2})\psi_i = 0, \quad i = 1, 3 \quad (82)$$

- Introducing the corresponding characteristic equation $\varepsilon^2 X^6 + \lambda\varepsilon^2 X^4 - \lambda X^2 - \lambda(\lambda - r^{-2}) = 0$, we have to solve the third order polynomial

$$p(m) = m^3 + \lambda m^2 - \frac{\lambda}{\varepsilon^2} m - \frac{\lambda}{\varepsilon^2} (\lambda - r^{-2}), \quad m = X^2 \quad (83)$$

If $\lambda > r^{-2}$ three real roots: $R_1(\lambda) < R_2(\lambda) < 0 < R_3(\lambda)$

$$R_1(\lambda), R_2(\lambda) = \frac{-(\lambda + R_3(\lambda)) \mp \sqrt{\Delta}}{2}, \quad R_3(\lambda) = 2\rho^{1/3} \cos\left(\frac{\theta}{3}\right) - \frac{\lambda}{3} \quad (84)$$

with

$$\begin{cases} \rho = \frac{-\lambda}{\varepsilon^2} - \frac{\lambda^2}{3} < 0, & q = \frac{\lambda}{27\varepsilon^2} (2\lambda^2\varepsilon^2 - 18\lambda + 27r^{-2}) \\ \rho = \left(-\frac{\rho}{3}\right)^{3/2}, & \theta = \arccos\left(\frac{-q}{2\rho}\right), \quad \Delta = (\lambda + R_3(\lambda))^2 + \frac{4r^{-1}}{R_3(\lambda)} > 0 \end{cases} \quad (85)$$

- Moreover, with respect to ε , we have

$$\begin{aligned} R_1^\varepsilon(\lambda) &= -\frac{\sqrt{\lambda}}{\varepsilon} - \frac{r^{-2}}{2} + O(\varepsilon), & R_3^\varepsilon(\lambda) &= \frac{\sqrt{\lambda}}{\varepsilon} - \frac{r^{-2}}{2} + O(\varepsilon), \\ R_2^\varepsilon(\lambda) &= -(\lambda - r^{-2}) + O(\varepsilon^2) \end{aligned} \quad (86)$$



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- $r_1(\lambda) = \sqrt{-R_1(\lambda)}$, $r_2(\lambda) = \sqrt{-R_2(\lambda)}$ and $r_3(\lambda) = \sqrt{R_3(\lambda)}$, there exists $A_i, B_i \in \mathbb{R}$, $i = 1, \dots, 6$ so that

$$K_1 = \frac{r_1(\lambda)^2 - \lambda}{r^{-1}r_1(\lambda)}, \quad K_2 = \frac{r_2(\lambda)^2 - \lambda}{r^{-1}r_2(\lambda)}, \quad K_3 = \frac{r_3(\lambda)^2 + \lambda}{r^{-1}r_3(\lambda)} \quad (87)$$

$$\begin{aligned} \begin{pmatrix} \psi_1(\xi) \\ \psi_3(\xi) \end{pmatrix} &= A_1 \begin{pmatrix} \cos(r_1(\lambda)\xi) \\ K_1 \sin(r_1(\lambda)\xi) \end{pmatrix} + A_2 \begin{pmatrix} \sin(r_1(\lambda)\xi) \\ -K_1 \cos(r_1(\lambda)\xi) \end{pmatrix} + A_3 \begin{pmatrix} \cos(r_2(\lambda)\xi) \\ K_2 \sin(r_2(\lambda)\xi) \end{pmatrix} \\ &+ A_4 \begin{pmatrix} \sin(r_2(\lambda)\xi) \\ -K_2 \cos(r_2(\lambda)\xi) \end{pmatrix} + A_5 \begin{pmatrix} e^{r_3(\lambda)\xi} \\ -K_3 e^{r_3(\lambda)\xi} \end{pmatrix} + A_6 \begin{pmatrix} e^{-r_3(\lambda)\xi} \\ K_3 e^{-r_3(\lambda)\xi} \end{pmatrix} \end{aligned} \quad (88)$$

