An iterative time reversal algorithm for solving initial data inverse problems

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Consider the linear infinite-dimensional system :

$$\begin{cases} \dot{z}(t) = Az(t), & t \in (0, \tau) \\ z(0) = x \end{cases} \qquad y(t) = Cz(t), \ t \in (0, \tau)$$

where

- $A = -A^* : \mathcal{D}(A) \longrightarrow X$ generates a C^0 group of isometries $\mathbb{T}$ on a Hilbert space X
- $C \in \mathcal{L}(X, Y)$ is a bounded observation operator.

If the pair (A, C) is exactly observable in time τ_{obs}, then

$$\Psi_\tau : x \in X \longmapsto y(t) = C\mathbb{T}_t x \in L^2(0, \tau)$$

has a bounded left-inverse for $\tau \geq \tau_{obs}$.

Problem

Find a practical and efficient way to compute the initial state x from the observation $y \in \text{Ran}\,\Psi_\tau$, i.e. how to solve : $\Psi_\tau x = y$?

A theoretical answer

Use the observability Gramian : $Q_\tau = \Psi_\tau^* \Psi_\tau = \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t \, dt$:

$$Q_\tau x = \Psi_\tau^* y.$$

Since (A, C) is exactly observable, $Q_\tau > 0$ for $\tau \geq \tau_{obs}$.

ill-conditioned (regularization) $+$ **computationally costly**

Kalman observers based methods

- D. Auroux & J. Blum : Data assimilation (2005-2008).
- K. Phung & X. Zhang : Kirchhoff plates (SIAP, 2008).

Kalman observer

Problem

Estimate the final state $z(\tau)$ from the partial observation $y(t)$.

Introduce the Kalman state observer

$$\begin{cases} \dot{v}(t) = Av(t) - C^*\Big[Cv(t) - y(t)\Big], & t \in (0,\tau) \\ v(0) = 0. \end{cases}$$

Then the error $e = v - z$ satisfies

$$\dot{e}(t) = Av(t) - C^*\Big[Cv(t) - Cz(t)\Big] - Az(t) = (A - C^*C)e(t).$$

Since (A, C) is exactly observable, $(A - C^*C)$ generates an exponentially stable C^0 semi-group on X :

$$\|v(\tau) - z(\tau)\| \leq Me^{-\omega\tau}\|z(0)\| \longrightarrow 0 \qquad \text{as } \tau \longrightarrow \infty.$$

Main Assumptions

- $A^* = -A$, $C \in \mathcal{L}(X, Y)$ and (A, C) is exactly observable.
- There exists a time reversal operator $\mathfrak{R}_\tau \in \mathcal{L}\left(L^2([0,\tau]; X)\right)$ associated with the pair (A, C) i.e. satisfying

$$\begin{cases} \mathfrak{R}_\tau^2 = I, & \\ \|(\mathfrak{R}_\tau v)(0)\| = \|v(\tau)\|, & \forall v \in C([0,\tau]; X) \\ C^* C(\mathfrak{R}_\tau v) = \mathfrak{R}_\tau(C^* C v), & \forall v \in C([0,\tau]; X) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{R}_\tau \mathbb{T}_t z_0) = -\mathfrak{R}_\tau \dfrac{\mathrm{d}}{\mathrm{d}t} \mathbb{T}_t z_0, & \forall z_0 \in X \\ A(\mathfrak{R}_\tau \mathbb{T}_t z_0) = -\mathfrak{R}_\tau(A \mathbb{T}_t z_0), & \forall z_0 \in X \end{cases}$$

Example 1 :
The wave equation with distributed observation on $\mathcal{O} \subset \Omega$

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad A_0 = -\Delta + B.C. \quad C \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_{2|\mathcal{O}}$$

$$(\mathfrak{R}_\tau z)(t) = \begin{bmatrix} z_1(\tau - t) \\ -z_2(\tau - t) \end{bmatrix}$$

