Energy Decay in a Timoshenko-type System with History in Thermoelasticity of Type III

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I. Introduction

In 1921, **Timoshenko** gave the following system of coupled hyperbolic equations

$$\begin{aligned} \rho u_{tt} &= (K(u_x - \varphi))_x \\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_x - \varphi) \end{aligned} \tag{1}$$

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- The coefficients ρ, I_ρ, are the density and the polar moment of inertia of a cross section,
- E, I and K are respectively ;Young's modulus, the moment of inertia, and the shear modulus

An important issue of research is to look for a **minimum dissipation** by which solutions of system (1) decay uniformly to the stable state as time goes to infinity. In this regards, several types of dissipative mechanisms have been introduced

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- Boundary damping

- memory damping $\int_0^t g(t-s)\varphi_{xx}(s)ds$ or history damping $\int_0^\infty g(t)\varphi_{xx}(t-s,.)ds$

- **Kim and Renardy** considered (1) together with two boundary control of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t), \quad \forall t \geq 0 \\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t), \quad \forall t \geq 0 \end{aligned}$$

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- **Raposo , Ferreira , Santos , and Castro**, studied (1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings. Precisely, they looked into the following system

$$\begin{array}{l} \rho_1 u_{tt} - K(u_x - \varphi)_x + u_t = 0, \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + K(u_x - \varphi) + \varphi_t = 0, \\ u(0,L) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \end{array} \tag{2}$$

  in $(0,L)\times(0,+\infty)$, and proved that the energy associated with (2) decays exponentially.

Soufyane and Wehbe showed that it is possible to stabilize uniformly (1) by using a unique locally distributed feedback. So, they considered

$$\begin{array}{l} \rho u_{tt} = (K(u_x - \varphi))_x, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, \\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \end{array} \tag{3}$$

in $(0, L) \times (0, +\infty)$ where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\, x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (3) holds if and only if the **wave speeds are equal**

$$\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$$

otherwise only the asymptotic stability has been proved.

Ammar-Khodja., Benabdallah , Muñoz Rivera . and Racke , considered a linear Timoshenko-type system with memory of the form

$$\begin{array}{l} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{array} \qquad (4)$$

in $(0, L) \times (0, +\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal

$$\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$$

and g decays uniformly. Precisely, they proved an **exponential** decay if g decays in an **exponential** rate and **polynomially** if g decays in a **polynomial** rate. They also required some extra technical conditions on both g' and g'' to obtain their result.

For Timenshinko systems in classical thermoelasticity, **Rivera and Racke** considered

$$\begin{array}{l} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta_x = 0 \\ \rho_3 \theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 \end{array} \qquad (5)$$

in $(0, \infty) \times (0, L)$ where φ, ψ,and θ are functions of (x, t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and non exponential stability result for the case of different wave speeds.

Recently, **Rivera and Fernández Sare** , concedered Timoshenko type system with past history acting only in one equation. More precisely they looked into the following problem

$$\begin{array}{l} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^\infty g(t)\psi_{xx}(t-s,.)ds + K(\varphi_x + \psi) = 0 \end{array} . \tag{6}$$

They showed that the dissipation given by the **history term** is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also, proved that the solution decays polynomially for the case of different wave speeds.

In the system

$$\begin{array}{l} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\theta_x = 0 \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 \end{array}$$

in $(0,\infty)\times(0,L)$ the **heat flux** is given by **Fourier's law**. As a result, this theory predicts an infinite speed of heat propagation. That is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed.

This phenomenon in dielectric crystals is called second sound. To overcome this physical paradox, many theories have merged such as **thermoelastcity of second sound** or **thermoelasticity type III**.

- We recall here that the **type III thermoelasticity** characterized by the following constitutive equations for the heat flux

$$q = -\kappa^* \tau_x - \widetilde{\kappa}\theta_x$$

where θ denotes the temperature, τ is the thermal displacement which satisfies $\tau_t = \theta$, and $\kappa^*, \widetilde{\kappa}$ are positives constants.

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} - \int_0^\infty \mu(s)\psi(x, t-s)\,ds + K(\varphi_x + \psi) + \beta\theta_x = 0 \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 \end{cases} . \quad (7)$$

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in $(0, \infty) \times (0, 1)$.

