Non-self-adjoint operators with small random perturbations

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1. Introduction

Consider an h-pseudodifferential operator $P = p(x, hD_x)$ where $p \in C^\infty(T^*X)$, X is either $\mathbf{R}^n$ or a compact manifold. Assume that p belongs to suitable symbol class and fulfills suitable ellipticity conditions at infinity (in (x, ξ)). Put

$$\Sigma = \overline{p(T^*X)},\ \Sigma_\infty = \{\lim_{j\to\infty} p(\rho_j);\ \rho_j \to \infty\}.$$

Using h-pseudodifferential calculus we can show that if $K \Subset \mathbf{C} \setminus \Sigma$, then for $h > 0$ small enough, the resolvent $(z - P)^{-1}$ exists and is uniformly bounded for $z \in K$. Further, if $\Omega \Subset \mathbf{C}$ is connected, not entirely contained in Σ and with closure disjoint from Σ_∞, then the spectrum of P in Ω is discrete and the number of eigenvalues in Ω is $\mathcal{O}(h^{-n})$.

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If $\Sigma \subset \{z;\, \Re z \geq 0\}$, then we can often define the semigroup $e^{-tP/h}$, $t \geq 0$ (by showing first that P is m-accretive) and establish the formula

$$e^{-tP/h} = \frac{1}{2\pi i} \int_{\gamma_a} e^{-tz/h} (z-P)^{-1} dz, \ \gamma_a = a + i\mathbf{R}$$

first for some $a < 0$.
By pushing the contour to the right (reaching positive values of a) we get (under additional assumptions)

$$e^{-tP/h} = \sum_{\{\lambda \in \sigma(P);\, \Re\lambda < a\}} e^{-t\lambda/h} \Pi_\lambda + e^{-ta/h} r(a,t),$$

where Π_λ is the spectral projection and

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Difficulty: General abstact theory only tells us that

$$\|(z-P)^{-1}\| \le \mathcal{O}(1)e^{\mathcal{O}(1)h^{-n}} \prod_{\lambda\in \mathrm{neigh}(z)\cap\sigma(P)} |z-\lambda|^{-1}, \qquad (1)$$

so Π_λ and $r(a,t)$ might have the same exponential growth and our expansion of $\exp(-tP/h)$ might be of interest only for $t \gg h^{-n}$. Also to have some profit from this formula one would like to know something more about the distribution of the eigenvalues.

Plan of this talk:

- Give a more precise version of bounds from [Dencker-Sj-Zworski] for the resolvent near the boundary of Σ.
- Explain that small random perturbations lead to Weyl asymptotics of the eigenvalues. ([Hager], [Hager-Sj], [Sj], [W. Bordeaux Montrieux], [WBM-Sj].)
- The random perturbations tend to get improved bounds on the resolvent [WBM]. These results should be further developed and can perhaps be considerably improved.

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2. Boundary estimates for the resolvent.

Assume for simplicity that $\Re p \geq 0$ and consider a point $z_0 \in (\Sigma \setminus \Sigma_\infty) \cap i\mathbf{R}$. Using the semi-classical version of the sharp Gårding inequlity we see that

$$\|(z-P)^{-1}\| \leq \mathcal{O}(\frac{1}{|\Re z|}),\ \Re z \leq -C_1 h,\ \Im z = \mathcal{O}(1).$$

Then we have the following slight improvement of one of the main results in [DeSjZw], where part 2) was first obtained in a model situation for $k = 2$ in [WBM, thesis]:

Theorem (1)