- Taking into account the 6 boundary conditions, we get that λ^ε solves the eigenvalue problem if

$$\det(D^\varepsilon(\lambda)) = 0 \quad (89)$$

with

$$D^\varepsilon(\lambda) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ \cos(r_1(\lambda)) & \sin(r_1(\lambda)) & \cos(r_2(\lambda)) & \sin(r_2(\lambda)) & e^{r_3(\lambda)} & e^{-r_3(\lambda)} \\ 0 & -K_1 & 0 & -K_2 & -K_3 & K_3 \\ K_1 \sin(r_1(\lambda)) & -K_1 \cos(r_1(\lambda)) & K_2 \sin(r_2(\lambda)) & -K_2 \cos(r_2(\lambda)) & -K_3 e^{r_3(\lambda)} & K_3 e^{-r_3(\lambda)} \\ K_1 r_1(\lambda) & 0 & K_2 r_2(\lambda) & 0 & -K_3 r_3(\lambda) & -K_3 r_3(\lambda) \\ K_1 r_1 \cos(r_1(\lambda)) & K_1 r_1 \sin(r_1(\lambda)) & K_2 r_2 \cos(r_2(\lambda)) & r_2 K_2 \sin(r_2(\lambda)) & -K_3 r_3 e^{r_3(\lambda)} & -K_3 r_3 e^{-r_3(\lambda)} \end{pmatrix} \quad (90)$$

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The adjoint time dependent solution may then be expanded as follows :

$$\phi(\xi, t) = \sum_{\lambda_k^\varepsilon \in \Lambda^+} \left(a_k \cos(\sqrt{\lambda_k^\varepsilon} t) + \frac{b_k}{\sqrt{\lambda_k^\varepsilon}} \sin(\sqrt{\lambda_k^\varepsilon} t) \right) \psi(\xi, \lambda_k), \quad \Lambda^+ = \{\lambda^\varepsilon > r^{-2}, \det(\mathbf{D}(\lambda^\varepsilon)) = 0\}$$
(91)

For $\lambda_k = r^{-2} + (k\pi)^2 \in \sigma(\mathbf{A}_M)$,

$$\det(\mathbf{D}^\varepsilon(\lambda_k)) = O(\sqrt{\varepsilon}) \rightarrow \lambda_k^\varepsilon = \lambda_k + O(\varepsilon^\alpha), \text{ for some } \alpha > 0$$
(92)

and

$$\sin(r_2(\lambda_k)) = O(\sqrt{\varepsilon})$$
(93)

BUT

$$\sigma(\mathbf{A}^\varepsilon) \not\rightarrow \sigma(\mathbf{A}_M) \text{ as } \varepsilon \rightarrow 0$$
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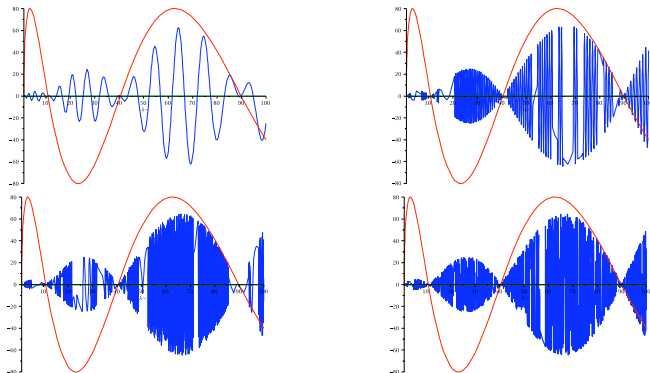


Figure: $\det(D(\lambda^\varepsilon))$ and $\sin(r_2(\lambda^\varepsilon)) - \lambda^\varepsilon > r^{-2} - \varepsilon = 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8} - r^{-1} = 1$

$$\lim_{\varepsilon \rightarrow 0} \overline{\sigma(\mathbf{A}^\varepsilon)} = \mathbb{R}^+ \quad (\text{convex hull of } \sigma_{\text{ess}}(\mathbf{A}_M)) \quad (95)$$

The roots of $\sin(r_2(\lambda^\varepsilon))$ converge to $\lambda_k = r^{-2} + (k\pi)^2 \in \sigma(\mathbf{A}_M)$. The spurious eigenvalues are related to $\sin(r_1(\lambda^\varepsilon)) \approx \sin(\frac{1}{\sqrt{\varepsilon}})$,