- with the following initial and boundary conditions

$$\begin{array}{rcl} \varphi\left(.,0\right) & = & \varphi_{0},\ \varphi_{t}\left(.,0\right) = \varphi_{1},\ \psi\left(t.,0\right) = \psi_{0},\ \psi_{1}\left(.,0\right) = \psi_{1}, \\ \theta\left(.,0\right) & = & \theta_{0},\ \theta_{t}\left(.,0\right) = \theta_{1} \end{array} \quad (8)$$

$$\varphi\left(0, t\right) = \varphi\left(1, t\right) = \psi\left(0, t\right) = \psi\left(1, t\right) = \theta\left(0, t\right) = \theta\left(1, t\right). \quad (9)$$

- we will show that, for

$$\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$$

the first energy **decays exponentially** (respectively **polynomially**) if g decays **exponentially** (respectively **polynomially**).

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- In the case of different wave speeds,

$$\left(\frac{K}{\rho_1} \neq \frac{b}{\rho_2}\right)$$

  we show that the decay is of **polynomial** type

- We introduce the new variable

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$$\begin{aligned} \eta^{t}(x,0) &= 0,\ \ \forall t\geq 0 && (11)\\ \eta^{t}(0,s) &= \eta^{t}(1,s)=0,\ \ \forall s,t\geq 0 && (12)\\ \eta^{0}(x,s) &= \eta_{0}(s),\ \ \forall s\geq 0. && (13) \end{aligned}$$

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- Clearly, (10) gives us

$$\eta^t_t(x,s) + \eta^t_s(x,s) = \psi_t(x,t). \tag{14}$$

- Concerning the kernel g, we assume the following hypothesis

$$g(t) > 0 : g^{'}(t) \leq -k_0 g(t)^p, \ \forall t \geq 0, \tag{15}$$

$$\widehat{b} = b - \int_0^\infty g\,(s)\,ds = b - g_0 > 0. \tag{16}$$

for a positive constant k_0 and for $1 \leq p < 3/2$.

$$G_0 = \int_0^\infty g(s)^{1/2} ds < \infty, \ G_p = \int_0^\infty g(s)^{2-p} ds < \infty, \ 1 \leq p < 3/2. \tag{17}$$

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- Under the above conditions , it is easy to verify that

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$\frac{\rho_1}{K} = \frac{\rho_2}{b}$

- Now, we are concerned with the following new problem

$$\begin{cases} \rho_1 \varphi_{tt} - K \left(\varphi_x + \psi \right)_x = 0 \\ \rho_2 \psi_{tt} - \widehat{b} \psi_{xx} - \int_0^\infty g \left(s \right) \eta_{xx} \left(x, s \right) ds + K \left(\varphi_x + \psi \right) + \beta \theta_x = 0 \\ \rho_3 \theta_{tt} - \delta \theta_{xx} + \gamma \psi_{ttx} - k \theta_{txx} = 0 \\ \eta_t^t \left(x, s \right) + \eta_s^t \left(x, s \right) - \psi_t \left(x, t \right) = 0 \end{cases} \tag{18}$$

in $(0, 1)$, for any $t \geq 0$ and any $s \geq 0$. This system is subjected to the initial and boundary conditions $(8) - (9)$ and $(11) - (13)$.

- In order to exhibit the dissipative nature of system (19) we introduce the new variables $\phi = \varphi_t$, $\Psi = \psi_t$, and $\widehat{\eta}^t = \widehat{\eta}_t$. Thus we are concerned with the following problem

$$\begin{cases} \rho_1\phi_{tt} - K\left(\phi_x + \Psi\right)_x = 0 \\ \rho_2\Psi_{tt} - \widehat{b}\Psi_{xx} - \int_0^\infty g\left(s\right)\widehat{\eta}^t_{xx}\left(x,s\right)ds + K\left(\phi_x + \Psi\right) + \beta\theta_{tx} = 0 \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\Psi_{tx} - k\theta_{txx} = 0 \\ \widehat{\eta}^t_t + \widehat{\eta}^t_s - \Psi_t = 0 \end{cases} \qquad (19)$$

where $x \in (0,1)$, $t \geq 0$ and $s \geq 0$

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where $x \in (0,1)$, $t \geq 0$ and $s \geq 0$

