An inverse problem of identifying locations of small volume perturbations of the refractive index for the acoustic equation at fixed frequency

Christian DAVEAU   Abdessatar KHELIFI
Anton SUSHCHENKO

Department of Mathematics, Cergy-Pontoise University, France
and
Department of Mathematics, Sciences University of Carthage, Tunisie

- I would like to give my special thanks to the organization and scientific comity and also for Dr. Habib Ammari for his help during my work.
- H. Ammari and G. Uhlmann, *Reconstruction of the potential from partial Cauchy data for the Schrodinger equation,* Indiana Univ. Math. J. 53, No. 1, (2004)

Problem formulation

Let $\Omega \subset R^3$ be a bounded domain with C^2 boundary, ν the unit-outer normal to $\partial\Omega$, Γ a smooth open subset of the boundary $\partial\Omega$ and Γ_c denotes $\partial\Omega \setminus \overline{\Gamma}$.

- The trace space $\widetilde{H}^{\frac{1}{2}}(\Gamma) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega), f \equiv 0 \text{ on } \Gamma_c \right\}$
- Ω contains a finite number of inhomogeneities $z_j + \alpha B_j$ and $\mathcal{B}_\alpha = \cup_{j=1}^{m}(z_j + \alpha B_j)$
- $z_j \in \Omega, j = 1, \dots, m$ satisfy:
  - $|z_j - z_l| \geq c_0 > 0, \forall\, j \neq l$ and $\text{dist}(z_j, \partial\Omega) \geq c_0 > 0, \forall\, j$

Problem formulation

The perturbed refractive index

$$\mathbf{n}_\alpha(x) = \begin{cases} \mathbf{n}(x), & x \in \Omega \setminus \bar{\mathcal{B}}_\alpha, \\ \mathbf{n}_j(x), & x \in z_j + \alpha B_j,\ j = 1 \dots m. \end{cases} \quad (1)$$

Consider the acoustic equation, at fixed frequency, in the presence of the inhomogeneities $\mathcal{B}_\alpha$

$$(\Delta + \omega^2 \mathbf{n}_\alpha) u_\alpha = 0 \text{ in } \Omega$$
$$u_\alpha|_{\partial\Omega} = f \in \widetilde{H}^{\frac{1}{2}}(\Gamma),$$

Define the local Dirichlet to Neumann map associated to $\mathbf{n}_\alpha$

$$\Lambda_{\mathbf{n}_\alpha}(f) = \frac{\partial u_\alpha}{\partial \nu}|_\Gamma, \forall f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$$

Identification procedure

Let $\mathbf{n}_0 \in L^\infty(\Omega)$ be a known function. Assume that $\mathbf{n} = \mathbf{n}_0$ almost everywhere in a neighborhood of $\partial\Omega$. Suppose $\Lambda_{\mathbf{n}} : \widetilde{H}^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ is known, then for any $u, v \in H^1(\Omega)$ satisfying:

$$\begin{array}{l} (\frac{1}{\omega^2}\Delta + \mathbf{n})u = 0 \text{ in } \Omega, (\frac{1}{\omega^2}\Delta + \mathbf{n}_0)v = 0 \text{ in } \Omega \\ u|_{\Gamma_c} = v|_{\Gamma_c} = 0 \end{array}$$

we obtain

$$\int_\Omega (\mathbf{n} - \mathbf{n}_0)uv \; dx = \int_\Gamma u(\Lambda_{\mathbf{n}} - \Lambda_{\mathbf{n}_0})v \; ds,$$

where $\Lambda_{\mathbf{n}_0}$ denotes the local Dirichlet to Neumann map associated to the refractive index $\mathbf{n}_0$.

Let $\mathbf{n}_0 \in L^\infty(\Omega)$ be known and $\mathbf{n} = \mathbf{n}_0$ almost everywhere in a neighborhood of $\partial\Omega$ and u_ρ is the solution of

$$\begin{array}{l} \Delta u_\rho = 0 \quad \text{in } R^3 \setminus \overline{\Omega}, \\ (\frac{1}{\omega^2}\Delta + \mathbf{n})u_\rho = 0 \quad \text{in } \Omega, \end{array}$$

Then $u_\rho|_\Gamma$ solves

$$\Lambda_{\mathbf{n}}(u_\rho|_\Gamma) + \oint_\Gamma \frac{\partial^2 g_\rho^D}{\partial\nu(x)\partial\nu(y)}(x,y)u_\rho|_\Gamma(y)\, ds(y) = \rho\cdot\nu(x)e^{x\cdot\rho}, \quad \forall\, x \in \Gamma \tag{2}$$