The above conditions are obviously satisfied. For instance

$$\begin{cases} A(\mathfrak{R}_\tau \mathbb{T}_t z_0) = A \begin{bmatrix} z_1(\tau - t) \\ -z_2(\tau - t) \end{bmatrix} = - \begin{bmatrix} z_2(\tau - t) \\ A_0 z_1(\tau - t) \end{bmatrix}, \\ \mathfrak{R}_\tau (A \mathbb{T}_t z_0) = \mathfrak{R}_\tau \begin{bmatrix} z_2(t) \\ -A_0 z_2(t) \end{bmatrix} = \begin{bmatrix} z_2(\tau - t) \\ A_0 z_1(\tau - t) \end{bmatrix}. \end{cases}$$

Example 2 : The Schrödinger equation with distributed observation

$$A = iA_0 \qquad\qquad Cz = z_{|\mathcal{O}}$$

$$(\text{Я}_\tau z)(t) = \overline{z(\tau - t)}.$$

$$\begin{cases} A(\text{Я}_\tau \mathbb{T}_t z_0) = iA_0\overline{z(\tau - t)}, \\ \\ \text{Я}_\tau (A\mathbb{T}_t z_0) = \text{Я}_\tau (iA_0 z(t)) = -iA_0\overline{z(\tau - t)}. \end{cases}$$

A time reversed Kalman observer

Let

$$\begin{cases} \dot{v}(t) = Av(t) - C^*\Big[Cv(t) - (\text{Я}_\tau y)(t)\Big], & t \in (0,\tau) \\ v(0) = 0, \end{cases}$$

and set

$$e(t) = v(t) - (\text{Я}_\tau z)(t).$$

Then, thanks to the properties of Я_τ, we easily get that

$$\dot{e}(t) = (A - C^*C)e(t).$$

Since (A, C) is exactly observable, we have

$$\|(\text{Я}_\tau e)(0)\| = \|e(\tau)\| \leq Me^{-\omega\tau}\|e(0)\|,$$

or equivalently

$$\|(\text{Я}_\tau v)(0) - x\| \leq Me^{-\omega\tau}\|x\| \longrightarrow 0 \qquad \text{as } \tau \longrightarrow \infty.$$

A time reversed Kalman observer

The wave equation with distributed observation

$$\begin{cases} \ddot{w}(t) + A_0 w(t) + \chi_{\mathcal{O}} \dot{w}(t) + \chi_{\mathcal{O}} \dot{w}(\tau - t) = 0, & t \in (0, \tau) \\ w(0) = \dot{w}(0) = 0, \end{cases}$$

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \lim_{\tau \longrightarrow +\infty} \begin{bmatrix} w(\tau) \\ -\dot{w}(\tau) \end{bmatrix}.$$

A time reversed Kalman observer

The Schrödinger equation with distributed observation

$$\begin{cases} \dot{v}(t) = iA_0 v(t) - \chi_{\mathcal{O}} v(t) + \chi_{\mathcal{O}} \overline{v(\tau - t)}, & t \in (0, \tau) \\ v(0) = 0, \end{cases}$$

$$x = \lim_{\tau \longrightarrow +\infty} \overline{v(\tau)}.$$

Idea

Solve the equation

$$(\Phi_\tau \Psi_\tau)x = \Phi_\tau y$$

where $\Phi_\tau \in \mathcal{L}\left(L^2([0,\tau];Y), X\right)$ is chosen such that $\Phi_\tau \Psi_\tau \in \mathcal{L}(X)$ is invertible.

