Transmission Eigenvalues in Scattering Theory

by

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JOINT STUDY WITH

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1. GENERAL SCATTERING THEORY A'LA L. HÖRMANDER

$$P_0(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \;;$$

$$D_j = -i \frac{\partial}{\partial x_j} \quad , \quad a_\alpha \in \mathbb{R}$$

P_0 is *simply characteristic*, if

$$(1.1) \qquad \sum_\alpha \left| P_0^{(\alpha)}(\xi) \right| \leq C \left(\sum_{|\alpha| \leq 1} |P_0^{(\alpha)}(\xi)| + 1 \right)$$

Remark i) Elliptic operators satisfy (1.1)

ii) Second order operators satisfy (1.1).

We assume all the time P_0 is simply characteristic. 2.

Then the resolvent

$$R_0(z) = (P_0(D) - zI)^{-1} \text{ exist}$$

for $z \in \mathbb{C}^{\pm} \setminus Z(P_0)$,

where $Z(P_0)$ is finite set of critical values i.e.

$$\lambda \in Z(P_0) \Leftrightarrow \exists\ \xi \in \mathbb{R}^n \text{ s.t.}$$

$$P_0(\xi) = \lambda \text{ and } \nabla P_0(\xi) = 0 .$$

Especially

$$R_0(\lambda \pm i0) \quad , \text{ for } \lambda \notin Z(P_0),$$

λ real.

$$R_0(\lambda \pm i0) : B \longrightarrow B^*$$

where B and B^* are dual Banach spaces satisfying

$$L^2_\delta \subset B \subset L^2 \subset B^* \subset L^2_{-\delta} \quad , \; \delta > 1/2$$

We define u is outgoing (incoming) if

$$u = R_0(\lambda \underset{(-)}{+} i0) f \quad , \; f \in B$$

2. SHORT RANGE POTENTIAL SCATTERING

We define first the Sobolev version of B^*:

$$B^*_{P_0} = \{ u \in B^* \mid P_0^{(\alpha)}(D)u \in B^*, \forall \alpha \}$$

$V(x,D)$ is short range w.r.t. P_0, if

$$V : B^*_{P_0} \longrightarrow B \quad \text{compactly.}$$

Intuitively: V is lower order than P_0 and coeff. $\to 0$ faster than $\frac{1}{|x|}$ (at least in elliptic case)

It holds

$$R_0(\lambda \pm i0) : B \longrightarrow B^*_{P_0}$$

is bounded.

Consider generalized Schrödinger equation

$$(2.1) \qquad (P_0(D) + V - \lambda)u = 0$$

<u>Ex</u> $\qquad P_0(D) = -\Delta$ and

$$V(x, D) = a(x)\cdot\nabla + b(x)$$

$$|a(x)| + |b(x)| \le \frac{C}{(1+|x|)^{1+\varepsilon}} .$$

Note

$$H^2(\mathbb{R}^n) \subset B^*_{P_0} \subset H^2_{-\delta}(\mathbb{R}^n),$$

for $P_0(D) = -\Delta$.

From now on assume V is both short range and symmetric.

2.1 Theorem If $u \in B^*_{P_0}$ sat. (2.1), then

$$u = u^\pm + R_0(\lambda \mp i0) V u$$

where $\hat{u}^\pm = v^\pm \delta(P_0 - \lambda) = v^\pm \frac{dS}{|P_0'|}$

and $v^\pm \in L^2(M_\lambda)$.

Here $M_\lambda = \{ \xi \mid P_0(\xi) = \lambda \}$ and dS its Lebesgue measure.

2.2. Theorem By denoting $d\sigma = \frac{dS}{|P_0'|}$ we have

$$\|v_+\|_{L^2(M_\lambda, d\sigma)} = \|v_-\|_{L^2(M_\lambda, d\sigma)}$$

We call v_- the incoming wave and v_+ the outgoing wave.

2.3 Definition The unitary map

$$\Sigma_\lambda : L^2(M_\lambda) \longrightarrow L^2(M_\lambda)$$

$$\Sigma_\lambda : v_- \longmapsto v_+$$

is called the scattering matrix and

$$A_\lambda := I - \Sigma_\lambda$$

the scattering amplitude.