1) If $p^{-1}(z_0)$ does not contain any maximal integral curve of $H_{\Re p}$, then $\exists\, C_0$ such that

$$\|(z-P)^{-1}\| \le \frac{\mathcal{O}(1)}{h}\exp(\frac{C_0\Re z}{h}),$$

for $|\Im z - \Im z_0| \le 1/C_0$, $C_1 h \le \Re z \le \mathcal{O}(1) h \ln \frac{1}{h}$.
2) If there exists $k \in 2\mathbf{N}$ such that for every $\rho \in p^{-1}(z_0)$, we have $H^j_{\Im p}\Re p(\rho) > 0$ for some $j \le k$, then

$$\|(z-P)^{-1}\| \le \frac{\mathcal{O}(1)}{h^{k/(k+1)}}\langle\frac{\Re z}{h^{k/(k+1)}}\rangle^{-\frac{k-1}{2k}}\exp(C_0\frac{(\Re z)^{(k+1)/k}}{h}),$$

for $|\Im z - \Im z_0| \le 1/C_0$, $-h^{\frac{k}{k+1}} \le \Re z \le \mathcal{O}(1)(h\ln\frac{1}{h})^{\frac{k}{k+1}}$.

To get 2) we here construct $e^{-tP/h}$ microlocally as a Fourier integral operator with complex phase ([MeSj], [Maslov], [Kucherenko], [Treves]) and show that the norm of this operator is $\le \mathcal{O}(1)\exp(-\frac{t^{k+1}}{Ch})$.

3. Weyl asymptotics and random perturbations

In the selfadjoint case, p will be real-valued (up to terms that are $\mathcal{O}(h)$ and we neglect for brevity) and under suitable additional assumptions, P will have discrete spectrum near some given interval I and we have the Weyl asymptotic distribution of the eigenvalues in the semiclassical limit:

$$\#(\sigma(P)\cap I)=\frac{1}{(2\pi h)^n}(\mathrm{vol}\,(p^{-1}(I))+o(1)),\ \ h\to 0.$$

In the non-self-adjoint case, we do not always have Weyl asymptotics. (In the semi-classical case this would be to replace intervals in the formula above by more general sets in $\mathbf{C}$, say with smooth boundary.)

Example 1. $P=hD_x+g(x)$ on S^1. The range of $p(x,\xi)=\xi+g(x)$ is the band $\{z\in\mathbf{C};\min\Im g\le\Im z\le\max\Im g\}$ while $\sigma(P)\subset\{z;\,\Im z=(2\pi)^{-1}\int_0^{2\pi}\Im g(x)dx\}$.

Example 2. $P=(hD_x)^2+ix^2$ with $p(x,\xi)=\xi^2+ix^2$. $\sigma(P)\subset e^{i\pi/4}[0,\infty[$ while the range of p is the closed first quadrant.

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The original 1D result of Hager was generalized in many ways by Hager–Sj (Math Ann 2008), we were able to count eigenvalues also near the boundary of the range of p. One weakness of this generalization was however that the random perturbations were no more multiplicative so the perturbed operator could not be a differential one but rather a pseudodifferential operator. By using an implicit complex analysis approach also to the probabilistic part, I was able to treat multiplicative random perturbations in any dimension and got the following result still a little technical to state.

Let X be a compact n-dimensional manifold,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x; h)(hD)^\alpha, \tag{2}$$

Assume

$$\begin{aligned} a_\alpha(x; h) &= a_\alpha^0(x) + \mathcal{O}(h) \text{ in } C^\infty, \\ a_\alpha(x; h) &= a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha| = m. \end{aligned} \tag{3}$$

Let

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \tag{4}$$

Assume that P is elliptic,

$$|p_m(x, \xi)| \geq \frac{1}{C} |\xi|^m, \tag{5}$$

and that $p_m(T^*X) \neq \mathbf{C}$.

