Control of the Heat Equation for a Discontinuous Diffusion Coefficient.

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THE PROBLEM

Recall : control problem for heat.

We search g such that the solution of the following problem verifies

$$\begin{cases} \partial_t u - \Delta u = g\mathbf{1}_\omega \\ u_{|t=0} = u_0 \\ u_{|\partial\Omega} = 0 \end{cases}$$

$u_{|t=T} = 0$.

$\Omega \subset \mathbb{R}^n$ an open set $\mathcal{C}^\infty$.

$\omega \subset \Omega$ an open set.

In this case we say that we have null controllability.

KNOWN RESULTS

-Russel ('73) (wave $\Rightarrow$ heat)

-Fursikov & Imanuvilov ('95), Lebeau & Robbiano ('95) (Laplacian with regular coefficients)

-Doubova, Osses & Puel ('02) (transmission problem)

-Fernandez-Cara & Zuazua ('02) (1D, BV coefficient)

-Alessandrini & Escauriaza ('06) (1D, L^∞ coefficients)

-Benabdallah, Dermenjian & Le Rousseau ('07) (1D, transmission)

-Le Rousseau ('07) (1D, BV coefficient, stratified problem multi-D)

-Le Rousseau & Robbiano ('08) (transmission problem)

-Non linear problem (heat, Navier-Stokes,...)

NOTATIONS & RESULT

Ω smooth bounded domain, $\Omega_1 \subset\subset \Omega$ smooth domain.

$\Omega_2 = \Omega \setminus \overline{\Omega}_1$, $\overline{\Omega}_1 \cap \overline{\Omega}_2 = S$ smooth hypersurface.

$c(x) > 0$ smooth function in Ω_1 and Ω_2, denote $c_j = c_{|\Omega_j}$.

$$\Delta_c = \sum_{j=1}^{n} \partial_{x_j} c \partial_{x_j}.$$

Theorem 1 *The null controllability is true for heat equation* $\partial_t - \Delta_c$ *with Dirichlet boundary condition.*

Remark 1 *Doubova, Osses and Puel, non linear result but assume*

$$\omega \subset \Omega_1, c_1 < c_2 \textit{ on } S$$

$$\omega \subset \Omega_2, c_2 < c_1 \textit{ on } S.$$

CARLEMAN ESTIMATE IN NEIGHBOROUGHT OF JUMP

Let $\psi(x)$, $-\partial_{n_1}\psi_{|\Gamma_1} >> \partial_{n_2}\psi_{|\Gamma_2}$

$\Gamma_1 = \Gamma_2 = \partial\Omega_1 \cap \partial\Omega_2$ $\phi_{|\Gamma_j} = \lim\limits_{x\in\Omega_j, x\to\Gamma_j} \phi(x)$

$\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K}$, $K > \sup\psi$.

Theorem 2 *$\exists h_0 > 0$, $C > 0$, $\forall u$ supported in D, a neighborhood of Γ_1 in Ω , u satisfying the transmission condition, $\forall h \in (0, h_0]$*

$$h^{-3}\left\|\left(\frac{1}{t(T-t)}\right)^{3/2} e^{\frac{\varphi}{ht(T-t)}} u\right\|^2 + \ \mathit{etc.} \leq C\|e^{\frac{\varphi}{ht(T-t)}}(\partial_t - \Delta_c)u\|^2$$

SOME REDUCTIONS AND GENERALITIES

-Writing transmission problem like a coupled system at boundary (following Bellassoued)

-Conjugate $v = e^{\varphi/h} u$, Denote $P_\varphi = e^{\varphi/h} P e^{-\varphi/h}$.

-Transformation $x_n \to -x_n$ when $x_n < 0$.

-Notations v_ℓ $(x_n < 0)$, v_r $(x_n > 0)$, and φ_ℓ, φ_r.

$$\begin{aligned} & P_{\varphi_\ell} v_\ell = f_\ell && \text{in } x_n > 0 \\ & P_{\varphi_r} v_r = f_r && \text{in } x_n > 0 \\ & v_\ell - v_r = k_1 && \text{on } x_n = 0 \\ & c_\ell (D_{x_n} + i \tfrac{1}{t(T-t)} \varphi'_{\ell x_n}) v_\ell && \\ & + c_r (D_{x_n} + i \tfrac{1}{t(T-t)} \varphi'_{r x_n}) v_r = k_2 && \text{on } x_n = 0 \end{aligned} \tag{1}$$

CARLEMAN APPROACH

-Usual method, $P_{\varphi_\ell} = A_\ell + iB_\ell$, $P_{\varphi_r} = A_r + iB_r$ (A, B self-adjoint).

$$\|P_{\varphi_\ell} v_\ell\|^2 = \|A_\ell v_\ell\|^2 + \|B_\ell v_\ell\|^2 + 2\,\mathrm{Re}(A_\ell v_\ell, iB_\ell v_\ell)$$

-Computation $\mathrm{Re}(A_\ell v, iB_\ell v_\ell)$ gives usual terms and boundary terms.

$$\|P_{\varphi_\ell} v_\ell\|^2 + \|P_{\varphi_r} v_r\|^2 = U(v) + \mathcal{B}(v)$$

-where $U(v)$ are the usual wellknown terms, “easy”.