3. BOUND STATES AND TRANSMISSION EIGENVALUES.

It holds that

$$P_0(D) \quad \text{and} \quad P_0(D)+V$$

are *essentially self-adjoint*.

Denote

$$H = \overline{P_0(D)+V} \quad \text{and} \quad H_0 = \overline{P_0(D)}.$$

Remark Let $W^{\pm}$ be the *wave operators*

$$W^{\pm} = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$$

and S the *scattering operator*

$$S = W_+^{-1} W_-$$

Since S commutes with H_0 its Fourier equivalent

$$\hat{S} = FSF^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

commutes with multiplication by $P_0(\xi)$.

The relation between S and Σ_λ is given by

$$\hat{S} f|_{M_\lambda} = \Sigma_\lambda (f|_{M_\lambda})$$

for $f \in L^2(\mathbb{R}^n)$.

3.1 Definition (i) $\lambda \in \sigma_p(H)$ is called a bound state if $\lambda \in \sigma_p(H)$, i.e. $\exists\, u \in L^2(\mathbb{R}^n)$ s.t.

$$Hu = \lambda u$$

(ii) $\lambda \in \sigma_{abs}(H)$ is called a transmission eigenvalue, if $\exists$ incoming wave $v_- \neq 0$ s.t.

$$A_\lambda v_- = 0 \iff$$

$$v_- = v_+$$

Remark

λ is a bound state $\Longleftrightarrow$ the total wave u is both incoming and outgoing.

λ is a TE $\Longleftrightarrow$ the scattered wave u_s is both incoming and outgoing.

Here

$$u_s := R_0(\lambda + i0)Vu.$$

4. CONNECTION TO THE INTERIOR TRANSMISSION PROBLEM

For simplicity (and this is all we can do) we assume

$$P_0(\xi) = \xi^2 \quad \text{and} \quad V = V(x),$$

where $V \in L^\infty \cap \mathcal{E}'$. Then

$$\sigma_{abs}(H) = (0, \infty).$$

We denote $\lambda = k^2$ and since $|P_0'(\xi)| = 2|\xi|$ we get

$$u(x,k) = \int_{|\xi| = k} e^{ix\cdot\xi} \, v_\pm \, \frac{dS}{2k} + R_0(k^2 \pm i0) V u$$

Especially, if $v_+ = v_-$

$u_s(x) = R_0(k^2+i0)Vu = R_0(k^2-i0)Vu$

is both incoming and outgoing.

Since $(\Delta + k^2)u_s = 0$ outside the support of V, we get by Rellich

$u_s(x,k) = 0$ in $\mathbb{R}^n \setminus \operatorname{supp} V$

4.1. Proposition $k>0$ is a TE, iff the interior transmission problem

$$(\mathrm{ITT})\quad \begin{cases} (\Delta+k^2)v=0 \\ (\Delta+k^2+V)w=0 \end{cases} \text{ in } D$$

$$\text{Cauchy } w = \text{Cauchy } v \text{ on } \partial D$$

D is a domain containing supp V, has a non-trivial solution $(v,w)\neq 0$.

__Proof__ If λ is TE, take

$$v = u_+|_D \qquad \text{and}$$

$$w = u|_D$$

Now $u_s = u - u_+$ vanish outside supp V $\boxed{}$

Question 1 : Is the set Λ of TE's discrete?

Question 2 : Is Λ non-empty?

We can answer these questions only in two special cases!