7

$$\{\lambda^\varepsilon \in \sigma(\mathbf{A}^\varepsilon), \lambda^\varepsilon > r^{-2}\} = \sigma^{\varepsilon,+} \cup \sigma^{\varepsilon,-} \quad (96)$$

with

$$\begin{aligned} \sigma^{\varepsilon,+} &= \{\lambda_k^\varepsilon > r^{-2}, \mathbf{A}^\varepsilon \psi_k^\varepsilon = \lambda_k^\varepsilon \psi_k^\varepsilon, \lambda_k^\varepsilon \rightarrow r^{-2} + (k\pi)^2 \in \sigma(\mathbf{A}_M), \psi_k^{\varepsilon,+} \rightarrow \mathbf{v}_k \text{ in } \mathbf{H}\}, \\ \sigma^{\varepsilon,-} &= \{\lambda_k^\varepsilon > r^{-2}, \mathbf{A}^\varepsilon \psi_k^{\varepsilon,-} = \lambda_k^\varepsilon \psi_k^{\varepsilon,-}, \lambda_k^\varepsilon \rightarrow \delta_k \notin \sigma(\mathbf{A}_M), \psi_k^{\varepsilon,-} \rightarrow \mathbf{0} \text{ in } \mathbf{H}\} \end{aligned} \quad (97)$$

In

$$\begin{aligned} \begin{pmatrix} \psi_1(\xi) \\ \psi_3(\xi) \end{pmatrix} &= A_1 \begin{pmatrix} \cos(r_1(\lambda)\xi) \\ K_1 \sin(r_1(\lambda)\xi) \end{pmatrix} + A_2 \begin{pmatrix} \sin(r_1(\lambda)\xi) \\ -K_1 \cos(r_1(\lambda)\xi) \end{pmatrix} + A_3 \begin{pmatrix} \cos(r_2(\lambda)\xi) \\ K_2 \sin(r_2(\lambda)\xi) \end{pmatrix} \\ &+ A_4 \begin{pmatrix} \sin(r_2(\lambda)\xi) \\ -K_2 \cos(r_2(\lambda)\xi) \end{pmatrix} + A_5 \begin{pmatrix} e^{r_3(\lambda)\xi} \\ -K_3 e^{r_3(\lambda)\xi} \end{pmatrix} + A_6 \begin{pmatrix} e^{-r_3(\lambda)\xi} \\ K_3 e^{-r_3(\lambda)\xi} \end{pmatrix} \end{aligned} \quad (98)$$

we take $A_4 = 1$ and solve

$$D(\lambda^\varepsilon)(A_1, A_2, A_3, 1, A_5, A_6)^T = 0 \quad (99)$$

leading to

$$A_1^\varepsilon, A_2^\varepsilon, A_3^\varepsilon, A_4^\varepsilon = O(\varepsilon^{\alpha/2}), \quad A_5^\varepsilon = O(e^{-1/\sqrt{\varepsilon}}), \quad A_6^\varepsilon = O(\varepsilon^{-\alpha/2}) \quad (100)$$

$$\phi(\xi, t) = \sum_{\lambda_k \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) \psi(\xi, \lambda_k) \quad (101)$$

We note $V^{\varepsilon,+}$ the space corresponding to the part $\sigma^{\varepsilon,+}$

$$\{\lambda^\varepsilon \in \sigma(\mathbf{A}^\varepsilon), \lambda^\varepsilon > r^{-2}\} = \sigma^{\varepsilon,+} \cup \sigma^{\varepsilon,-} \quad (96)$$

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$$A_1^\varepsilon, A_2^\varepsilon, A_3^\varepsilon, A_4^\varepsilon = O(\varepsilon^{\alpha/2}), \quad A_5^\varepsilon = O(e^{-1/\sqrt{\varepsilon}}), \quad A_6^\varepsilon = O(\varepsilon^{-\alpha/2}) \quad (100)$$

$$\phi(\xi, t) = \sum_{\lambda_k \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) \psi(\xi, \lambda_k) \quad (101)$$

We note $V^{\varepsilon,+}$ the space corresponding to the part $\sigma^{\varepsilon,+}$

$$\{\lambda^\varepsilon \in \sigma(\mathbf{A}^\varepsilon), \lambda^\varepsilon > r^{-2}\} = \sigma^{\varepsilon,+} \cup \sigma^{\varepsilon,-} \quad (96)$$

with

$$\begin{aligned} \sigma^{\varepsilon,+} &= \{\lambda_k^\varepsilon > r^{-2}, \mathbf{A}^\varepsilon \psi_k^\varepsilon = \lambda_k^\varepsilon \psi_k^\varepsilon, \lambda_k^\varepsilon \rightarrow r^{-2} + (k\pi)^2 \in \sigma(\mathbf{A}_M), \psi_k^{\varepsilon,+} \rightarrow \mathbf{v}_k \text{ in } \mathbf{H}\}, \\ \sigma^{\varepsilon,-} &= \{\lambda_k^\varepsilon > r^{-2}, \mathbf{A}^\varepsilon \psi_k^{\varepsilon,-} = \lambda_k^\varepsilon \psi_k^{\varepsilon,-}, \lambda_k^\varepsilon \rightarrow \delta_k \notin \sigma(\mathbf{A}_M), \psi_k^{\varepsilon,-} \rightarrow \mathbf{0} \text{ in } \mathbf{H}\} \end{aligned} \quad (97)$$

