Carleman Estimates for Nonhomogeneous Parabolic Problems

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Stokes system

The main goal is to obtain in a more direct way a global Carleman estmate for the Stokes system

$$
\begin{aligned}
&\frac{\partial y}{\partial t} - \Delta y + \nabla p = f \text{ in } \Omega \times (0, T),\\
&\operatorname{div} y = 0 \text{ in } \Omega \times (0, T),\\
&y = 0 \text{ on } \Gamma \times (0, T),\\
&y(0) = y_0.
\end{aligned}
$$

Nonhomogeneous elliptic problems

Previous work (Imanuvilov-Puel, IMRN 2003) on nonhomogeneous elliptic problems.

$$-\Delta y = f_0 + \sum_{i=0}^{N} \frac{\partial f_i}{\partial x_i} \text{ in } \Omega,$$
$$y = g \text{ on } \Gamma,$$

with $f_0, f_i \in L^2(\Omega)$, and $g \in H^{\frac{1}{2}}(\Gamma)$.
We take ω a non empty open subset of Ω and weights ψ and β such that

- $\psi \in C^2(\bar{\Omega})$, $\psi > 0$ in Ω, $\psi = 0$ on Γ.
- $|\nabla \psi| \geq c_0 > 0$ in $\Omega \setminus \bar{\omega}$.
- $\beta = e^{\lambda \psi}$.

We obtain the following estimate

Theorem

There exists $s_0 > 0$, $\lambda_0 > 0$ *and a constant* C *such that for every* $s \geq s_0$ *for every* $\lambda \geq \lambda_0$

$$\int_\Omega e^{2s\beta}|\nabla y|^2 dx + s^2\lambda^2 \int_\Omega \beta^2 e^{2s\beta}|y|^2 dx \leq$$

$$C(s^{\frac{1}{2}} e^{2s}||g||^2_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{s\lambda^2}\int_\Omega \frac{e^{2s\beta}}{\beta}|f_0|^2 dx + \sum_{i=1}^{N} s\int_\Omega \beta e^{2s\beta}|f_i|^2 dx$$

$$+s^2\lambda^2 \int_\omega \beta^2 e^{2s\beta}|y|^2 dx).$$

Result for parabolic equations

Now we consider a general parabolic equation

$$Ly = \frac{\partial y}{\partial t} - \Delta y + (\text{l.o.t.}) = f_0 + \sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} \text{ in } Q = \Omega \times (0, T),$$

$$y = g \text{ on } \Gamma \times (0, T),$$

$$y_0 = y_0$$

where $f_0, f_j \in L^2(\Omega \times (0, T))$ and
$g \in H^{\frac{1}{4},\frac{1}{2}}(\Gamma \times (0, T)) = H^{\frac{1}{4},\frac{1}{2}}(\Sigma)$. ($y_0$ is not important).
In order to give the result we need to define the correct weights. If $l(t) = t(T - t)$ we define for $k \geq 2$

$$\varphi(x, t) = \frac{e^{\lambda\psi(x)}}{l^k(t)}$$

and

$$\alpha(x, t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda|\psi|_\infty}}{l^k(t)}.$$

Theorem

There exists $\hat{\lambda}$ such that for every $\lambda \geq \hat{\lambda}$, there exists $C > 0$ and there exists $s_0(\lambda)$ such that for every $s \geq s_0(\lambda)$, if $y \in L^2(0,T;H^1(\Omega))$ with $\frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$ is solution of the parabolic problem, we have

$$\frac{1}{s}\int_Q \frac{1}{\varphi} e^{2s\alpha}|\nabla y|^2 dxdt + s\int_Q \varphi e^{2s\alpha}|y|^2 dxdt \leq$$

$$C(s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}} e^{s\alpha} g||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}|\varphi^{-\frac{1}{4}+\frac{1}{k}} e^{s\alpha} g|^2_{L^2(\Sigma)}$$

$$+\frac{1}{s^2}\int_Q \frac{e^{2s\alpha}}{\varphi^2}|f_0|^2 dxdt + \sum_{j=1}^{N}\int_Q e^{2s\alpha}|f_j|^2 dxdt$$

$$+s\int_{Q_\omega} \varphi e^{2s\alpha}|y|^2 dxdt),$$

where $Q_\omega = \omega \times (0,T)$.