5. SPHERICALLY SYMMETRIC CASE

$V(x) \equiv 0 \implies$ no waves are scattered $\implies \Lambda = \mathbb{R}_+$

To exclude this assume

$$V(x) = V(r) > 0$$

and that

$$n(r) := 1 + \frac{V(r)}{k^2}$$

satisfies

$$n(r) > 0$$
$$n(r) = 1 \quad , \text{ for } r > a$$

(3.1) $$\frac{1}{a}\int_0^a [n(r)]^{1/2}\, dr \neq 1$$

5.1. Theorem (Colton – P. – Sylvester, 2006)
Assume $V \in C^2$ and (3.1) holds.
Then Λ is infinite and discrete.

Idea of Proof: Writing

$$v(r) = a_0 j_0(kr)$$

$$w(r) = b_0 \frac{y(r)}{r}$$

one gets

$$y'' + k^2 n(r) y = 0$$

$$y(0) = 0, \quad y'(0) = 1$$

Boundary conditions $\Rightarrow$

$$d := \det \begin{pmatrix} \frac{y(a)}{a} & -j_0(ka) \\ \frac{d}{dr} \frac{y(r)}{r}\Big|_{r=a} & -k j_0'(ka) \end{pmatrix} = 0$$

if k is TE.

Liouville - transform and Volterra - estimate $\Rightarrow$ 18.

$$d = \frac{1}{ak'} \underbrace{\left[B \sin k'\delta \cos k' - C \cos k'\delta \sin k' \right]}_{=: f(k')} + O\left(\frac{1}{k'^2}\right),$$

where

$$B = \left(n(0)n(a)\right)^{-1/4}, \quad C = \left(\frac{n(a)}{n(0)}\right)^{1/4}$$

and

$$\delta = \frac{1}{a} \int_0^a [n(\rho)]^{1/2} \, d\rho .$$

Now f takes both positive and negative value and is almost periodic $\Rightarrow$ claim

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6. THE CASE $V(x)$ IS POSITIVE

Assumptions

$V \in L^\infty(\mathbb{R}^n)$

$V(x) > c_0 > 0$ in D

$V(x) = 0$ outside D

6.1. Proposition The following are equivalent

(i) $\exists\ (v,w) \neq 0$ solving ITP

(ii) $\exists\ u \in H^2_0(D)$ solving

$$(4.1)\quad \left(P(x,D)-\lambda\right)\frac{1}{v}\left(P_0(D)-\lambda\right)u=0 \quad \text{in } \Omega$$

Proof Note first $u \in H^2_0(D)$ means

$$(4.2)\qquad u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

(i) $\Rightarrow$ (ii) : Assume (v,w) satisfies

$$\left(P_0(D)-\lambda\right)v = 0 \quad , \left(P(x,D)-\lambda\right)w = 0$$

$$v = w \ , \ \frac{\partial}{\partial\nu}v = \frac{\partial}{\partial\nu}w \qquad \text{on } \partial D$$

Define

$$u = v - w$$

21.

Then $(P_0(D)-\lambda)u = -(P_0(D)-\lambda)w = -Vw$

$\Rightarrow (P(x,D)-\lambda)\frac{1}{V}(P_0(D)-\lambda)u$

$= -(P(x,D)-\lambda)w = 0$

(ii) $\Rightarrow$ (i):

$$(P(x,D)-\lambda)\underbrace{\frac{1}{V}(P_0(D)-\lambda)u}_{=:w} = 0 \iff$$

$$(P_0(D)-\lambda)\underbrace{\frac{1}{V}(P(x,D)-\lambda)u}_{=:v} = 0$$

▨

6.1 Theorem (Colton - P. - Kirsch -89, Rynne - Sleeman, -91) If $V > c_0 > 0$ then Λ is discrete.

6.2 Theorem (Colton - P. - Sylvester, -07)

$$\Lambda \subset \{\lambda \mid \lambda > \lambda_0(D) - \|V\|_\infty\}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

Denote $m = \frac{V}{k^2}$ and $\mu_p(D)$ the p'th Dirichlet eigenvalue of Δ^2.