For all $\xi \in L^2([0,\tau];Y)$, let

$$\begin{cases} \dot{v}(t) = Av(t) + C^*\left[Cv(t) - (\text{Я}_\tau \xi)(t)\right], & t \in (0,\tau) \\ v(0) = 0. \end{cases}$$

and set

$$\Phi_\tau \xi = (\text{Я}_\tau v)(0).$$

The one-shot method shows that

$$\|I - \Phi_\tau \Psi_\tau\|_{\mathcal{L}(X)} \leq M e^{-\omega\tau} \longrightarrow 0, \qquad \text{as } \tau \longrightarrow \infty.$$

Theorem

Given $\tau \geq \tau_{obs}$, *there exists* $0 < \delta < 1$ *such that*

$$\|I - \Phi_\tau \Psi_\tau\|_{\mathcal{L}(X)} \leq \delta.$$

Moreover, if we define the sequence (s_n)

$$\begin{cases} s_0 = \Phi_\tau y, \\ s_{n+1} = s_n - \Phi_\tau \Psi_\tau s_n, \qquad \forall n \geq 0. \end{cases}$$

then

$$\left\| x - \sum_{n=0}^{N} s_n \right\| \leq \delta^{N+1} \|x\|.$$

Proof

$$x \in X \quad \xrightarrow{\Psi_\tau} \quad y(t) = C\mathbb{T}_t x \quad \xrightarrow{\Phi_\tau} \quad (Я_\tau v)(0) \in X$$

where

$$\begin{cases} \dot{v}(t) = Av(t) - C^*\Big[Cv(t) - (Я_\tau y)(t)\Big], & t \in (0,\tau) \\ v(0) = 0, \end{cases}$$

Setting $e(t) = v(t) - (Я_\tau \mathbb{T}_t x)(t)$, we have once again

$$\dot{e}(t) = (A - C^*C)e(t).$$

Since (A, C) exactly observable in time τ_{obs}, the semigroup $\mathbb{S}$ generated by $(A - C^*C)$ satisfies $\|\mathbb{S}_\tau\| < 1$ for all $\tau \geq \tau_{obs}$.

$$\|(Я_\tau e)(0)\| = \|e(\tau)\| \leq \delta\|e(0)\| \qquad (\text{with } \delta := \|\mathbb{S}_\tau\|)$$

or equivalently

$$\|x - \Phi_\tau \Psi_\tau x\| \leq \delta\|x\|, \qquad \forall\, x \in X.$$

Sketch of proof

The iterative algorithm follows from a Neumann series argument :

$$\begin{aligned} x &= (\Phi_\tau \Psi_\tau)^{-1}(\Phi_\tau y) \\ &= \Big[I - (I - \Phi_\tau \Psi_\tau)\Big]^{-1}(\Phi_\tau y) \\ &= \sum_{n=0}^{\infty} s_n, \qquad\qquad s_n = (I - \Phi_\tau \Psi_\tau)^n (\Phi_\tau y). \end{aligned}$$

We also have the error estimate

$$\left\| x - \sum_{n=0}^{N} s_n \right\| \leq \delta^{N+1} \|x\|.$$

■

If s_n is known, the computation of $s_{n+1} = s_n - \Phi_\tau \Psi_\tau s_n$ requires to solve successively 2 problems :

Conservative System

$\Psi_\tau s_n$ is obtained by solving on $(0, \tau)$:

$$\begin{cases} \dot{z}_n(t) = A z_n(t), \\ z_n(0) = s_n \end{cases} \Longrightarrow \quad \Psi_\tau s_n = C z_n := y_n,$$

Damped System

$\Phi_\tau \Psi_\tau s_n = (Я_\tau v_n)(0)$ is obtained by solving on $(0, \tau)$:

$$\begin{cases} \dot{v}_n(t) = A v(t) - C^* \Big[C v_n(t) - (Я_\tau y_n)(t) \Big], \\ v_n(0) = 0. \end{cases}$$

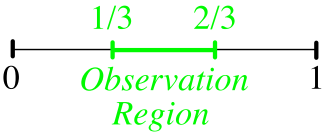
$$s_{n+1} = s_n - (Я_\tau v_n)(0).$$

Remark

Obviously, the previous results still hold if we use a Kalman observer with gain γ :

$$\begin{cases} \dot{v}(t) = Av(t) - \gamma C^* \Big[Cv(t) - (Я_\tau y)(t) \Big], & t \in (0, \tau), \\ v(0) = 0. \end{cases}$$