Stokes system written in vorticity velocity

Usual functional spaces for the Stokes system

$$H = \{v \in (L^2(\Omega))^N, \operatorname{div} v = 0, (v.\nu)_{/\Gamma} = 0\}$$

$$V = \{v \in (H_0^1(\Omega))^N, \operatorname{div} v = 0\}.$$

For $f \in L^2(0, T; (L^2(\Omega))^N)$ and $y_0 \in V$ let y be the solution of the Stokes system

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \Delta y + \nabla p = f \text{ in } \Omega \times (0, T), \\
&\operatorname{div} y = 0 \text{ in } \Omega \times (0, T), \\
&y = 0 \text{ on } \Gamma \times (0, T), \\
&y(0) = y_0.
\end{aligned}$$

Then $y \in L^2(0, T; (H^2(\Omega))^N \cap V$, $\frac{\partial y}{\partial t} \in L^2(0, T; (L^2(\Omega))^N)$ and $p \in L^2(0, T; H^1(\Omega))$.

Defining $w = \mathrm{curl}\, y$, we have as $\mathrm{div}\, y = 0$

$$\frac{\partial w}{\partial t} - \Delta w = \mathrm{curl}\, f \text{ in } \Omega \times (0,T),$$
$$\Delta y(t) = \mathrm{curl}\, w(t) \text{ in } \Omega, \text{ a.e. in } t \in (0,T),$$
$$y(t) = 0 \text{ on } \Gamma, \text{ a.e. in } t \in (0,T).$$

Apply successively the parabolic estimate for w with parameter s, then the elliptic estimate for y with parameter $\frac{s}{l^k(t)}$. We notice that

$$\beta = e^{\lambda\psi},\ \varphi = \frac{\beta}{l^k(t)},$$

$$\alpha = \frac{\beta - e^{2\lambda|\psi|_{L^\infty}}}{l^k(t)} = \frac{\beta}{l^k(t)} - C(t).$$

Parabolic estimate :

$$\frac{1}{s}\int_Q \frac{e^{2s\alpha}}{\varphi}|\nabla w|^2 dxdt + s\int_Q \varphi e^{2s\alpha}|w|^2 dxdt \leq$$
$$C(s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}}e^{s\alpha}w||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}|\varphi^{-\frac{1}{4}+\frac{1}{k}}e^{s\alpha}w|^2_{L^2(\Sigma)}$$
$$+\int_Q e^{2s\alpha}|f|^2 dxdt + s\int_{Q_\omega}\varphi e^{2s\alpha}|w|^2 dxdt).$$

On the boundary, as $y = 0$ we have

$$|w| \sim |\frac{\partial y}{\partial \nu}|.$$

Elliptic estimate : for a.e. $t \in (0, T)$,

$$\frac{s^2}{l^{2k}(t)}\lambda^2 \int_\Omega \beta^2 e^{2s\alpha}|y|^2 dx \leq$$

$$C(\frac{s}{l^k(t)} \int_\Omega \beta e^{2s\alpha}|w|^2 dx + \frac{s^2}{l^{2k}(t)}\lambda^2 \int_\omega \varphi^2 e^{2s\alpha}|y|^2 dx.$$

Now we fix λ and integrate in time to obtain

$$s^2 \int_Q \varphi^2 e^{2s\alpha}|y|^2 dxdt \leq$$

$$C(s \int_Q \varphi e^{2s\alpha}|w|^2 dxdt + s^2 \int_{Q_\omega} \varphi^2 e^{2s\alpha}|y|^2 dxdt).$$

Combining the estimates on w and on y we obtain

$$\frac{1}{s}\int_Q \frac{e^{2s\alpha}}{\varphi}|\nabla w|^2 dxdt + s\int_Q \varphi e^{2s\alpha}|w|^2 dxdt + s^2\int_Q \varphi^2 e^{2s\alpha}|y|^2 dxdt$$
$$\leq C(s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}}e^{s\alpha}\frac{\partial y}{\partial \nu}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}|\varphi^{-\frac{1}{4}+\frac{1}{k}}e^{s\alpha}\frac{\partial y}{\partial \nu}|^2_{L^2(\Sigma)}$$
$$+\int_Q e^{2s\alpha}|f|^2 dxdt + s\int_{Q_\omega} \varphi e^{2s\alpha}|w|^2 dxdt + s^2\int_{Q_\omega} \varphi^2 e^{2s\alpha}|y|^2 dxdt).$$

We need to estimate the first two terms in the right hand side by a regularity argument.