-$\mathcal{B}(v)$ are terms on boundary like $(L_0 D_{x_n} v_r, D_{x_n} v_r)_0$, $(L_1 v_r, D_{x_n} v_r)_0$, $(L_2 v_r, v_r)_0$, L_j are differential tangential operators of order j. ($(\cdot,\cdot)_0$ means inner product on boundary).

-We can use boundary conditions to analyse $\mathcal{B}(v)$ but in general $\mathcal{B}(v) \not\geq 0$. In case Doubova-Osses-Puel $\mathcal{B}(v) \geq 0$

MODEL PROBLEM

“Formal” solution $\Delta u = 0$ in $x_n > 0$ with boundary conditions.

Fourier with respect $x' = (x_1, \cdots, x_{n-1})$, we have

$(\partial^2_{x_n} - |\xi'|^2)\hat{u}(\xi', x_n) = 0.$

The solution $\hat{u}$ is given by

$\hat{u}(\xi', x_n) = A(\xi')e^{-x_n|\xi'|} + B(\xi')e^{x_n|\xi'|}.$

but $e^{x_n|\xi'|} \notin \mathcal{S}'$ if $x_n > 0$ → “false solution” then

$\hat{u}(\xi', x_n) = A(\xi')e^{-x_n|\xi'|}.$

In particular $\hat{u}(\xi', 0) = A(\xi')$, $\partial_{x_n}\hat{u}(\xi', 0) = -|\xi'|A(\xi')$ then

$|\xi'|\hat{u}(\xi', 0) + \partial_{x_n}\hat{u}(\xi', 0) = 0$

It follows : **one** trace determine the **two** traces.

Dirichlet-to-Neumann, Neumann-to-Dirichlet etc.

ELLIPTIC BOUNDARY PROBLEM

$Pv = f$ in $x_n > 0$

$$\text{Denote } \underline{v} = \begin{cases} v \text{ if } x_n > 0 \\ 0 \text{ if } x_n < 0 \end{cases}$$

$P\underline{v} = \underline{f} + \gamma_0 \delta'_{x_n=0} + \gamma_1 \delta_{x_n=0}$

where γ_j depend of $v_{|x_n=0}$ and $D_{x_n} v_{|x_n=0}$.

Choose Q a parametrix of $P \longrightarrow QP = Id + R$ where R smoothing.

$\underline{v} = Q\underline{f} + C_0\gamma_0 + C_1\gamma_1 + R\underline{v}$ where C_j are operators on traces.

On $x_n = 0^+$

$v_{|x_n=0} = Q\underline{f}_{|x_n=0} + C'_0\gamma_0 + C'_1\gamma_1 + R\underline{v}_{|x_n=0}$

Does this give relation between $v_{|x_n=0}$ and $D_{x_n} v_{|x_n=0}$?

CALDERON OPERATORS I

C_0, C_1 have the following form (principal symbol)

$$\int e^{ix'\xi'}\hat{\gamma}(\xi')\int e^{ix_n\xi_n}\frac{\xi_n^{\mu}}{p(x,\xi)}d\xi_n d\xi'$$

KEY POINT $\longrightarrow$ roots in ξ_n of $p(x,\xi',\xi_n)$.

WHY? $\longrightarrow$ residues formula.

$p(x,\xi',\xi_n)$ is the principal symbol of P.

ξ_n^{μ} comes from Fourier transform of $\delta^{(\mu)}_{x_n=0}$.

CALDERON OPERATORS II

Let $p(x,\xi) = (\xi_n - r_1(x,\xi'))(\xi_n - r_2(x,\xi'))$ → 3 cases

- If $\operatorname{Im} r_1(x,\xi') < 0$ and $\operatorname{Im} r_2(x,\xi') < 0$ then

$$\int e^{ix_n\xi_n} \frac{\xi_n^\mu}{p(x,\xi)} d\xi_n = 0$$

f determine the traces of u (analoguous computation for $\partial_n u$).