In

$$\begin{aligned} \begin{pmatrix} \psi_1(\xi) \\ \psi_3(\xi) \end{pmatrix} &= A_1 \begin{pmatrix} \cos(r_1(\lambda)\xi) \\ K_1 \sin(r_1(\lambda)\xi) \end{pmatrix} + A_2 \begin{pmatrix} \sin(r_1(\lambda)\xi) \\ -K_1 \cos(r_1(\lambda)\xi) \end{pmatrix} + A_3 \begin{pmatrix} \cos(r_2(\lambda)\xi) \\ K_2 \sin(r_2(\lambda)\xi) \end{pmatrix} \\ &+ A_4 \begin{pmatrix} \sin(r_2(\lambda)\xi) \\ -K_2 \cos(r_2(\lambda)\xi) \end{pmatrix} + A_5 \begin{pmatrix} e^{r_3(\lambda)\xi} \\ -K_3 e^{r_3(\lambda)\xi} \end{pmatrix} + A_6 \begin{pmatrix} e^{-r_3(\lambda)\xi} \\ K_3 e^{-r_3(\lambda)\xi} \end{pmatrix} \end{aligned} \quad (98)$$

we take $A_4 = 1$ and solve

$$D(\lambda^\varepsilon)(A_1, A_2, A_3, 1, A_5, A_6)^T = 0 \quad (99)$$

leading to

$$A_1^\varepsilon, A_2^\varepsilon, A_3^\varepsilon, A_4^\varepsilon = O(\varepsilon^{\alpha/2}), \quad A_5^\varepsilon = O(e^{-1/\sqrt{\varepsilon}}), \quad A_6^\varepsilon = O(\varepsilon^{-\alpha/2}) \quad (100)$$

$$\phi(\xi, t) = \sum_{\lambda_k \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right) \psi(\xi, \lambda_k) \quad (101)$$

We note $V^{\varepsilon,+}$ the space corresponding to the part $\sigma^{\varepsilon,+}$

$$\phi_{1,1}(0, t) = \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k^\varepsilon} t) + \frac{b_k}{\sqrt{\lambda_k^\varepsilon}} \sin(\sqrt{\lambda_k^\varepsilon} t) \right) [r_{1,k}^\varepsilon A_{2,k}^\varepsilon + r_{2,k}^\varepsilon + A_{5,k}^\varepsilon r_{3,k}^\varepsilon - A_{6,k}^\varepsilon r_{3,k}^\varepsilon] \quad (102)$$

$$\phi_{3,11}(0, t) = \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} \left(a_k \cos(\sqrt{\lambda_k^\varepsilon} t) + \frac{b_k}{\sqrt{\lambda_k^\varepsilon}} \sin(\sqrt{\lambda_k^\varepsilon} t) \right) [K_1 (r_{1,k}^\varepsilon)^2 A_{2,k}^\varepsilon + (r_{2,k}^\varepsilon)^2 K_2 + A_{5,k}^\varepsilon K_3 (r_{3,k}^\varepsilon)^2 + A_{6,k}^\varepsilon K_3 (r_{3,k}^\varepsilon)^2] \quad (103)$$

Applying Ingham on $\phi_{1,1}(0, t)$ and on $\phi_{3,11}(0, t)$ plus tedious computations, we get that

Proposition

For any $\varepsilon > 0$, there exist a T^{***} and $C_1 > 0$ independent of ε , such that for any $T \geq T^{***} (r^{-1})$, the following inequality holds :

$$C_1 E^\varepsilon(0, \phi) \leq \int_0^T (\phi_{1,1}^2 + \varepsilon^2 \phi_{3,11}^2)(0, t) dt, \quad (104)$$

where ϕ is the solution of the adjoint system with initial condition

$$\begin{aligned} \phi(\xi, 0) &= \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} a_k \psi(\xi, \lambda_k^\varepsilon) \in H_0^1(\omega) \times H^2(\omega), \\ \phi'(\xi, 0) &= \sum_{\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}} b_k \psi(\xi, \lambda_k^\varepsilon) \in L^2(\omega) \times L^2(\omega) \end{aligned} \quad (105)$$

- A priori, IMPOSSIBLE, because we don't know explicitly $V^{\varepsilon,+}$
- A way is to compute the first N $\lambda_k^\varepsilon \in \sigma^{\varepsilon,+}$ and corresponding ψ_k^ε (duable but not straightforward due to the densification)
- However, the approximate controllability is very likely possible because the spurious modes converge to zero with ε .