$$\hat{\alpha}(t) = \alpha(t)_{/\Gamma} = \frac{1 - e^{2\lambda|\psi|_\infty}}{l^k(t)}.$$

Define

$$(u,q) = l(t)(ye^{s\hat{\alpha}(t)}, pe^{s\hat{\alpha}(t)}).$$

We have

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u + \nabla q = l(t)e^{s\hat{\alpha}(t)} f + s\hat{\alpha}'(t)u + l'(t)e^{s\hat{\alpha}(t)} y \text{ in } Q,\\
&\operatorname{div} u = 0 \text{ in } Q,\\
&u = 0 \text{ on } \Sigma,\\
&u(0) = 0 \text{ in } \Omega.
\end{aligned}$$

By regularity on Stokes equation we have

$$||u||^2_{H^{1,2}(Q)} \leq C(|l(t)e^{s\hat{\alpha}(t)} f|^2_{L^2(Q)} + s^2 |\frac{l'(t)}{l^{k+1}(t)} u|^2_{L^2(Q)} + |l'(t)e^{s\hat{\alpha}(t)} y|^2_{L^2(Q)}).$$

As $l'(t)$ is bounded, $\varphi \sim \frac{1}{l^k(t)}$ and $\hat{\alpha} \leq \alpha$, we obtain

$$||u||^2_{H^{1,2}(Q)} \leq C(|e^{s\alpha} f|^2_{L^2(Q)} + s^2 |\varphi e^{s\alpha} y|^2_{L^2(Q)}).$$

On the other we can show that

$$||\frac{\partial u}{\partial \nu}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \leq C||u||^2_{H^{1,2}(Q)}$$

so that, as

$$\frac{\partial u}{\partial \nu} = l(t) e^{s\hat{\alpha}(t)} \frac{\partial y}{\partial \nu} \sim \varphi^{-\frac{1}{k}} e^{s\hat{\alpha}} \frac{\partial y}{\partial \nu},$$

we have

$$s^{-\frac{1}{2}} ||\varphi^{-\frac{1}{k}} e^{s\hat{\alpha}} \frac{\partial y}{\partial \nu}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \leq C(s^{-\frac{1}{2}} |e^{s\alpha} f|^2_{L^2(Q)} + s^{\frac{3}{2}} |\varphi e^{s\alpha} y|^2_{L^2(Q)}).$$

Now we have, taking

$$k = 8$$

$$s^{-\frac{1}{2}}||\varphi^{-\frac{1}{4}}e^{s\alpha}\frac{\partial y}{\partial \nu}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}|\varphi^{-\frac{1}{4}+\frac{1}{k}}e^{s\alpha}\frac{\partial y}{\partial \nu}|^2_{L^2(\Sigma)} \leq$$

$$Cs^{-\frac{1}{2}}||\varphi^{-\frac{1}{8}}e^{s\hat{\alpha}}\frac{\partial y}{\partial \nu}||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \leq$$

$$C(s^{-\frac{1}{2}}|e^{s\alpha}f|^2_{L^2(Q)} + s^{\frac{3}{2}}|\varphi e^{s\alpha}y|^2_{L^2(Q)}).$$

Result for the Stokes system

Combining all these estimates, we obtain a Carleman estimate for the Stokes system.

Theorem

Let T be positive and ω be a non empty open subset of Ω. There exists $s_0 > 0$ and a constant $C > 0$ such that if y is the solution of the Stokes system with right hand side $f \in L^2(0,T;L^2(\Omega))$ and if we write $w = \mathrm{curl}\, y$, then we have

$$\frac{1}{s}\int_Q \frac{e^{2s\alpha}}{\varphi}|\nabla w|^2 dxdt + s\int_Q \varphi e^{2s\alpha}|w|^2 dxdt$$
$$+s^2\int_Q \varphi^2 e^{2s\alpha}|y|^2 dxdt \le C(\int_Q e^{2s\alpha}|f|^2 dxdt)$$
$$+C(s^2\int_{Q_\omega} \varphi^2 e^{2s\alpha}|y|^2 dxdt + s\int_{Q_\omega} \varphi e^{2s\alpha}|w|^2 dxdt).$$

Localization in space

Covering of Ω :

$$\Omega \subset \Omega_0 \bigcup_{i=1}^{I} B(\tilde{x}^i, \delta),$$

$$\Omega_0 \subset \Omega,\ \omega \subset \Omega_0,\ \tilde{x}^i \in \Gamma,\ B(\tilde{x}^i, \delta) \cap \omega = \emptyset.$$