Let $p = \sum_{|\alpha|\le m} a_\alpha^0(x)\xi^\alpha$ be the semi-classical principal symbol. We make the symmetry assumption

$$P^* = \Gamma P \Gamma, \tag{6}$$

where P^* denotes the complex adjoint with respect to some fixed smooth positive density of integration and Γ is the antilinear operator of complex conjugation; $\Gamma u = \overline{u}$. Notice that this assumption implies that

$$p(x,-\xi) = p(x,\xi). \tag{7}$$

Let $V_z(t) := \mathrm{vol}\left(\{\rho \in T^*X;\ |p(\rho)-z|^2 \le t\}\right)$. For $\kappa \in]0,1]$, $z \in \mathbf{C}$, we consider the non-flatness property that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \le t \ll 1. \tag{8}$$

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Random potential:

$$q_\omega(x) = \sum_{0<h\mu_k\le L} \alpha_k(\omega)\epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \le R, \tag{9}$$

where ϵ_k is the orthonormal basis of eigenfunctions of $\widetilde{R}$, where $\widetilde{R}$ is an h-independent positive elliptic 2nd order operator on X with smooth coefficients. Moreover, $\widetilde{R}\epsilon_k = \mu_k^2\epsilon_k$, $\mu_k > 0$.

We choose $L = L(h)$, $R = R(h)$ in the interval

$$h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \le h^{-M}, \qquad M \ge \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}, \tag{10}$$

$$\frac{1}{C}h^{-(\frac{n}{2}+\epsilon)M+\kappa-\frac{3n}{2}} \le R \le Ch^{-\widetilde{M}}, \qquad \widetilde{M} \ge \frac{3n}{2}-\kappa+(\frac{n}{2}+\epsilon)M,$$

for some $\epsilon \in]0, s-\frac{n}{2}[$, $s > \frac{n}{2}$. Put $\delta = \tau_0 h^{N_1+n}$, $0 < \tau_0 \le \sqrt{h}$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \tag{11}$$

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The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{12}$$

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$\mathbb{P}(d\alpha) = C(h) e^{\Phi(\alpha;h)} L(d\alpha), \tag{13}$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{14}$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0, R)$ is equal to 1.) We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \tag{15}$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small.

Theorem (Sj 2008)

Let $\Gamma \Subset \mathbf{C}$ have smooth boundary, let $\kappa \in]0,1]$ be the parameter in (9), (10), (15) and assume that (8) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then, for $C^{-1} \geq r > 0$, $\widetilde{\epsilon} \geq C\epsilon_0(h)$ we have with probability

$$\geq 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_5+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \quad (16)$$

that:

$$|\#(\sigma(P_\delta)\cap\Gamma) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma))| \leq \quad (17)$$

$$\frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + C\left(r + \ln(\frac{1}{r})\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r)))\right)\right).$$

Here $\#(\sigma(P_\delta)\cap\Gamma)$ denotes the number of eigenvalues of P_δ in Γ, counted with their algebraic multiplicity.

Explain the choice of parameters!

Some ideas in the proofs

In the proofs of the semi-classical theorems, a common feature is to identify the eigenvalues of the operator with the zeros of a holomorphic function with exponential growth and to show that with probability close to 1 this function really is exponentially large at finitely many points distributed nicely along the boundary of Γ, then apply a result about the the number of zeros of such functions.

In the one dimensional results by Hager (and Bordeaux-Montrieux for matrix-valued operators) this is done via a Grushin (Feschbach) problem that makes use of the Davies-Hörmander quasimodes for P and P^* and we get quite a concrete holomorphic function.

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First we construct a symbol $\widetilde{p}$, equal to p outside a compact set such that $\widetilde{p} - z \neq 0$ for $z \in \mathrm{neigh}\,(\Gamma)$, and put on the operator level: $\widetilde{P} = P + (\widetilde{p} - p)$. Then $\widetilde{P} - z$ has a bounded (pseudodifferential) inverse for every z in some simply connected neighborhood of Γ. The eigenvalues of P coincide with the zeros of the holomorphic function,

$$z \mapsto \det(\widetilde{P} - z)^{-1}(P - z) = \det(1 - (\widetilde{P} - z)^{-1}(\widetilde{P} - P)).$$