6.3 Theorem (P. – Sylvester, –07)

If

$$\left\|\frac{1}{m}\right\|_\infty \geq 4\frac{\mu_p^{1/2}}{\lambda_0} + \frac{\mu_p}{\lambda_0^2}$$

then $\exists$ $p+1$ TE's λ with

$$\lambda \leq \mu_0 \left(\frac{1 + 2\left\|\frac{1}{m}\right\|_\infty}{1 + \frac{2}{\|m\|_\infty}} \right)^2$$

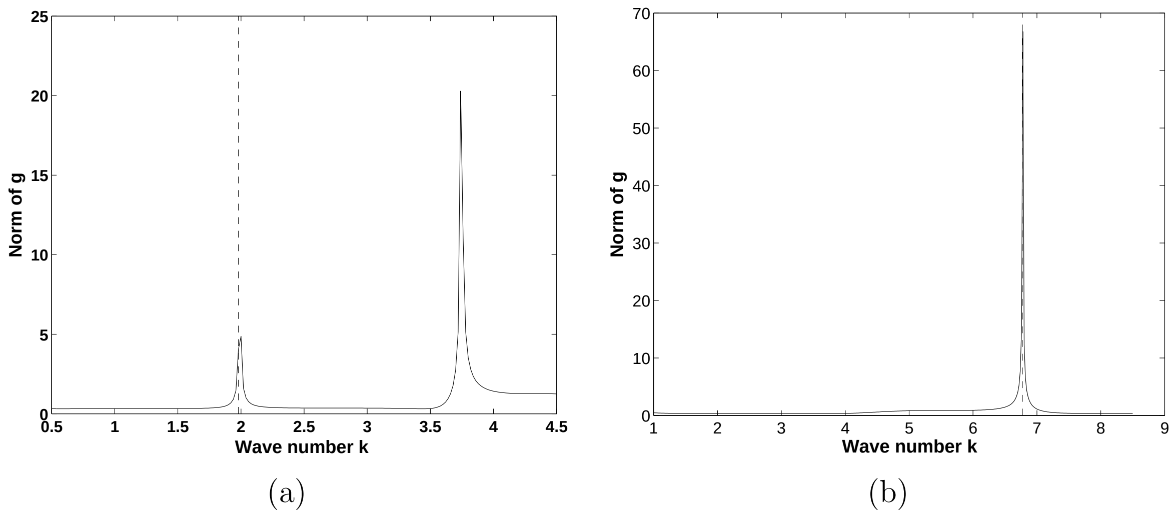

Figure 1. The first eigenvalue can be detected from the far field pattern. In the left panel we show a graph of $\|g\|_{L^2(\Omega)}$ against k for the circle with $n = 16$ using far field data computed using the finite element method. The left most peak is a good candidate for the lowest transmission eigenvalue and is confirmed using the exact value determined by the determinant criterion (57) and marked as a dashed line. The right panel shows the same result for $n = 4$ where the lowest transmission eigenvalue has increased markedly.

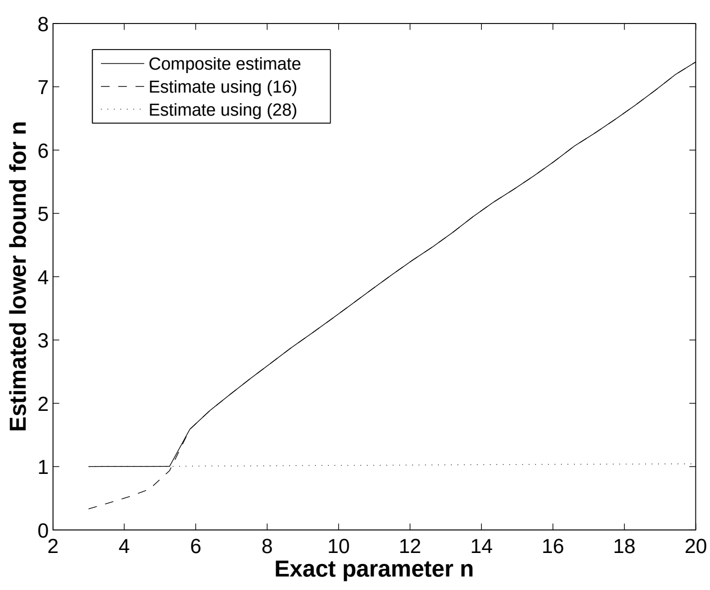

Figure 2. Use of the two estimates of this paper to provide a lower bound for n in the case of a circle. Here the exact value of n is varied from $n = 3$ to $n = 20$. For each n, the lowest transmission eigenvalue is computed from (57) and the two estimates (16) and (28) evaluated. The composite curve is the maximum of the two estimates assuming $n > 1$. Clearly for low n, neither estimate works well whereas for larger values of n, estimate (16) gives an increasing lower bound that underestimates the true value of n by approximately 1/3.