Corresponding partition of unity : $(e_i)_{i=0}^{I}$.
$y_i = y.e_i$: solution of similar problem with compact support in space in Ω_0 or in $B(\tilde{x}^i, \delta)$.
Case $i = 0$: Already done in Imanuvilov-Yamamoto.
Case $i = 1, ..., I$: We drop the index i. In $B(\tilde{x}, \delta)$, we have $|\nabla\psi| \neq 0$ and we can suppose $|\frac{\partial\psi}{\partial x_N}| \neq 0$.
New coordinates :

$$\hat{x}_N = \psi(x_1, \cdots, x_N),\ \hat{x}_i = x_i - \tilde{x}_i,\ i = 1, \cdots, N-1,$$
$$\hat{y}(t, \hat{x}_1, \cdots, \hat{x}_N) = y(t, x_1, \cdots, x_N).$$

We now have with unambiguous notations (B' is a ball in $\mathbf{R}^{N-1}$) :

$$\begin{aligned}
&\hat{\varphi}(\hat{x}) = e^{\lambda \hat{x}_N} \text{ (new weight)},\\
&\hat{L}\hat{y} = \frac{\partial \hat{y}}{\partial t} - \frac{\partial^2 \hat{y}}{\partial \hat{x}_N^2} - \sum_{j=1}^{N-1} a_{Nj} \frac{\partial^2 \hat{y}}{\partial \hat{x}_N \partial \hat{x}_j} - \hat{A}\hat{y} + l.o.t. = \hat{f}_0 + \sum_{j=1}^{N} \frac{\partial \hat{f}_j}{\partial \hat{x}_j},\\
&\hat{y}(t, \hat{x}', 0) = \hat{g}(t, \hat{x}', 0),\\
&\text{Supp}\hat{y} \subset (0, T) \times B'(0, \delta) \times [0, \delta[,\\
&\hat{A}\hat{y} = \sum_{i,j=1}^{N-1} \hat{a}_{i,j} \frac{\partial^2 \hat{y}}{\partial \hat{x}_i \partial \hat{x}_j}.
\end{aligned}$$

We drop the $\hat{.}$ notation. We still have an ellipticity condition

$$\forall \xi \in \mathbf{R}^N,\ \xi_N^2 + \sum_{j=1}^{N-1} a_{Nj}\xi_N\xi_j + \sum_{j,k=1}^{N} a_{jk}\xi_j\xi_k \geq \gamma|\xi|^2.$$

Localization in time

Take $\tilde{\psi} \in C_0^\infty(]\frac{1}{2}, 2[)$ such that $\forall s,\ \sum_{j=-\infty}^{+\infty} \tilde{\psi}(2^{-j}s) = 1$. Define

$$\mu_j(t) = \tilde{\psi}(\frac{2^{-j}}{l^k(t)}),$$

$$w_j = \mu_j w = \mu_j e^{s\alpha} y.$$

Notice that on $Supp(\mu_j)$, we have $2^{-j-1} < l^k(t) < 2^{-j+1}$. We then write

$$L(t, x, D_t, D_x + is\nabla_x \alpha) w_j = e^{s\alpha} L(\mu_j(t) y)$$

so that

$$L(t, x, D_t, D_x + is\nabla_x\alpha) w_j = \frac{\partial \mu_j}{\partial t} w + F \text{ in } Supp(\mu_j) \times B'(0, \delta) \times]0, \delta$$

$$w_j(t, x', 0) = \mu_j(t) e^{s\alpha} g,$$

$$Supp(w_j) \subset G = Supp(\mu_j) \times B'(0, \delta) \times [0, \delta[,$$

$$F = e^{s\alpha} f_0 - \sum_{j=1}^{N} s \frac{\partial \alpha}{\partial x_j} e^{s\alpha} f_j - s \frac{\partial \alpha}{\partial t} w + \sum_{j=1}^{N} \frac{\partial (e^{s\alpha} f_j)}{\partial x_j}.$$