Identification procedure

Define the double layer potential

$$N_\rho(f) = \oint_\Gamma \frac{\partial^2 g_\rho^D}{\partial \nu(x) \partial \nu(y)}(x, y) f|_\Gamma(y)\, ds(y), \quad \forall\, f \in \widetilde{H}^{\frac{1}{2}}(\Gamma),$$

and set

$$\rho_1 = \frac{\eta}{2} + i(\frac{k+l}{2})$$
$$\rho_2 = -\frac{\eta}{2} + i(\frac{k-l}{2})$$

where $\eta, k, l \in R^3$ such that $\eta \cdot k = \eta \cdot l = k \cdot l = 0$, and $|\eta|^2 = |k|^2 + |l|^2$

Identification procedure

Define the double layer potential

$$N_\rho(f) = \oint_\Gamma \frac{\partial^2 g_\rho^D}{\partial\nu(x)\partial\nu(y)}(x,y) f|_\Gamma(y)\, ds(y), \quad \forall\, f \in \widetilde{H}^{\frac{1}{2}}(\Gamma),$$

and set

$$\rho_1 = \frac{\eta}{2} + i\left(\frac{k+l}{2}\right)$$
$$\rho_2 = -\frac{\eta}{2} + i\left(\frac{k-l}{2}\right)$$

where $\eta, k, l \in R^3$ such that $\eta \cdot k = \eta \cdot l = k \cdot l = 0$, and $|\eta|^2 = |k|^2 + |l|^2$

Lemma 1 *Assume that* 0 *is not a Dirichlet eigenvalue of* $(\frac{1}{\omega^2}\Delta + \mathbf{n})$ *in* Ω*. Then, there is a unique solution* $u_\rho|_\Gamma \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ *of (2)*

$$u_\rho|_\Gamma = (\Lambda_{\mathbf{n}} + N_\rho)^{-1}(\rho_1 \cdot \nu(x) e^{x\cdot\rho_1}|_\Gamma)$$

Identification procedure

Proposition 1 *Let* $\mathbf{n}_0 \in L^\infty(\Omega)$ *be a given function. Assume that* 0 *is not a Dirichlet eigenvalue of* $(\frac{1}{\omega^2}\Delta + \mathbf{n})$ *in* Ω *and* $\mathbf{n} = \mathbf{n}_0$ *almost everywhere in a neighborhood of* $\partial\Omega$. *Then*

$$\mathcal{F}(\mathbf{n} - \mathbf{n}_0)(-k) = \frac{1}{\omega^2} \lim_{|l| \to +\infty} \int_\Gamma (\Lambda_\mathbf{n} + N_{\rho_1})^{-1}(\rho_1 \cdot \nu(x) e^{x \cdot \rho_1}|_\Gamma)(\Lambda_\mathbf{n} -$$

$$\Lambda_{\mathbf{n}_0})(\Lambda_{\mathbf{n}_0} + N_{\rho_2})^{-1}(\rho_2 \cdot \nu(x) e^{x \cdot \rho_2}|_\Gamma)\, ds(x)$$

Identification procedure

Proposition 1 *Let* $\mathbf{n}_0 \in L^\infty(\Omega)$ *be a given function. Assume that* 0 *is not a Dirichlet eigenvalue of* $(\frac{1}{\omega^2}\Delta + \mathbf{n})$ *in* Ω *and* $\mathbf{n} = \mathbf{n}_0$ *almost everywhere in a neighborhood of* $\partial\Omega$*. Then*

$$\mathcal{F}(\mathbf{n} - \mathbf{n}_0)(-k) = \frac{1}{\omega^2} \lim_{|l| \to +\infty} \int_\Gamma (\Lambda_{\mathbf{n}} + N_{\rho_1})^{-1}(\rho_1 \cdot \nu(x) e^{x\cdot\rho_1}|_\Gamma)(\Lambda_{\mathbf{n}} - \Lambda_{\mathbf{n}_0})(\Lambda_{\mathbf{n}_0} + N_{\rho_2})^{-1}(\rho_2 \cdot \nu(x) e^{x\cdot\rho_2}|_\Gamma)\, ds(x)$$

Lemma 2 *Let* ρ_1 *and* ρ_2 *be given, then the following asymptotic behavior holds :*

$$N_{\rho_i} = |l| L_i + O(\frac{1}{|l|}) \text{ as } |l| \to +\infty,$$

where, for $i \in \{1, 2\}$, L_i *is a well defined integral operator on* $\widetilde{H}^{\frac{1}{2}}(\Gamma)$.