Lemma If $\lambda_0 = \lambda_0(D)$ and $\mu_0 = \mu_0(D)$ are the first Dirichlet eigenvalues of $-\Delta = D^2$ and Δ^2, then

(i) $$\lambda_0 = \inf_{u \in H^1_0} \frac{\int |\nabla u|^2}{\|u\|^2}$$ and

(ii) $$\mu_0 = \inf_{u \in H^2_0} \frac{\int |\Delta u|^2}{\|u\|^2} \geq \lambda_0^2$$

Theorem i) If $m>0$ is constant, then

$$\tau \le \frac{\lambda_0(D)}{m+1} \implies \tau \text{ is not TE}$$

ii) If $\dfrac{\left(1+\frac{m}{2}\right)^2}{1+m} \ge \dfrac{\mu_p}{\lambda_0^2} \ge 1$

then $\exists$ $p+1$ TE's τ with

$$\tau \le \left(\frac{m+2}{m+1}\right)\lambda_0$$

Proof If $f_\tau(u) = (u, T_\tau u)$ and $u \in H_0^2(D)$ and $\|u\| = 1$

$$m f_\tau(u) = \|(\Delta + \tau(1+m))u\|^2$$

$$- m\tau \int \bar{u}(\Delta + \tau(1+m))u$$

$$= (m+1)\tau^2 - 2\left(1+\frac{m}{2}\right)\|\nabla u\|^2 \tau + \|\Delta u\|^2$$

$$\le (m+1)\tau^2 - 2\left(1+\frac{m}{2}\right)\lambda_0^2 + \|\Delta u\|^2$$

By choosing $\tau = \dfrac{\left(1+\frac{m}{2}\right)}{1+m}$ we get

$$m\, t_\tau(u) \le -\frac{\left(1+\frac{m}{2}\right)}{1+m} + \|\Delta u\|^2$$

Now if $\Delta^2 u = \mu_k u$, $\mu_k \le \mu_p$

we have

$$\|\Delta u\|^2 \le \mu_k \le \mu_p$$

$\Rightarrow$ $$m\, t_\tau(u) < 0$$

But $m\, t_0 > 0$.

$\Rightarrow$ $p+1$ lowest eigenvalues of T_τ must be negative.

Each of them go through 0 as τ^* decreases from τ to 0

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Conclusions:

We have shown

1. If $m > 0$ is large enough then $\exists$ TE's, with upper and lower bound for their location

2. The upper and lower bounds depend only on lowest eigenvalues of Dirichlet Δ and Δ^2. Dependence of D for these bounds are through $\lambda_0(D)$ and $\mu_0(D)$.

NOTES

- The existence and discreteness of TE's was first shown by Colton and Monk 1988 in the case V is spherically symmetric and positive
- In 2007-paper with Sylvester and Colton we removed the positivity assumption
- McLaughlin and Polyakova 1994 showed that TE's determine V in spherical case.

- The 1988 paper with Colton and Kirsch showed the discreteness in case of positive V
- 1989 the Rynne and Sleeman gave a simplified proof of this
- The existence was shown with Sylvester 2007
- This was extended to Maxwell case by Kirsch 2008.
- In the Born approximation there are NO TE's.

Open Problem 1. Is Lemma 4.1 valid in general case i.e. does the equivalence of existence of a TE and a non-trivial solution of ITP hold for general P_0 and compactly supported V?

Open Problem 2. Assume V can change sign. Is

$$\Lambda \neq \emptyset \; ?$$

Is Λ discrete?

Open Problem 3. Assume $P_0 = -\Delta$, V positive, but does not have compact support.

Is $\Lambda \neq \emptyset$?

Is Λ discrete ?