Take $t_j \in Supp(\mu_j)$ so that $l^k(t_j) \sim 2^{-j}$ and define

$$\beta = \alpha l^k(t_j), \ \tau = \frac{s}{l^k(t_j)}.$$

We have

$$L(t,x,D_t,D_x + i\tau\nabla_x\beta)w_j = \frac{\partial\mu_j}{\partial t}w + F \text{ in } Supp(\mu_j)\times B'(0,\delta)\times]0,\delta[,$$

and it is enough to prove the following estimate : For $\tau \geq \tau_0$,

$$\frac{1}{\tau}|\nabla w_j|^2_{L^2(G)} + \tau|w_j|^2_{L^2(G)} \leq C(\tau^{-\frac{1}{2}}||e^{s\alpha}g||^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)}$$

$$+\tau^{-\frac{1}{2}+\frac{1}{k}}|e^{s\alpha}g|^2_{L^2(\Sigma)} + \frac{1}{\tau^2}|\frac{\partial\mu_j}{\partial t}w + e^{s\alpha}f_0|^2_{L^2(G)} + \sum_{j=1}^{N}|e^{s\alpha}f_j|^2_{L^2(G)}).$$

We drop the index j.

Principal symbol of L :

$$L_2(t,x,\xi,\tau) = i\xi_0 + (\xi_N + i\frac{\partial\beta}{\partial x_N})^2 + \sum_{j=1}^{N-1} a_{Nj}(\xi_N + i\frac{\partial\beta}{\partial x_N})\zeta_j + \sum_{j,k=1}^{N-1} a_{k,j}\zeta_j\zeta_k$$

where

$$\zeta = \xi + i\nabla_x\beta.$$

Roots in ξ_N :

$$\xi_N = -i\tau\frac{\partial\beta}{\partial x_N} - \frac{1}{2}\sum_{j=1}^{N-1} a_{Nj}\zeta_j \pm \sqrt{\underbrace{\frac{1}{4}(\sum_{j=1}^{N-1} a_{Nj}\zeta_j)^2 - (i\xi_0 + \sum_{j,k=1}^{N-1} a_{jk}\zeta_j\zeta_k)}_{Z}}.$$

Parabolic normalization

$$M(\xi', \tau) = (\xi_0^2 + \sum_{j=1}^{N-1} \xi_j^4 + \tau^4)^{\frac{1}{4}}.$$

-Regularization of the square root near $M = 0$

-Separation in two regions with corresponding partition of unity $\chi_0, \ \chi_1$:

- Region 0 : Neighborhood of $\{M = 1, \ Z \in \mathbf{R}_+\}$. Because of ellipticity, near the boundary we must have $\tau \sim 1$ and $\xi_j \sim 0, \ j = 0, \cdots, N-1$.
- Region 1 : Outside this neighborhood with still $M = 1$.

Then extend χ_k as a C^∞ function for $M < 1$ and for $M > 1$ by :

$$\chi_k(\xi', \tau) = \chi_k(\frac{\xi_0}{M^2}, \frac{\xi_1}{M}, \cdots, \frac{\xi_{N-1}}{M}, \frac{\tau}{M})$$

$$w_k = \chi_k(D', \tau)w.$$

Factorization

We concentrate on w_1 and drop the index 1.

$$r^+ = \tau\frac{\partial\beta}{\partial x_N} - \frac{1}{2}\sum_{j=1}^{N-1} a_{Nj}\zeta_j + i\sqrt{\frac{1}{4}(\sum_{j=1}^{N-1} a_{Nj}\zeta_j)^2 - (i\xi_0 + \sum_{j,k=1}^{N-1} a_{jk}\zeta_j\zeta_k)}$$

$$r^- = \tau\frac{\partial\beta}{\partial x_N} - \frac{1}{2}\sum_{j=1}^{N-1} a_{Nj}\zeta_j - i\sqrt{\frac{1}{4}(\sum_{j=1}^{N-1} a_{Nj}\zeta_j)^2 - (i\xi_0 + \sum_{j,k=1}^{N-1} a_{jk}\zeta_j\zeta_k)}.$$

We have

$$Re(r^-) \geq C.M, \quad -Re(r^+) \geq C.M.$$

Corresponding operators : R^+ and R^-.