1D wave equation

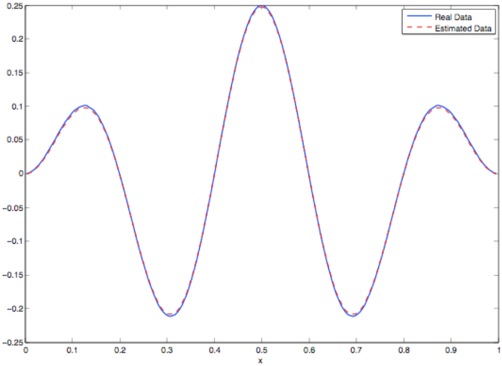


$$\begin{cases} \ddot{w}(x,t) - w''(x,t) = 0, & x \in (0,1),\ t \in (0,\tau), \\ w(0,t) = w(1,t) = 0, & t \in (0,\tau), \\ w(x,0) = w_0(x), & x \in (0,1), \\ \dot{w}(x,0) = w_1(x), & x \in (0,1), \end{cases}$$

$$y(x,t) = \dot{w}(x,t), \qquad x \in (1/3, 2/3),\ t \in (0,\tau).$$

We use a centered finite difference scheme with $\Delta t / \Delta x = 0.5$.

1D wave equation



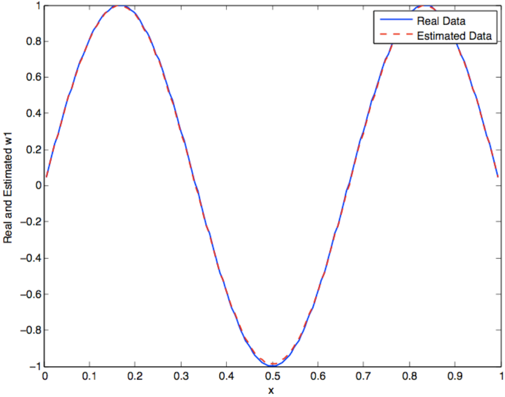
$$w_0(x) = x(1-x)\sin(5\pi x)$$

Real and Estimated w_0 after 6 iterations, for $\tau = 3\tau_{obs}$

Relative error less then 1.5%

(200 points of discretization)

1D wave equation



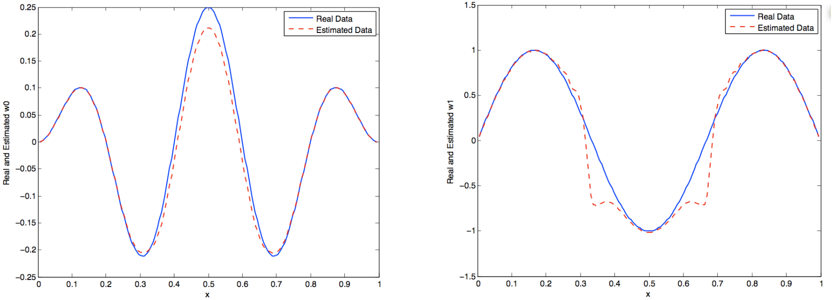
$w_1(x) = \sin(3\pi x)$

Real and Estimated w_1 after 6 iterations, for $\tau = 3\tau_{obs}$

Relative error less then 1.5%

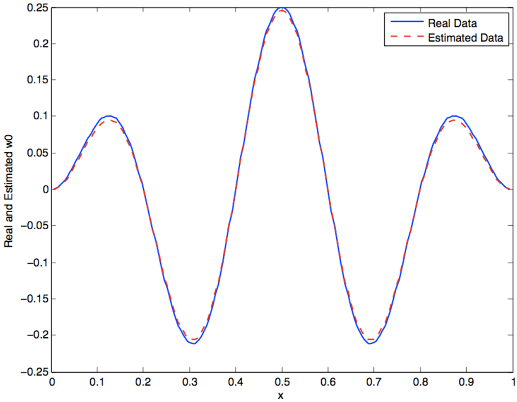
(200 points of discretization and $\Delta t/h = 0.5$)

Influence of τ



Real and Estimated Initial Data after 100 iterations, for $\tau = \tau_{obs}$.