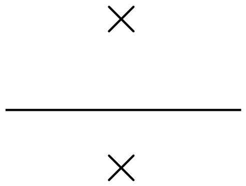
- If $\operatorname{Im} r_1(x,\xi') > 0$ and $\operatorname{Im} r_2(x,\xi') < 0$ then

$$\int e^{ix_n\xi_n} \frac{\xi_n^\mu}{p(x,\xi)} d\xi_n = 2i\pi e^{ix_n r_1(x,\xi')} \frac{r_1^\mu(x,\xi')}{r_1(x,\xi') - r_2(x,\xi')}$$

We find the model case and we have a relation between the two traces

- If $\operatorname{Im} r_1(x,\xi') > 0$ and $\operatorname{Im} r_2(x,\xi') > 0$ no relations between the two traces

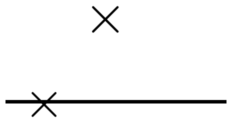
BOUNDARY CONDITIONS AND ROOTS LOCALISATION

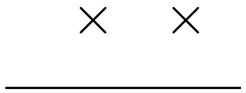


One relation between the two traces $u_{|x_n=0}$ and $D_{x_n}u_{|x_n=0}$.

The two traces $u_{|x_n=0}$ and $D_{x_n}u_{|x_n=0}$ are determined.

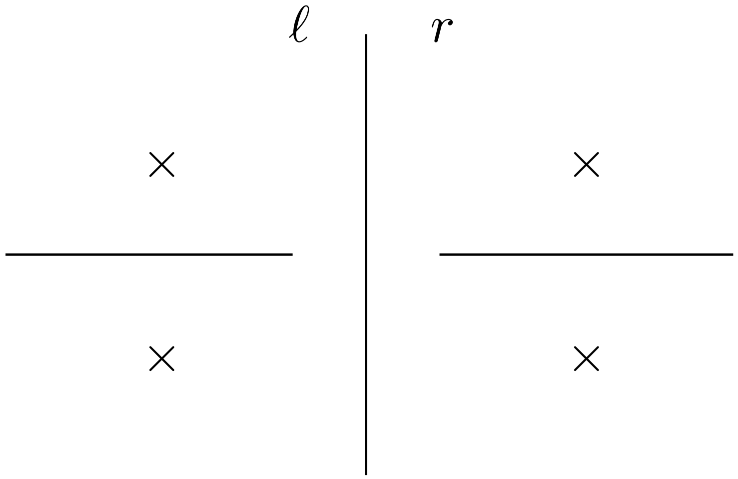
Carleman method $\longrightarrow \simeq$ One relation between the two traces.





No relation between two traces $u_{|x_n=0}$ and $D_{x_n}u_{|x_n=0}$

RETURN TO COUPLED BOUNDARY PROBLEM



- This case exists if $|\xi'|$ large.

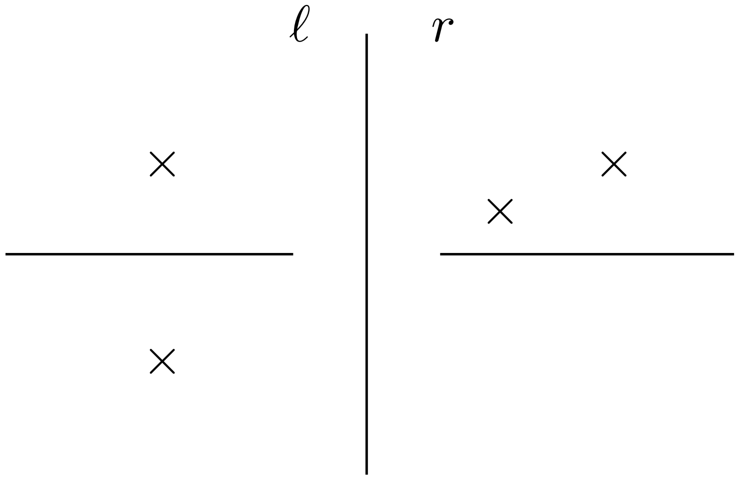
- One relation between the traces at left.