- We consider the following initial and boundary conditions

$$\begin{aligned} \phi(.,0) &= \phi_0,\ \phi_t(.,0) = \phi_1,\ \Psi(t.,0) = \Psi_0,\ \Psi_1(.,0) = \Psi_1, \\ \theta(.,0) &= \theta_0,\ \theta_t(.,0) = \theta_1 \end{aligned} \tag{20}$$

- In order to use the Poincaré inequality for θ, let us introduce

$$\overline{\theta} = \theta\left(x, t\right) - t \int_0^1 \theta_1\left(x\right) dx - \int_0^1 \theta_0\left(x\right) dx$$

Then by $(19)_3$ we have

$$\int_0^1 \overline{\theta}\left(x, t\right) dx = 0$$

for all $t \geq 0$,

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for all $t \geq 0$,

- in this case the Poincaré inequality for $\overline{\theta}$ is applicable.
- In the sequel we shall work with $\overline{\theta}$ but for simplicity we write θ instead of $\overline{\theta}$. Then the associated energy of first order is defined by

$$\begin{aligned} E(t) &= \frac{\gamma}{2}\int_0^1 \left(\rho_1\phi_t^2 + \rho_2\Psi_t^2 + K\left|\phi_x + \Psi\right|^2 + \widehat{b}\Psi_x^2\right)dx \\ &\quad + \frac{\beta}{2}\int_0^1 \left(\rho_3\theta_t^2 + \delta\theta_x^2\right)dx + \frac{\gamma}{2}\int_0^1\int_0^\infty g(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx. \end{aligned} \tag{21}$$

Theorem 2.1

Suppose that

$$\frac{\rho_1}{K} = \frac{\rho_2}{b} \tag{22}$$

and

$$\phi_0, \Psi_0,\ \theta_0, \widehat{\eta}_0^t \in H_0^1(0,1),\ \widehat{\eta}_0^t \in L_g^2\left(\mathbb{R}^+; H_0^1(0,1)\right),\ \phi_1, \Psi_1,\ \theta_1 \in L^2(0,1).$$

Then there exist two positive constants C *and* ξ *such that*

$$E(t) \leq Ce^{-\xi t},\ \forall t > 0,\ p = 1 \tag{23}$$

$$E(t) \leq \frac{C}{(1+t)^{1/(p-1)}},\ p > 1. \tag{24}$$

Lemma 2.1

Let $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* (19). *Then we have*

$$\frac{dE(t)}{dt} = -\beta k \int_0^1 \theta_{tx}^2 dx + \frac{\gamma}{2} \int_0^1 \int_0^\infty g'(s) \left|\widehat{\eta}_x^t(s)\right|^2 dsdx \qquad (25)$$

Proof.

Multiplying equation $(19)_1$ by $\gamma\phi_t$, $(19)_2$ by $\gamma\Psi_t$ and $(19)_3$ by $\beta\theta_t$, integrating over $(0,1)$ and summing up we obtain (25). □

Lemma 2.1

- *Let* $\left(\phi, \Psi, \theta, \widehat{\eta}^{t}\right)$ *be a solution of* (19). *Then we have for* $1 < p < 3/2$

$$\left(\int_0^1 \int_0^{\infty} g\left(s\right) \left|\widehat{\eta}_x^t\left(s\right)\right|^2 dsdx\right)^{2-p} \leq C_0 \int_0^1 \int_0^{\infty} g\left(s\right)^p \left|\widehat{\eta}_x^t\left(s\right)\right|^2 dsdx \tag{26}$$

for a constant $C_0 > 0$.

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for a constant $C_0 > 0$.

- *For* $1 \leq p < 3/2$, *we have*

$$\int_0^1 \left(\int_0^\infty g\left(s\right) \widehat{\eta}_x^t\left(s\right) ds\right)^2 dx \leq G_p \int_0^1 \int_0^\infty g\left(s\right)^p \left|\widehat{\eta}_x^t\left(s\right)\right|^2 dsdx. \tag{27}$$

- Let
$$I_1 := \int_0^1 (\rho_2 \Psi_t \Psi + \rho_1 \phi_t \omega)\, dx. \quad (28)$$
where ω is the solution of
$$-\omega_{xx} = \Psi_x, \quad \omega(0) = \omega(1) = 0, \quad (29)$$