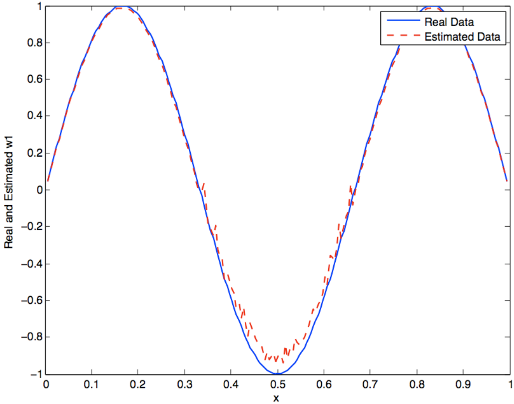
Noisy Data (5% of noise)



$$w_0(x) = x(1-x)\sin(5\pi x)$$

Real and Estimated w_0 after 6 iterations, for $\tau = 3\tau_{obs}$.

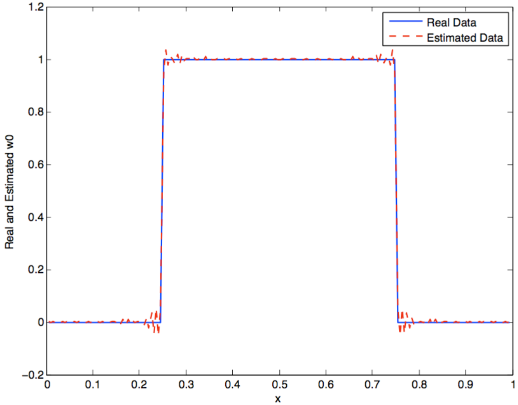
Noisy Data (5% of noise)



$$w_1(x) = \sin(3\pi x)$$

Real and Estimated w_1 after 6 iterations, for $\tau = 3\tau_{obs}$.

Non Smooth Data

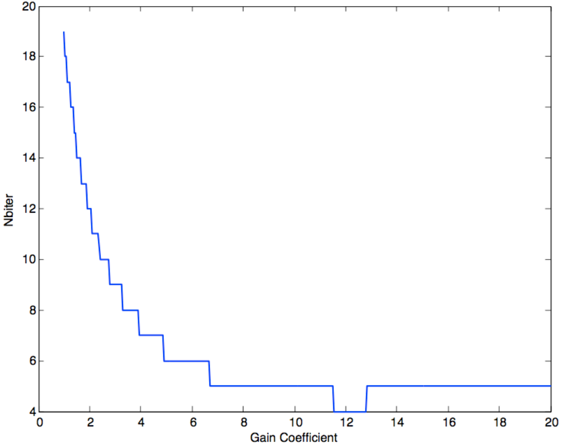


Real and Estimated w_0 after 15 iterations, for $\tau = 3\tau_{obs}$

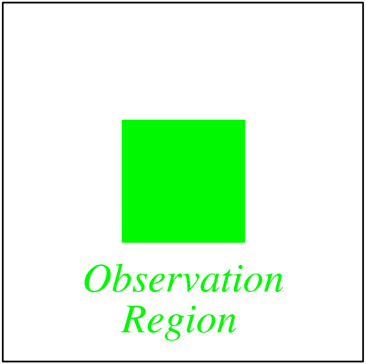
Relative error less then 2%

(200 points of discretization and $\Delta t/h = 0.5$)