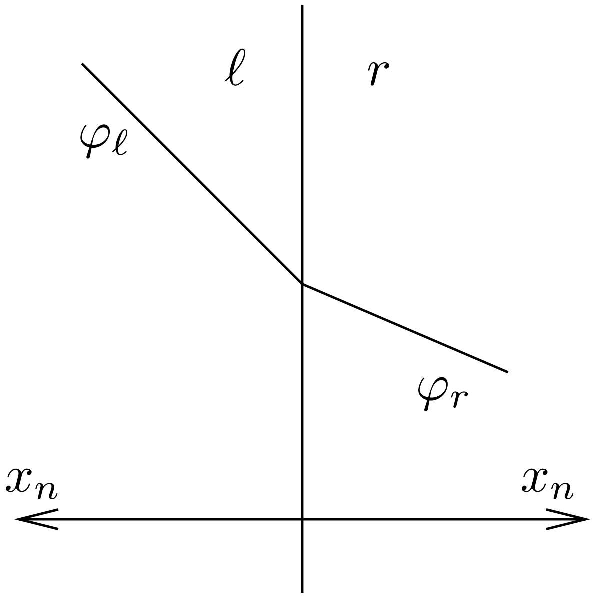
- One relation between the traces at right.

- Two transmission conditions.

$\longrightarrow$ 4 equations for 4 unknown and ... OK (algebraic checking)

A BAD CASE





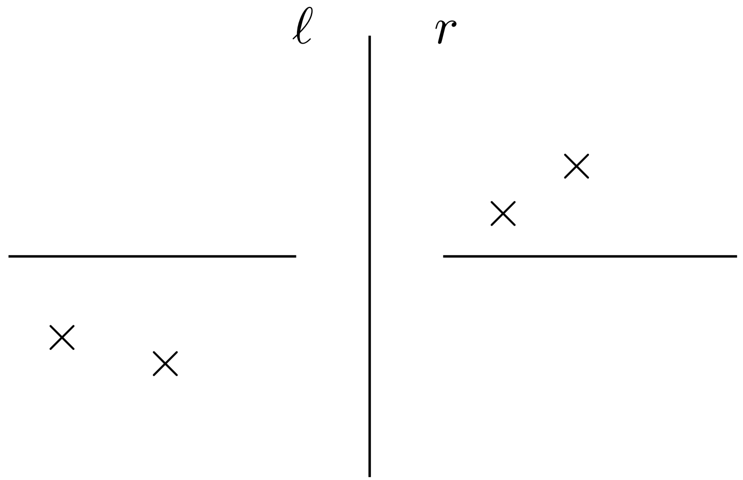
- One relation between the traces at left.

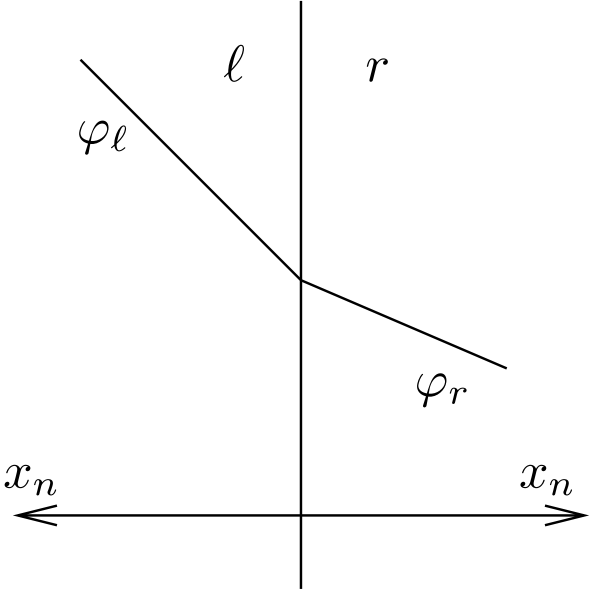
- No condition between the traces at right.

- Two transmission conditions.

$\longrightarrow$ 3 equations for 4 unknown !!!! Impossible to solve.

AVOID BAD CASE





Choose parameters such that the two roots at the left are in $\mathrm{Im}\, z < 0$ when the two roots at the right are in $\mathrm{Im}\, z > 0$.