Identification procedure

The kernel of the operator N_{ρ_1} is $\frac{\partial^2 g^D_{\rho_1}}{\partial\nu(x)\partial\nu(y)}(x,y)$ where $g^D_{\rho_1}(x,y) = e^{x\cdot\rho_1} G^D_{\rho_1}(x,y)$ and $G^D_{\rho_1}(x,y)$ is a solution of the following first kind integral equation :

$$-G_{\rho_1}(x) = \int_{\partial\Omega} G_{\rho_1}(x,y) \frac{\partial G^D_{\rho_1}}{\partial\nu(y)}(x,y)\, ds(y)$$

And G_{ρ_1} is given by

$$G_{\rho_1}(x) = \int_{R^3} \frac{e^{ix\cdot\xi}}{\xi^2 + 2i\rho_1\cdot\xi}\, d\xi.$$

$$\frac{\partial^2 g^D_{\rho_1}}{\partial\nu(x)\partial\nu(y)}(x,y) = \exp(x\cdot\rho_1)\left[(\rho_1\cdot\nu(x))\frac{\partial G^D_{\rho_1}}{\partial\nu(y)} + \frac{\partial^2 G^D_{\rho_1}}{\partial\nu(x)\partial\nu(y)}\right]$$

$$\rho_1\cdot\xi = i\frac{|l|}{2}\left[|\xi|\cos(\widehat{l,\xi}) - \frac{(-k\cdot\xi + i\eta\cdot\xi)}{|l|}\right]$$

$$\rho_1\cdot\nu(x) = \frac{|l|}{\sqrt{2}}\cos(\widehat{\eta,\nu(x)}) + O(\frac{1}{|l|})$$

Identification procedure

Theorem 1 *Let* $\mathbf{n}_0 \in L^\infty(\Omega)$ *be a given function. Assume that* 0 *is not a Dirichlet eigenvalue of* $(\frac{1}{\omega^2}\Delta + \mathbf{n})$ *in* Ω *and* $\mathbf{n} = \mathbf{n}_0$ *almost everywhere in a neighborhood of* $\partial\Omega$*. Then*

$$\mathcal{F}(\mathbf{n} - \mathbf{n}_0)(-k) = -\frac{\sqrt{2}}{\omega^2} \int_\Gamma \frac{1}{|\eta|^2} \left(\eta \cdot \nu(x)|_\Gamma\right)^3 e^{ik\cdot x} \, ds(x),$$

where $\eta, k \in R^3$ *such that* $\eta \cdot k = 0$

Identification procedure

Theorem 1 *Let* $\mathbf{n}_0 \in L^\infty(\Omega)$ *be a given function. Assume that* 0 *is not a Dirichlet eigenvalue of* $(\frac{1}{\omega^2}\Delta + \mathbf{n})$ *in* Ω *and* $\mathbf{n} = \mathbf{n}_0$ *almost everywhere in a neighborhood of* $\partial\Omega$*. Then*

$$\mathcal{F}(\mathbf{n} - \mathbf{n}_0)(-k) = -\frac{\sqrt{2}}{\omega^2} \int_\Gamma \frac{1}{|\eta|^2} \left(\eta \cdot \nu(x)|_\Gamma\right)^3 e^{ik\cdot x}\, ds(x),$$

where $\eta, k \in R^3$ *such that* $\eta \cdot k = 0$

Ideas of the proof:

$$N_{\rho_i} = |l| L_i + O(\frac{1}{|l|}) \text{ as } |l| \to +\infty$$

$$(\Lambda_{\mathbf{n}} + N_{\rho_1})^{-1} = (\Lambda_{\mathbf{n}} + |l| L_1 + O(\frac{1}{|l|}))^{-1} = \frac{L_1^{-1}}{|l|} + O(\frac{1}{|l|})$$

$$(\Lambda_{\mathbf{n}_0} + N_{\rho_2})^{-1} = (\Lambda_{\mathbf{n}_0} + |l| L_2 + O(\frac{1}{|l|}))^{-1} = \frac{L_2^{-1}}{|l|} + O(\frac{1}{|l|})$$

$$\mathcal{F}(\mathbf{n}-\mathbf{n}_0)(-k)=\frac{1}{\omega^2}\lim_{|l|\to+\infty}\int_\Gamma\frac{1}{|l|^2}(L_1^{-1}(\rho_1\cdot\nu(x)e^{x\cdot\rho_1}|_\Gamma)(\Lambda_{\mathbf{n}}-$$

$$\Lambda_{\mathbf{n}_0})(L_2^{-1}(\rho_2\cdot\nu(x)e^{x\cdot\rho_2}|_\Gamma)\,ds(x)$$