If $P_\delta = P + \delta Q_\omega$, put $\widetilde{P}_\delta := \widetilde{P} + \delta Q_\omega$ which has no spectrum in near Γ. The eigenvalues of P_δ in that region are the zeros of

$$z \mapsto \det(\widetilde{P}_{\delta,z}),$$

where

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- Step 1. Show that with probability close to 1, we have for all z in a neighborhood of $\partial\Gamma$ with $p_z = (\widetilde{p} - z)^{-1}(p - z)$:

$$\ln|\det P_{\delta,z}| \leq \frac{1}{(2\pi h)^n}\Big(\int \ln|p_z(\rho)|d\rho + o(1)\Big). \qquad (18)$$

- Step 2. Show that for each z in a neighborhood of $\partial\Gamma$ we have with probability close to one that

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- Step 3. Apply results ([Ha, HaSj]) about counting zeros of holomorphic functions: Roughly, if $u(z) = u(z;\widetilde{h})$ is holomorphic with respect to z in a neighborhood of $\overline{\Gamma}$, $|u(z)| \leq \exp(\phi(z)/\widetilde{h})$ near $\partial\Gamma$ and we have a reverse estimate $|u(z_j)| \geq \exp((\phi(z_j) - \text{"small"})/\widetilde{h})$ for a finite set of points, distributed "densely" along the boundary, then the number of zeros of u in Γ is equal to $(2\pi\widetilde{h})^{-1}(\int\int_\Gamma \Delta\phi(z)d\Re z d\Im z + \text{"small"})$. This is applied with $\widetilde{h} = (2\pi h)^n$, $\phi(z) = \int \ln|p_z(\rho)|d\rho$.

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Step 1 can be carried out using microlocal analysis for the unperturbed operator (cf Melin–Sj) and the fact that the perturbation is small in a suitable sense.

Step 2 is the delicate one. In the other results, (Hager, Bordeaux M., Hager–Sj) with the Gaussianity assumption we are lead to the problem of finding lower bounds on the determinant of a random matrix which is close to a Gaussian one, but in the case of multiplicative perturbations in higher dimension, this does not seem to work. Instead, we forget about Gaussianity, and make a complex analysis argument in the α-variables[1]. Then we come down to the task of constructing (for each fixed z near $\partial\Gamma$) at least one perturbation of the requested form for which we have a nice lower bound on the determinant. This is again a delicate problem.

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Theorem

Let $\Gamma \Subset \mathbf{C}$ open, $\gamma := \partial\Gamma$, $r : \gamma \to]0,1[$ Lipschitz: $|r(x) - r(y)| \le \frac{1}{2}|x-y|$ and assume that for each $z \in \gamma$, $D(z, r(z)) \cap \gamma$ is the graph of a Lipschitz function after a translation and rotation, uniformly with respect to x. Let $z_1, z_2, ..., z_N$ run through γ with cyclic convention; "$N+1=1$" such that $C^{-1}r(z_k) \le |z_{k+1} - z_k| \le \frac{1}{2}r(z_k)$. Let ϕ be continuous and subharmonic in a neighborhood of the closure of $\gamma_r := \cup_{x\in\gamma} D(x, r(x))$. Then $\exists$ $\widetilde{z}_j \in D(z_j, \frac{1}{C}r(z_j))$ such that:

If $u = u_{\widetilde{h}}$, $0 < \widetilde{h} \le 1$ is holomorphic in $\Gamma \cup \gamma_r$ such that $\widetilde{h} \ln |u| \le \phi$ on γ_r, $\widetilde{h} \ln |u(\widetilde{z}_j)| \ge \phi(\widetilde{z}_j) - \epsilon_j$, $j = 1, ..., N$,

then with $\mu := \Delta\phi L(dz)$ (where ϕ denotes any extension from γ_r to $\Gamma \cup \gamma_r$):

$$|\#(u^{-1}(0) \cup \Gamma) - \frac{1}{2\pi\widetilde{h}}\mu(\Gamma)| \le \frac{\widetilde{C}}{\widetilde{h}}(\mu(\gamma_r) + \sum \epsilon_j)$$