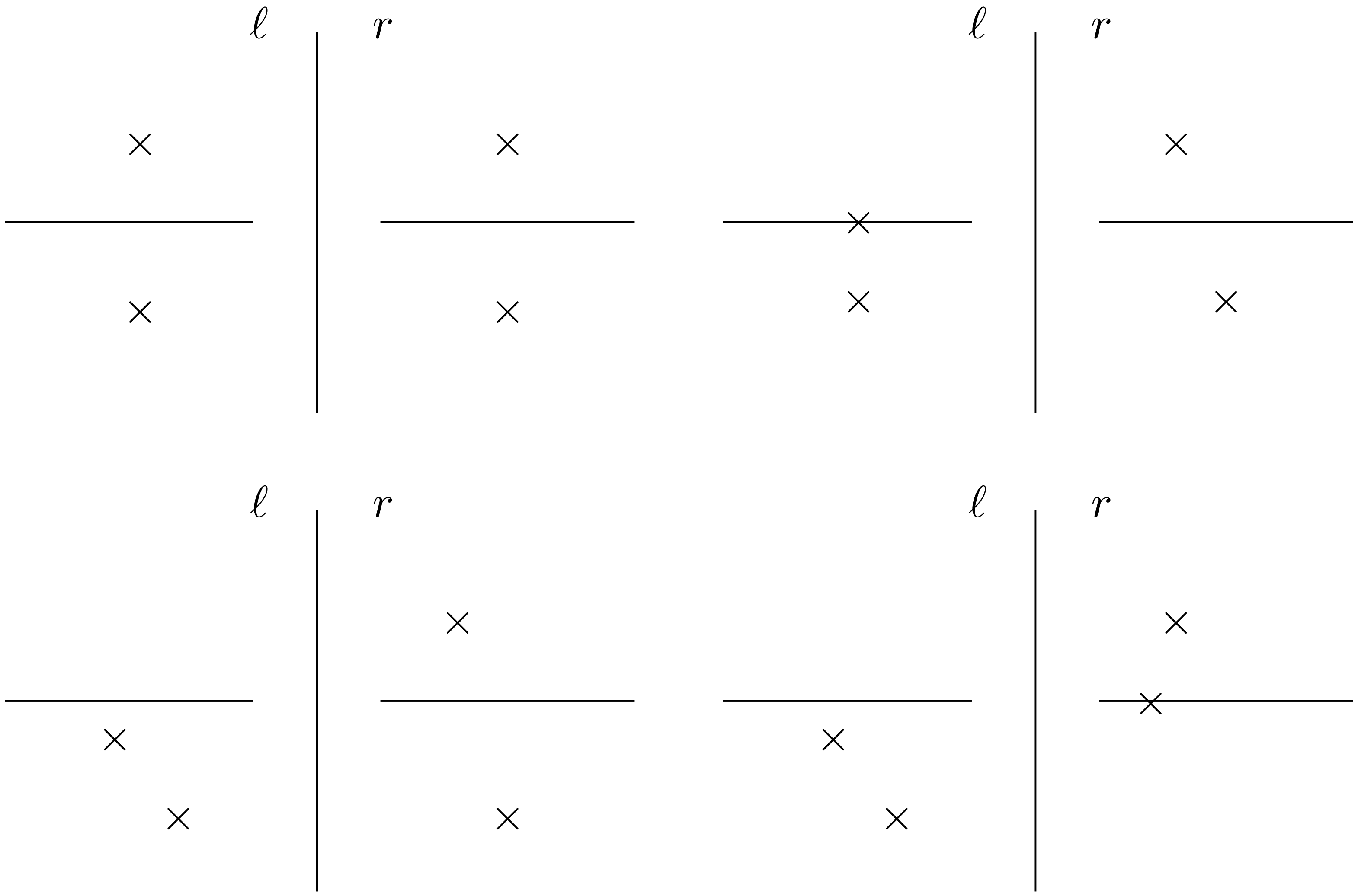
- The traces at left are known $\simeq$ two equations

- No condition between the traces at right.

- Two transmission conditions

$\longrightarrow$ 4 equations for 4 unknown and ... OK.

DIFFERENT CASES, MOVIE



COMPUTATIONS WITH FURSIKOV-IMANUVILOV WEIGHTS

Computation $e^{\frac{\varphi}{ht(T-t)}}P(e^{\frac{-\varphi}{ht(T-t)}}u)$ où $P = h^2(\partial_t - \Delta)$.

Symbol rule $\xi \leftrightarrow \frac{h}{i}\partial_x$ et $\tau \leftrightarrow \frac{h^2}{i}\partial_t$.

$$\text{Symbol} = i\tau + |\xi|^2 + 2i\frac{\xi\varphi'}{t(T-t)} - \frac{|\varphi'|^2}{t^2(T-t)^2} - h\frac{(T-2t)\varphi}{t^2(T-t)^2}.$$

Recall : we want distinguish elliptic zone and “Carleman” zone.

→ Symbolic calculus in τ and ξ.

Each term must be of the same “strength” τ, $|\xi|^2$ et $\dfrac{1}{t^2(T-t)^2}$.

$$\text{Weight } \left(|\tau| + |\xi|^2 + \frac{1}{t^2(T-t)^2}\right)^{1/2}$$

“Bad symbolic calculus ” in Hörmander sense : singularities in $t = 0$ et $t = T$.