$$L^+ = \frac{\partial}{\partial x_N} - R^+, \quad L^- = \frac{\partial}{\partial x_N} - R^-$$

$$Q = \frac{1}{2}(L^+ + L^{+*}) = Q^*, \quad P = \frac{1}{2}(L^+ - L^{+*}) = -P^*.$$

$$Lw = L^-.L^+ + Kw.$$

We write

$$L^+w = z,$$
$$L^-z = F - Kw,$$
$$z(t, x', \delta) = 0.$$

L^- is a "good" backward (in x_N) operator. If

$$||y||^2_{H^{\frac{1}{2},1,\tau}} = ||y||^2_{H^{\frac{1}{2},1}} + \tau^2|y|^2_{L^2}$$

and

$$H^{-\frac{1}{2},-1,\tau} = (H^{\frac{1}{2},1,\tau})'$$

we have the first estimate (standard)

$$|z|^2_{L^2} \leq C(||F||^2_{H^{-\frac{1}{2},-1,\tau}} + ||Kw||^2_{H^{-\frac{1}{2},-1,\tau}}).$$

Second estimate :

$$L^+ w = Qw + Pw = z$$

so that

$$|Qw|^2_{L^2} + |Pw|^2_{L^2} + 2Re(Qw, Pw)_{L^2} = |z|^2_{L^2}$$

$$|Qw|^2_{L^2} + |Pw|^2_{L^2} + ([Q,P]w, w)_{L^2} - 2Re(Q(t,x',0)h, h)_{L^2(\mathbf{R}^N)} = |z|^2_{L^2}$$

where $h = w(t,x',0) = e^{s\alpha}g$.
We have $Re(\{Q,P\} \geq C.M$ if $Q = 0$ so that

$$|Qw|^2_{L^2} + |Pw|^2_{L^2} + ([Q,P]w, w)_{L^2} \geq C||w||^2_{H^{\frac{1}{4},\frac{1}{2},\tau}} - C|w|^2_{L^2}$$

-First case : $Re(Q(t,x',0)h, h)_{L^2(\mathbf{R}^N)} \leq 0$.
We see immediately that here we have

$$||w||^2_{H^{\frac{1}{4},\frac{1}{2},\tau}} \leq |z|^2_{L^2} \leq C(||F||^2_{H^{-\frac{1}{2},-1,\tau}} + ||Kw||^2_{H^{-\frac{1}{2},-1,\tau}}).$$

-Second case : $Re(Q(t,x',0)h,h)_{L^2(\mathbf{R}^N)} \geq 0$. We then have

$$(1+\tau)|h|^2_{L^2(\mathbf{R}^N} \leq C||h||^2_{H^{\frac{1}{4},\frac{1}{2}}(\mathbf{R}^N)}$$

Then we can show that there exists a pair (p,v) satisfying

$$L^{+*}p = (1+\tau)w + v$$

$$\sqrt{1+\tau}\int(|p|^2 + m_\tau^2|v|^2)dxdt + \int m_\tau^2|p(t,x',0)|^2 dtdx' \leq C(1+\tau)^{\frac{3}{2}}\int |w|^2 dtdx$$

where

$$m_\tau = 1 \text{ in } Suppw, \;\; m_\tau = \frac{1}{(1+\tau)} \text{ outside.}$$

This estimate is non standard but similar to what is done in the elliptic case (Imanuvilov-Puel).

Taking the scalar product with w we obtain (essentially)

$$
\begin{aligned}
&(1+\tau)|w|_{L^2}^2 = (w, L^{+*}p - v)_{L^2} = \\
&(L^+ w, p)_{L^2} - (w, v)_{L^2} + (h, p(0))_{L^2(\mathbf{R}^N)} = \\
&(z, p)_{L^2} - (w, v)_{L^2} + (h, p(0))_{L^2(\mathbf{R}^N)} \leq \\
&|z|_{L^2}|p|_{L^2} + |h|_{L^2(\mathbf{R}^N)}(1+\tau)^{\frac{3}{4}}|w|_{L^2} + |m_\tau v|_{L^2}|w|_{L^2} \leq \\
&C\sqrt{(1+\tau)}|w|_{L^2}(|z|_{L^2} + \frac{1}{(1+\tau)^{\frac{1}{4}}}||h||_{H^{\frac{1}{4},\frac{1}{2}}(\mathbf{R}^N)} + |w|_{L^2})
\end{aligned}
$$

so that

$$
\sqrt{(1+\tau)}|w|_{L^2} \leq C(\frac{1}{(1+\tau)^{\frac{1}{4}}}||h||_{H^{\frac{1}{4},\frac{1}{2}}(\mathbf{R}^N)} + ||F||^2_{H^{-\frac{1}{2},-1,\tau}} + |w|_{L^2}).
$$

We end up with energy estimates for the problem in w and we have to put together all the estimates we have in different regions to obtain the desired Carleman estimate for the complete function w.