Lemma

Let $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* (19). *Then we have, for any* $\lambda_1 > 0$

$$\frac{dI_1(t)}{dt} \le \left(-\frac{\widehat{b}}{2} + \lambda_1\right) \int_0^1 \Psi_x^2 dx + \varepsilon_1 \rho_1 \int_0^1 \phi_t^2 dx + \left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) \int_0^1 \Psi_t^2 dx$$
$$+ \frac{\beta^2}{2\widehat{b}} \int_0^1 \theta_{tx}^2 dx + \frac{G_p}{4\lambda_1} \int_0^1 \int_0^\infty g(s)^p \left|\widehat{\eta}_x^t(s)\right|^2 dsdx. \quad (30)$$

- Next, let us define the following functional

$$I_2 := -\rho_2 \int_0^1 \Psi_t(x,t) \int_0^\infty g(s)\,\widehat{\eta}^t(s)\,ds. \tag{31}$$

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Lemma

Let $(\phi, \Psi, \theta, \widehat{\eta}^t)$ be a solution of (19). Then we have

$$\begin{aligned} \frac{d}{dt} I_2(t) \leq & -\frac{\rho_2 g_0}{2} \int_0^1 \Psi_t^2 dx + \varepsilon_2 \widehat{b}^2 \int_0^1 \Psi_x^2 dx \\ & + \varepsilon_2 K^2 \int_0^1 (\phi_x + \Psi)^2 dx + \frac{\beta^2}{2} \int_0^1 \theta_{tx}^2 \\ & + G_p \left(1 + \frac{1}{4\varepsilon_2} + \frac{C^*}{2} + \frac{C^*}{4\varepsilon_2}\right) \int_0^1 \int_0^\infty g(s)^p \left|\widehat{\eta}_x^t(s)^2\right| dsdx \\ & - \frac{C^* g(0)}{2\rho_2} \int_0^1 \int_0^\infty g(s)' \left|\widehat{\eta}_x^t(s)^2\right| dsdx \qquad (32) \end{aligned}$$

where C^* is the Poincaré constant.

Let us define

$$J(t) := \rho_2 \int_0^1 \Psi_t (\phi_x + \Psi)\, dx + \frac{\rho_1 \widehat{b}}{K} \int_0^1 \Psi_x \phi_t dx + \frac{\rho_1}{K} \int_0^1 \phi_t(t) \int_0^\infty g(s)\, \widehat{\eta}_x^t(s)\, ds dx \tag{33}$$

Lemma

Let $(\phi, \Psi, \theta, \widehat{\eta})$ *be a solution of* (19). *Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{\widehat{b} + g_0}. \tag{34}$$

Then we have

$$\begin{aligned} \frac{dJ(t)}{dt} \leq & \left[\phi_x \left(b\Psi_x + \int_0^\infty g(s)\, \widehat{\eta}_x^t(x,s) \right)\right]_{x=0}^{x=1} \\ & -\frac{K}{2}\int_0^1 (\phi_x + \Psi)^2\, dx + \rho_2 \int_0^1 \Psi_t^2 dx \\ & +\varepsilon_3 \int_0^1 \phi_t^2 dx + \frac{\beta^2}{2K}\int_0^1 \theta_{tx}^2 dx \\ & +C(\varepsilon_3)\, g_0 \int_0^1 \int_0^\infty g(s)' \left|\widehat{\eta}_x^t(s)\right|^2 dsdx \end{aligned}$$

Lemma

Let $(\phi, \Psi, \theta, \widehat{\eta})$ *be a solution of* (19). *Then we have, for any* $\varepsilon_3 > 0$