GOOD SYMBOLIC CALCULUS

Transformation $s(t) = \tan(\frac{\pi t}{T} - \frac{\pi}{2})$

goal $t \longrightarrow T^-$ then $s \longrightarrow +\infty$ and $t \longrightarrow 0^+$ then $s \longrightarrow -\infty$.

$\partial_t \longrightarrow a(s)\partial_s$ where $a(s) \sim \langle s\rangle^2$

$\eta(t) = \frac{1}{t(T-t)}$ in variable "s" we have $\eta(s) \sim \langle s\rangle$.

Symbol $= ia(s)\tau + |\xi|^2 + 2i\xi\varphi'\eta(s) - |\varphi'|^2\eta^2(s) - h\varphi\eta^2(s)$.

Roughly speaking $i\langle s\rangle^2\tau + |\xi|^2 + 2i\xi\langle s\rangle - \langle s\rangle^2$

Adapted weight $\Lambda^4 = \langle s\rangle^4 + \langle s\rangle^4\tau^2 + |\xi|^4$.

Symbol of order Λ^2.

Calculus in $S(\Lambda^2, g)$ where $g = \frac{ds^2}{\langle s\rangle^2} + dx^2 + \frac{\langle s\rangle^4 d\tau^2}{\Lambda^4} + \frac{d\xi^2}{\Lambda^2}$.

Good analysis for elliptic points.

CARLEMAN AND SYMBOLIC CALCULUS

Fact : previous calculus are not adapted for Carleman method.

Real part $= |\xi|^2 - |\varphi'|^2\eta^2(s) - h\varphi\eta^2(s)$.

Roughly speaking $|\xi|^2 - \langle s\rangle^2$.

Imaginary part $= a(s)\tau + 2\xi\varphi'\eta(s)$. Roughly speaking $\langle s\rangle^2\tau + \xi\langle s\rangle$

We must balance the two terms of real part and imaginary part.

Goal search a weight giving the same "strength" to $|\xi|^2$ and $\langle s\rangle^2$

and the same "strength" to $\langle s\rangle^2\tau$ and $\langle s\rangle\xi$

Adapted weight : $M^2 = \langle s\rangle^2 + \langle s\rangle^2\tau^2 + |\xi|^2$.

Good calculus $S(M^2, \tilde{g})$ where $\tilde{g} = \frac{ds^2}{\langle s\rangle^2} + dx^2 + \frac{\langle s\rangle^2 d\tau^2}{M^2} + \frac{d\xi^2}{M^2}$.

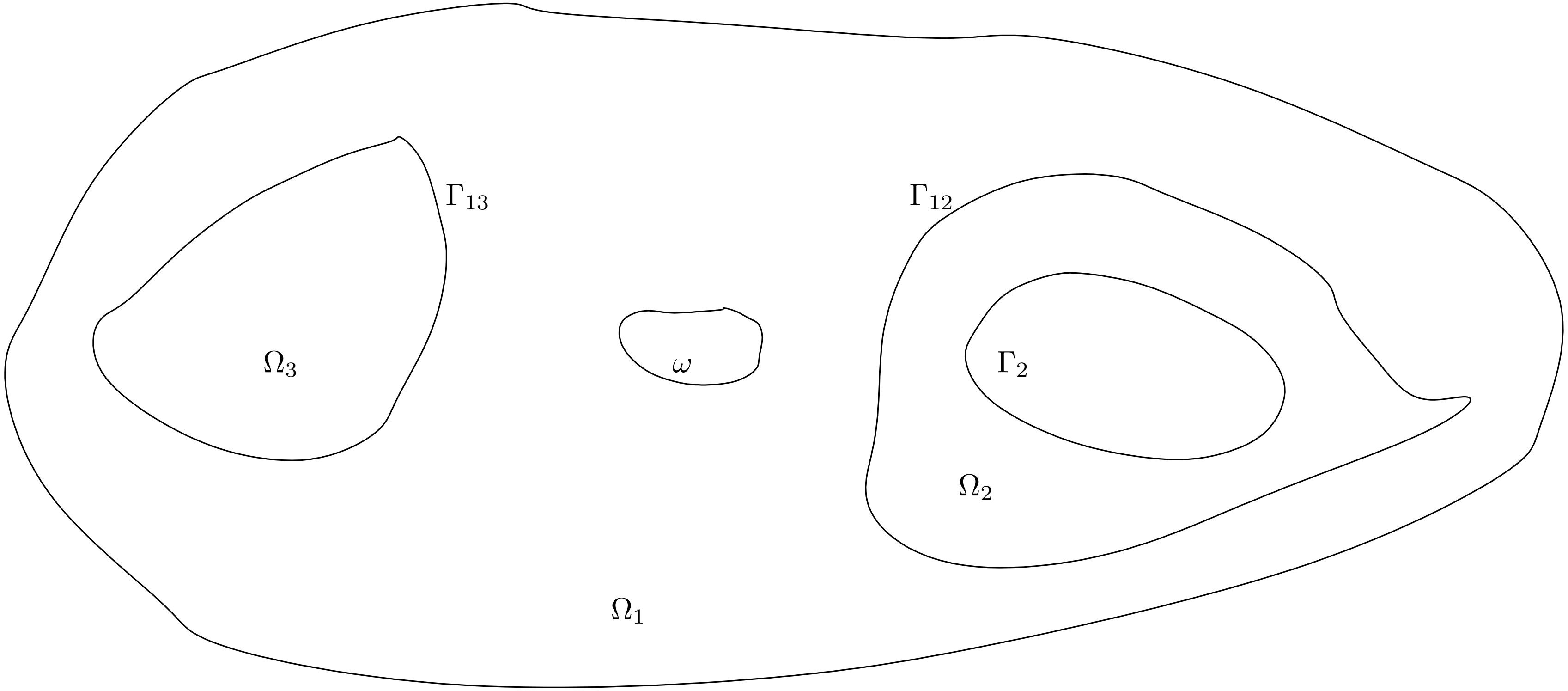
Real part of order M^2, imaginary part of order $\langle s\rangle M$.