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$\omega = (0, 1)^2$, $(\xi_1, \xi_2) \in \omega$ - Two controls $y_\alpha = v_\alpha$ for the three components $\mathbf{y} = (y_1, y_2, y_3)$.

$$\mathbf{A}_M \mathbf{y} = \begin{pmatrix} -a^{1111}(y_{1,11} + r^{-1}y_{3,1}) - a^{1212}y_{1,22} - (a^{1212} + a^{1122})y_{2,21} \\ -a^{1212}y_{2,11} - a^{2222}y_{2,22} - (a^{2211} + a^{1212})y_{1,12} - a^{2211}r^{-1}y_{3,2} \\ r^{-1}(a^{1111}(y_{1,1} + r^{-1}y_3) + a^{2211}y_{2,2}) \end{pmatrix}; \quad \mathbf{A}_M = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & a^{1111}r^{-2}I \end{pmatrix}. \quad (106)$$

$$a^{1111} = a^{2222} = \frac{8\mu(\lambda + \mu)}{\lambda + 2\mu}; \quad a^{1122} = a^{2211} = \frac{4\lambda\mu}{\lambda + 2\mu}, \quad a^{1212} = 2\mu \quad (107)$$

- $\text{Ker} \mathbf{A}_M = \{\phi \in \mathbf{V} = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \phi_{1,1} + r^{-1}\phi_3 = 0, \phi_{2,2} = 0, \phi_{1,2} + \phi_{2,1} = 0\} = \{\mathbf{0}\}$
- $\sigma_{\text{ess}}(\mathbf{A}_M) = [0, \frac{2(3\lambda+2\mu)\mu}{\lambda+\mu}r^{-2}] \neq \emptyset \implies$ Non uniform controllability in $(L^2(\omega))^3 \times V'$
- $a^{1111}r^{-2} \in \sigma(\mathbf{A}_M) \setminus \sigma_{\text{ess}}(\mathbf{A}_M)$ corresponding to $(0, 0, 1)$
- $\lambda \in \sigma(\mathbf{A}_M)$ s.t. $\lambda \neq a^{1111}r^{-2}$ fulfills

$$\left(\mathbf{A} + \frac{r^{-2}}{\lambda - a^{1111}r^{-2}} \mathbf{B}\mathbf{B}^* \right) (\phi_1, \phi_2) = \lambda(\phi_1, \phi_2), \quad (\phi_1, \phi_2) \in H_0^1(\omega) \times H_0^1(\omega) \quad (108)$$

What about the 2-D case ? $\varepsilon = 0$ - Cylindrical shell

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- $\sigma_{\text{ess}}(\mathbf{A}_M) = [0, \frac{2(3\lambda+2\mu)\mu}{\lambda+\mu}r^{-2}] \neq \emptyset \implies$ Non uniform controllability in $(L^2(\omega))^3 \times \mathbf{V}'$
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Work in progress by Karine Mauffrey (PhD, Besancon).

$$\Omega = (0, 1)^2$$

$$\begin{cases} u'' = \Delta u + \alpha v_x, & \Omega \times (0, T), \\ v'' = -\alpha u_x - av, & \Omega \times (0, T), \\ u = f, & \partial\Omega \times (0, T), \\ + \text{Initial conditions} \end{cases} \quad (109)$$

$$\lambda_{p,q}^{\pm} = \frac{1}{2} \left((p^2 + q^2)\pi^2 + a \pm \sqrt{((p^2 + q^2)\pi^2 - a)^2 + 4\alpha^2 p^2 \pi^2} \right) \quad (110)$$

$$e_{p,q}^{\pm}(x, y) = \left(\frac{2(\lambda_{p,q}^{\pm} - a)}{\sqrt{(\lambda_{p,q}^{\pm} - a)^2 + \alpha^2 p^2 \pi^2}} \sin(p\pi x) \sin(q\pi y), \frac{2\alpha p\pi}{\sqrt{(\lambda_{p,q}^{\pm} - a)^2 + \alpha^2 p^2 \pi^2}} \cos(p\pi x) \sin(q\pi y) \right) \quad (111)$$

$$\sigma_{\text{ess}} = [a - \alpha^2, a] \quad (112)$$

Thank you for your attention

Once again, many thanks to Florence, Franck, Jérôme, Michel and Patricia.