Influence of the gain coefficient on the speed of convergence

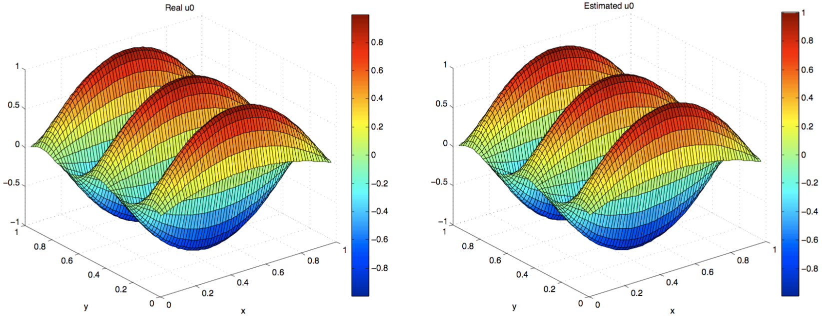


2D wave equation



$$w^0(x) = 4x(1-x)\sin(5\pi y)$$
$$w^1(x) = 16x(1-x)y(1-y)$$

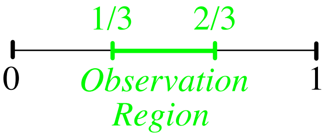
2D wave equation



Real and Estimated w_0 after 9 iterations ($\tau = 3\tau_{aobs}$, $\gamma = 20$).

Relative error less then 1%.

(50 points of discretization in x and y and $\Delta t/h = 0.5$)

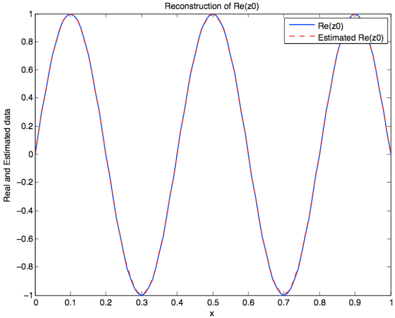


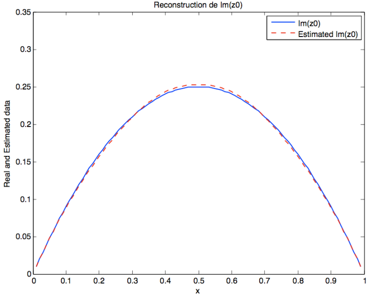
$$
\begin{cases}
\dot{z}(x,t) + iz''(x,t) = 0, & x \in (0,1), \ t \in (0,\tau), \\
z(0,t) = z(1,t) = 0, & t \in (0,\tau), \\
z(x,0) = z_0(x), & x \in (0,1),
\end{cases}
$$

$$
y(x,t) = z(x,t), \qquad x \in (1/3, 2/3), \ t \in (0,\tau).
$$

We use a Crank-Nicolson finite difference scheme.

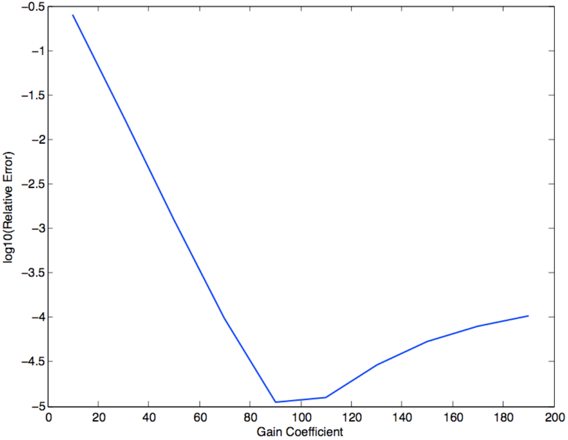
The 1D Schrödinger equation





Real and Estimated z_0 after 1 iteration for $\tau = 0.2$, for $\gamma = 100$.
Relative error less then 1%.

Influence of the gain coefficient on the error



Concluding comments

- Φ_τ can be seen as a preconditioner for Ψ_τ. The equation $\Phi_\tau \Psi_\tau x = \Phi_\tau y$ can then be solved using a GMRES algorithm.
- Our method can be extended to the more technical case of unbounded observation operators C (boundary observation) and/or non skew-adjoint generators A (work in progress with M. Tucsnak and G. Weiss).
- The method can be coupled to a solver of Volterra equations to solve inverse source problems (see Alvez, Silvestre, Takahashi and Tucsnak, SICON, 2009).