The Poisson's bracket (symbol of the commutator) : order $\langle s\rangle M^2$.

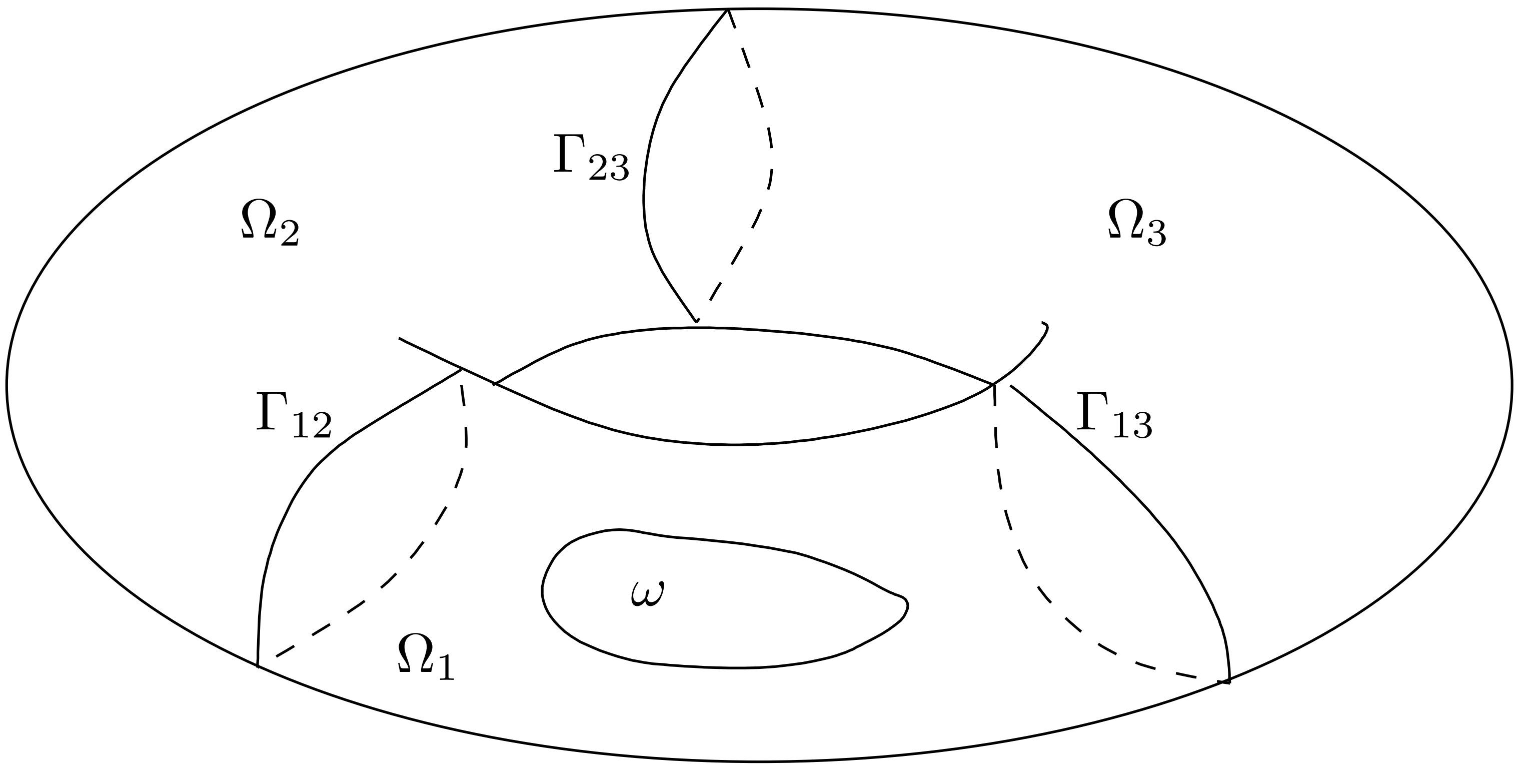
SOME PROBLEMS TO FINISH THE CALCULUS

- Managed to kind of estimates, elliptic