$$\begin{aligned}
&\left[\phi_x \left(\widehat{b}\Psi_x + \int_0^\infty g(s)\,\widehat{\eta}_x^t(s,x)\right)\right]_{x=0}^{x=1} \\
\leq\;& \frac{-\varepsilon_3}{K}\frac{d}{dt}\int_0^1 \rho_1 q\phi_t\phi_x dx + 3\varepsilon_3\int_0^1 \phi_x^2 dx + \frac{2\rho_1\varepsilon_3}{K}\int_0^1 \phi_t^2 dx \\
&-\frac{1}{4\varepsilon_3}\frac{d}{dt}\int_0^1 \rho_2 q(x)\,\Psi_t\left(\widehat{b}\Psi_x + \int_0^\infty g(s)\,\widehat{\eta}_x^t(s)\,ds\right) \\
&+\frac{1}{\varepsilon_3}\left(\widehat{b}^2 + \frac{\widehat{b}^2}{8\lambda_3} + \frac{\widehat{b}^2\lambda_3}{2} + \varepsilon_3^2\right)\int_0^1 \Psi_x^2 dx + \frac{\beta^2}{4\varepsilon_3\lambda_3}\int_0^1 \theta_{tx}^2 dx \\
&-\frac{G_0}{4\varepsilon_3}\left(4 + \frac{1}{2\lambda_3} + 2\lambda_3\right)\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx \qquad (35) \\
&+\frac{\rho_2}{4\varepsilon_3}(2b+\varepsilon_3)\int_0^1 \Psi_t^2 dx + \frac{K^2\lambda_3}{\varepsilon_3}\int_0^1 (\phi_x + \Psi)^2 dx \\
&-\rho_2 g(0)\,C(\varepsilon_3)\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx.
\end{aligned}$$

where $q(x) = 2 - 4x$.

- Let's introduce the functional

$$\mathcal{K}(t) := -\rho_1 \int_0^1 \phi_t \phi dx - \rho_2 \int_0^1 \Psi_t \Psi dx.$$

It easily follows, by using the Poincaré's inequality $\int_0^1 \Psi^2 dx \leq \int_0^1 \Psi_x^2 dx$

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) \leq & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_2 \int_0^1 \Psi_t^2 dx + \left(\widehat{b} + \frac{3}{2}\right) \int_0^1 \Psi_x^2 dx \\ & + K \int_0^1 \phi_x^2 dx + \frac{\beta^2}{2} \int_0^1 \theta_{tx}^2 dx \\ & - \int_0^1 \Psi_x \int_0^\infty g(s) \widehat{\eta}_x^t(s) \, ds dx \end{aligned} \quad (36)$$

Then,

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) \leq & -\rho_1 \int_0^1 \phi_t^2 dx - \rho_2 \int_0^1 \Psi_t^2 dx \\ & + \left(\widehat{b} + \frac{3}{2} + \varepsilon_3 \right) \int_0^1 \Psi_x^2 dx \\ & + K \int_0^1 \phi_x^2 dx + \frac{\beta^2}{2} \int_0^1 \theta_{tx}^2 dx \\ & + C(\varepsilon_3) G_p \int_0^1 \int_0^\infty g(s)^p \left| \widehat{\eta}_x^t(s) \right|^2 dsdx \end{aligned} \quad (37)$$

- Next, let

$$\Theta(t) := \int_0^1 \left(\rho_3 \theta_t \theta + \frac{k}{2} \theta_x^2 + \gamma \Psi_x \theta \right) dx,$$

Lemma

Let $\left(\phi, \Psi, \theta, \widehat{\eta}^{t}\right)$ *be a solution of* (19). *Then we have, for any* $\varepsilon_2 > 0$

$$\frac{d}{dt}\Theta\left(t\right) \leq -\delta\int_0^1 \theta_x^2 dx + \left(\rho_3 + \frac{\gamma^2}{4\varepsilon_2}\right)\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \Psi_x^2 dx. \quad (38)$$

- Now we are able to define the Lyapunov functional $\mathcal{L}$ as follows

$$\begin{aligned}\mathcal{L}\left(t\right) \quad : \quad &= NE\left(t\right) + N_1 I_1 + N_2 I_2 + J\left(t\right) + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\phi_t\phi_x dx \\ &+\frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q\left(x\right)\Psi_t\left(\widehat{b}\Psi_x + \int_0^\infty g\left(s\right)\widehat{\eta}_x^t\left(s\right)ds\right) \\ &+\mu\mathcal{K}\left(t\right) + \Theta\left(t\right). \qquad (39)\end{aligned}$$

- Consequently we have the following result:

Lemma

There exist C_1 and C_2 two positive constants such that

$$C_1 E(t) \leq \mathcal{L}(t) \leq C_2 E(t). \tag{40}$$

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Lemma

There exist C_1 and C_2 two positive constants such that

$$C_1 E(t) \leq \mathcal{L}(t) \leq C_2 E(t). \tag{40}$$

- By using all the above Lemmas and the following two inequalities

$$\int_0^1 \theta_t^2 dx \leq \int_0^1 \theta_{tx}^2 dx$$

and

$$\int_0^1 \phi_x^2 dx \leq 2\int_0^1 (\phi_x + \Psi)^2 dx + 2\int_0^1 \Psi_x^2 dx$$

we find

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq\ & -[N\beta k - C_1]\int_0^1 \theta_{tx}^2 dx + \Lambda_1 \int_0^1 \Psi_x^2 dx + \Lambda_2 \int_0^1 \phi_t^2 dx \\ & +\Lambda_3 \int_0^1 \Psi_t^2 dx + \Lambda_4 \int_0^1 (\phi_x + \Psi)^2 dx \qquad (41) \\ & + \left(N\frac{\gamma}{2} - C_2\right)\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx - \delta\int_0^1 \theta_x^2 dx \\ & + \left(\rho_3 + \frac{\gamma^2}{4\varepsilon_2}\right)\int_0^1 \theta_t^2 dx + C_3\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx,\end{aligned}$$

where C_1, C_2, C_3 are all positives constants.

We choose the above constants carefully, and by using the inequality $g^{'}(t) \leq -k_0 g(t)^p$, we obtain finally

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq & -\sigma_1 \left[\int_0^1 \theta_t^2 dx + \int_0^1 \theta_x^2 dx + \int_0^1 \Psi_x^2 dx + \int_0^1 \phi_t^2 dx + \int_0^1 \Psi_t^2 dx \right. \\ & \left. \int_0^1 (\phi_x + \Psi)^2 dx + \int_0^1 \int_0^\infty g^p(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx\right]\end{aligned}$$

Here we have two cases:

- **Case 1:** $p = 1$

the following inequality

$$\frac{d}{dt}\mathcal{L}(t) \leq -c\mathcal{L}(t), \quad \forall t \geq 0 \tag{42}$$

holds.
Consequently we obtain the exponential decay of $\mathcal{L}(t)$ and since $\mathcal{L}(t)$ and $E(t)$ are equivalent the exponential decay of the energy $E(t)$ also holds.

- **Case 2:** $p > 1$

In this case we have the following inequality

$$\frac{d}{dt}\mathcal{L}(t) \leq -cE(t)^{2p-1} \leq -c\mathcal{L}(t)^{2p-1}.$$

Then, a simple integration of the above inequality leads to

$$\mathcal{L}(t) \leq \frac{c}{(t+1)^{1/(2p-2)}}.$$

By doing some calculations then the second part of Theorem 2.1 holds.

Polynomial decay

$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$

- In this section, we show that in the case of different wave-speed propagation
$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$$
the solution energy $E(t)$ decays at a polynomial rate even if the relaxation function g decays exponentially provided that the initial data are regular enough.

$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$

- In this section, we show that in the case of different wave-speed propagation
$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$$
the solution energy $E(t)$ decays at a polynomial rate even if the relaxation function g decays exponentially provided that the initial data are regular enough.
- Let's define the second-order energy by
$$E_2(t) = E_1\left(\Psi_t, \phi_t, \theta_t, \widehat{\eta}_t^t(s)\right).$$
Then, we have

Theorem 3.1

Suppose that

$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b} \tag{43}$$

and

$$\begin{aligned} \phi_0, \Psi_0,\ \theta_0, \widehat{\eta}_0^t &\in H^2 \cap (0,1)\,,\ \widehat{\eta}_0^t \in L_g^2\left(\mathbb{R}^+; H^2 \cap H_0^1(0,1)\right), \\ \phi_1, \Psi_1,\ \theta_1 &\in H_0^1(0,1)\,. \end{aligned}$$

Then there exists a positive constants C *such that, for all* $t \geq 0$*, we have*

$$E(t) \leq Ct^{-1/(2p-1)}$$

- In the thermoelasticity of second sound, **S.A. Messaoudi, M. Pokojovy, and B. Said-Houari** proved that for some kinds of dissipations the hypothesis

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}$$

is unnecessary. Can we get the same result for thermoelasticity of type III?

some open problems

- In the thermoelasticity of second sound, **S.A. Messaoudi, M. Pokojovy, and B. Said-Houari** proved that for some kinds of dissipations the hypothesis

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  is unnecessary. Can we get the same result for thermoelasticity of type III?
- Inspired by a recent works of **N.Tatar** and **V. Pata** , can we get the stability results by assuming that our kernel function is not necessary decaying?

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