4. Improved resolvent estimates under random perturbation

That such improvements arrive follows from the proof of Weyl asymptotics. We here describe some first results from the thesis of WBM. Further (possibly very substantial) improvements should be possible. The general setting of WBM is here the earlier work [HaSj] where the random perturbation is of the form δQ_ω,

$$Q_\omega = \widehat{S} \circ \sum_{j,k} \alpha_{j,k}(\omega) e_j f_k^* \circ \widetilde{S}$$

where $\widehat{S}, \widetilde{S}$ are elliptic h-pseudodifferential operators of Hilbert Schmidt class, one of them even of trace class, $\{e_j\}$ and $\{f_k\}$ are arbitrary orthonormal bases and $\alpha_{j,k}(\omega)$ are independent complex $\mathcal{N}(0,1)$ laws. By examining the proofs in [HaSj], W Bordeaux Montrieux established:

Theorem

If $0 < \delta \ll h^{3n+\frac{1}{2}}$*, then with probability*

$$\geq 1 - \frac{C_1}{h(\ln \frac{1}{\delta})^{1/\kappa}} \delta^{\widetilde{C} h^{\kappa - n}} - \frac{1}{\widetilde{C} h (\ln \frac{1}{\delta})^{1/\kappa}} e^{-C_2 h^{-2n}},$$

we have for all $z \in \Gamma$*:*

$$\|(P + \delta Q_\omega - z)^{-1}\| \leq$$

$$C_0 h^{\kappa - n - \frac{1}{2}} \delta^{-C_0 h^{\kappa - n}} \prod_{w \in \sigma(P + \delta Q_\omega) \cap D(z, C h^{\frac{1}{2}} (\ln \frac{1}{\delta})^{\frac{1}{2\kappa}})} \frac{1}{|z - w|}.$$

When $\delta \geq h^{N_0}$, this is better than the estimate (1).

In the interior of Σ we may have $\kappa = 1$ (for instance when $\{p, \overline{p}\} \neq 0$ on $p^{-1}(z)$. When that is the case and $n = 1$, we get the polynomial bound

$$\leq C_0 h^{-1/2} \delta^{-C_0} \prod_{w} \frac{1}{|z - w|}.$$

Near the boundary of Σ we cannot hope to have $\kappa = 1$. In this situation WBM analyzed a model operator: $P = hD_x + g(x)$ on S^1, with g smooth and : $inf \Im g = \Im g(a)$ where a is the unique point of minimum and $\Im g''(a) > 0$, so that we are in the situation $k = 2$ in 2) in Theorem 1. Now

$$Q_\omega = \sum_{j,k} a_{j,k}(\omega) e_j e_k^*, \; e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx},$$

where $a_{j,k} \in \mathcal{N}(0, \sigma_j \sigma_k)$ are independent, $\sigma_j \asymp \langle j \rangle^{-\rho}$ for some $\rho > 1$.

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Put $\Gamma_\alpha = \{z \in \mathbf{C};\ |\Re z| \le C,\ C_1 \le \frac{\Im z}{\alpha} \le C_2\}$. Assume that

$$h^\beta \le \delta \le \frac{h^{1/2}\alpha^{1/4}h^{2\rho}}{(\ln \frac{1}{h})^3},$$

for some $\beta > 0$ and that $\alpha \gg (h \ln \frac{1}{h})^{2/3}$. Then with probability close to 1:

$$\|(P + \delta Q_\omega - z)^{-1}\| \le \frac{C}{\sqrt{h}\alpha^{1/4}h^M} \prod_{|w_j - z| \le r} \frac{1}{|z - w_j|}, z \in \Gamma_\alpha,$$

where w_j are the eigenvalues of $P + \delta Q_\omega$ and $r = (\sqrt{\alpha} h \ln \frac{1}{h})^{1/2}$.