Then using the estimations:

$$\rho_1\cdot\nu(x)=\frac{|l|}{\sqrt{2}}\cos(\widehat{\eta,\nu(x)})+O(\tfrac{1}{|l|})$$
$$\rho_2\cdot\nu(x)=\frac{-|l|}{\sqrt{2}}\cos(\widehat{\eta,\nu(x)})+O(\tfrac{1}{|l|})$$

we obtain

$$\mathcal{F}(\mathbf{n}-\mathbf{n}_0)(-k)=-\frac{\sqrt{2}}{\omega^2}\int_\Gamma\frac{1}{|\eta|^2}\left(\eta\cdot\nu(x)|_\Gamma\right)^3e^{ik\cdot x}\,ds(x),$$

Identification procedure

Theorem 2 *Assume that* 0 *is not a Dirichlet eigenvalue of* $(\frac{1}{\omega^2}\Delta + \mathbf{n}_\alpha)$ *in* Ω *and* $\mathbf{n}_\alpha = \mathbf{n}$ *almost everywhere in a neighborhood of* $\partial\Omega$ *(for* α *sufficiently small). Then, the following identification holds:*

$$\mathcal{F}(\mathbf{n}_\alpha - \mathbf{n})(-k) = \frac{1}{\omega^2} \lim_{|l|\to+\infty} \int_\Gamma (\Lambda_{\mathbf{n}_\alpha} + N_{\rho_1})^{-1}(\rho_1 \cdot \nu(x) e^{x\cdot\rho_1}|_\Gamma)(\Lambda_{\mathbf{n}_\alpha} - \Lambda_{\mathbf{n}})(\Lambda_{\mathbf{n}} + N_{\rho_2})^{-1}(\rho_2 \cdot \nu(x) e^{x\cdot\rho_2}|_\Gamma)\, ds(x) = \frac{\alpha^3}{\omega^2} \sum_{j=1}^{m} (\mathbf{n}(z_j) - \mathbf{n}_j(z_j))|B_j| e^{ik\cdot z_j} + o(\alpha^3)$$

Corollary 1 *Suppose that we have the hypothesis of Theorem 2. Then, the following identification holds:*

$$-\sqrt{2}\int_\Gamma \frac{1}{|\eta|^2}\left(\eta\cdot\nu(x)|_\Gamma\right)^3 e^{ik\cdot x}\, ds(x) =$$

$$\alpha^3\sum_{j=1}^m(\mathbf{n}(z_j)-\mathbf{n}_j(z_j))|B_j|e^{ik\cdot z_j}+o(\alpha^3),$$

where $\eta, k \in R^3$ *such that* $\eta\cdot k = 0$.

The locations $\{z_j\}_{j=1}^m$ could be obtained as the supports of the inverse Fourier transform of

$$-\sqrt{2}\int_\Gamma \frac{1}{|\eta|^2}\left(\eta\cdot\nu(x)|_\Gamma\right)^3 e^{ik\cdot x}\, ds(x)$$

Conclusion and Perspectives:

- The explicit formulation for $\mathcal{F}(\mathbf{n}_\alpha - \mathbf{n})(-k)$ was obtained.
- The theoretical base for the precision reconstruction method of the refractive index has been developed.
- The method could be easily adopted for the different geometry of the inhomogeneities.
- The numerical algorithms based on our reconstruction method could be useful for the industrial applications.

Bibliography

G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal., 27 (1988), 153-172.

H. Ammari, S. Moskow, and M. Vogelius, *Boundary integral formulas for the reconstruction of electromagnetic imperfections of small diameter*, ESAIM: Cont. Opt. Calc. Var. 9 (2003), 49-66.

H. Ammari and A. Ramm, *Recovery of small electromagnetic inhomogeneities from boundary measurements on part of the boundary*, C. R. Acad. Sci. II., 330 (2002), 199-205.

A. L. Bukhgeim and G. Uhlmann, *Recovering a potential from partial Cauchy data*, Commun. Part. Diff. Equat.27 (2002), 653-668.

R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements*, Commun. Pure Appl. Math., 37 (1984), 289-298.

Bibliography

- A. Nachmann, *Reconstructions from boundary measurements,* Ann. Math., 128 (1988), 531-587.
- J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem,* Ann. Math. 125 (1987), 153-169.