zone and “Carleman” zone.
- Glued estimates → need several weight.

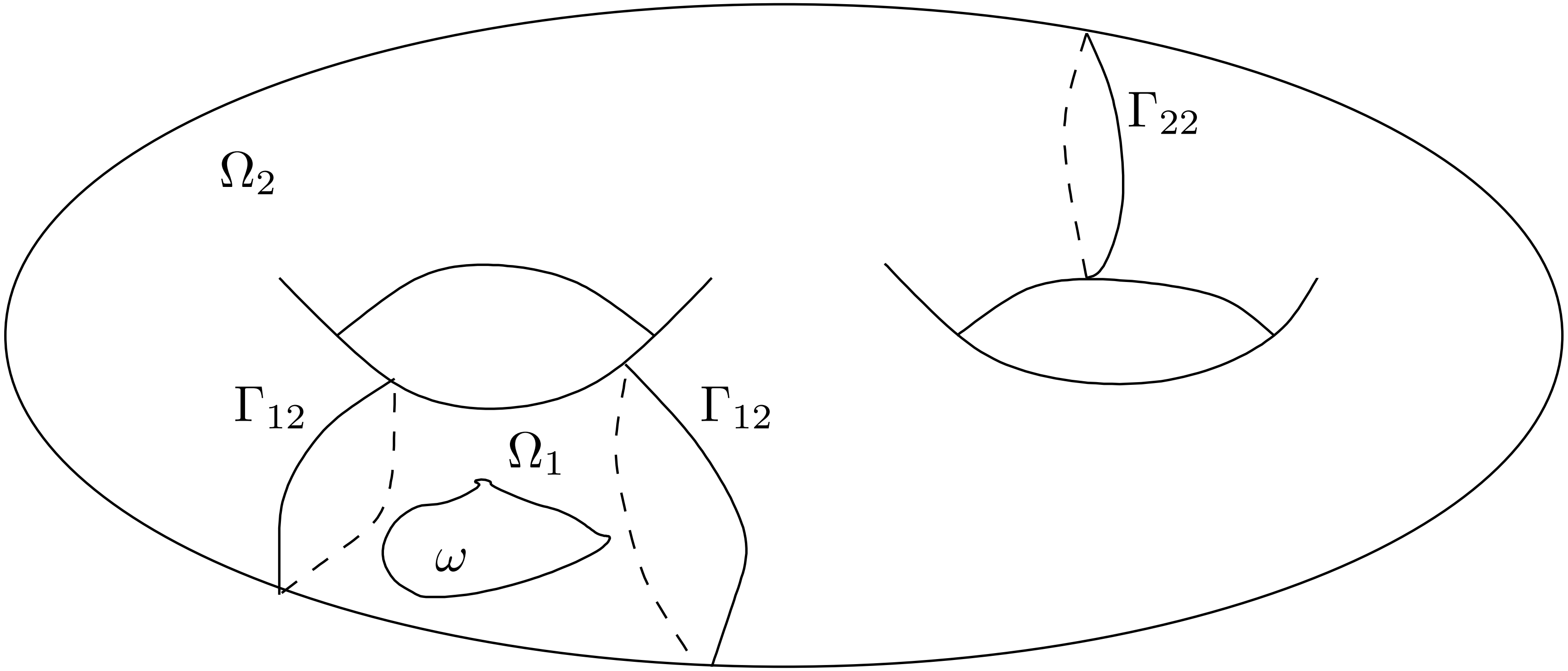
EXAMPLE : AN OPEN



TORUS WITH ONE HOLE



TORUS WITH 2 HOLES



OPEN PROBLEM