Exponential stability of the wave equation with boundary time varying delay

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Motivation

Stabilization

Assure the decay of the energy of solutions by a feedback law.

Degrees of stability

- Strong stability : $E(t) \to 0, \quad \text{when } t \to +\infty.$
- Exponential stability : $E(t) \leq Ce^{-\nu t}, \quad \forall t > 0.$
- Polynomial stability : $E(t) \leq \frac{C}{t^\alpha}, \quad \forall t > 0.$

Outline of the talk

1. The problem
   - The problem
   - Assumptions
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4. Nonlinear case
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The problem

We consider the wave equation with a time-varying delay term in the boundary condition in a bounded domain $\Omega \subset \mathbb{R}^n$ with a boundary Γ of class C^2 such that $\Gamma = \Gamma_D \cup \Gamma_N$, with $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \\ u(x,t) = 0 \quad \text{on} \quad \Gamma_D \times (0,+\infty) \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x, t-\tau(t)) \quad \text{on} \quad \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \\ u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)) \quad \text{in } \Gamma_N \times (0, \tau(0)), \end{cases} \tag{1}$$

where $\tau(t)$ is the time-varying delay, $\mu_1, \mu_2 > 0$.

Assumptions on the delay and on the domain

We assume that the delay satisfies

$$0 \leq \tau(t) \leq M, \quad \forall\ t > 0, \tag{2}$$

$$\tau'(t) \leq d < 1, \quad \forall\ t > 0, \tag{3}$$

and

$$\tau \in W^{2,\infty}([0, T]), \quad \forall\ T > 0. \tag{4}$$

We assume that there exists $x_0 \in \mathbb{R}^n$ such that, denoting by m the standard multiplier

$$m(x) := x - x_0,$$

we have

$$m(x) \cdot \nu(x) \leq 0 \quad \text{on} \quad \Gamma_D \tag{5}$$

and,

$$m(x) \cdot \nu(x) \geq \delta > 0 \quad \text{on} \quad \Gamma_N. \tag{6}$$

Case without delay and with constant delay: some references

- In absence of delay (i.e. $\mu_2 = 0$), the energy decays exponentially (see [Chen, 1981], [Lagnese, 1983, 1988], [Lasiecka-Triggiani, 1987], [Komornik-Zuazua, 1990], [Komornik, 1991, 1994]).
- If the delay is constant in time (i.e. $\tau(t) = \tau$), the energy decays also exponentially under some assumptions (see [Xu-Yung-Li, 2006], [Nicaise-Pignotti, 2006]).
- For time-varying delay, we refer to [Nicaise-Valein-Fridman, 2009] for one space dimension.

The main difficulty with time-varying delay

To prove the exponential decay in constant delay case (or without delay case), the authors use an observability inequality of type:

$$\exists T > 0,\ \exists C_T > 0,\ E(0) \leq C_T \int_0^T \int_{\Gamma_N} (u_t^2(x,t) + u_t^2(x, t-\tau))\, d\Gamma\, dt.$$

This implies the exponential decay of the energy since the system is invariant by translation in time.
However, in our case (for time-varying delay), this method can not be applied due to the loss of the time-translation invariance $\rightsquigarrow$ other method: Lyapunov functional.

Outline of the talk

Transformation of the system

We set $z(x,\ \rho,\ t) = u_t(x, t - \tau(t)\rho)$ for $x \in \Gamma_N,\ \rho \in (0,\ 1)$ and $t > 0$. Then (1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \\ \tau(t) z_t(x,\rho,t) + (1 - \tau'(t)\rho) z_\rho(x,\rho,t) = 0 \text{ in } \Gamma_N \times (0,1) \times (0,+\infty) \\ u(x,t) = 0 \quad \text{on} \quad \Gamma_D \times (0,+\infty) \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 z(x,1,t) \quad \text{on} \quad \Gamma_N \times (0,+\infty) \\ z(x,0,t) = u_t(x,t) \quad \text{on} \quad \Gamma_N \times (0,\infty) \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \\ z(x,\rho,0) = f_0(x, -\rho\tau(0)) \quad \text{in} \quad \Gamma_N \times (0,\tau(0)). \end{cases}$$

If we denote by $U = (u,\ u_t,\ z)^T$, then U satisfies

$$U' = (u_t,\ u_{tt},\ z_t)^T = \left(u_t,\ \Delta u,\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \right)^T.$$

First order system

The system (1) can be rewritten as

$$\left\{ \begin{array}{l} U'(t) = \mathcal{A}(t)U(t) \\ U(0) = (u_0,\ u_1, f^0(., -.\tau(0)))^T, \end{array} \right. \tag{7}$$

where the operator $\mathcal{A}(t)$ depends on time and is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain (independent of time) $D(\mathcal{A}(t)) = \Big\{(u, v, z) \in (E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_D}(\Omega)) \times H^1_{\Gamma_D}(\Omega) \times L^2(\Gamma_N; H^1(0, 1));$ $v = z(., 0),\ \frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(., 1) \text{ on } \Gamma_N \Big\}$. We introduce the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_D}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N \times (0, 1)).$$

Choice of weights

If the delay is constant in time,

- if $\mu_1 = 0$, the system is unstable [Dakto-Lagnese-Polis, 1986],
- if $\mu_2 \geq \mu_1$, some unstabilities may appear [Nicaise-Pignotti, 2006, Nicaise-Valein, 2007],
- but if $\mu_2 < \mu_1$, the system is dissipative.

Here, with time-varying delay, we impose:

Condition between μ_1 and μ_2

$$\mu_2 \leq \sqrt{1-d}\mu_1, \tag{8}$$

where

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

Inner product on $\mathcal{H}$

We choose $q > 0$ such that

$$\frac{\mu_2}{\sqrt{1-d}} \leq q \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}.$$

The difficulty to prove the well-posedness of (1) is that $\mathcal{A}(t)$ depends on t. We use the variable norm technique of Kato and we define the time dependent inner product on $\mathcal{H}$:

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_\Omega \left(\nabla u(x) \cdot \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\right) dx$$

$$+ q\tau(t) \int_{\Gamma_N} \int_0^1 z(x,\rho)\tilde{z}(x,\rho) d\rho d\Gamma.$$

A result from Kato

Theorem ([Kato, 1976])

Assume that
(i) $Y = \mathcal{D}(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$,
(ii) $D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \forall t > 0$,
(iii) for all $t \in [0,\ T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,\ T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e.* $(S_t(s))_{s\geq 0}$ *satisfies* $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, $\forall u \in \mathcal{H}$, $s \geq 0$*),*
(iv) $\partial_t \mathcal{A} \in L^\infty_*([0,\ T], B(Y, \mathcal{H}))$, *the space of equivalent classes of essentially bounded, strongly measure functions from* $[0,\ T]$ *into the set* $B(Y, \mathcal{H})$.
Then, problem (7) has a unique solution
$U \in C([0,\ T], Y) \cap C^1([0,\ T], \mathcal{H})$ *for any initial datum in* Y.

Well-posedness

In a first time, we assume that

$$0 < \tau_0 \leq \tau(t) \leq M, \quad \forall t > 0.$$

Theorem

For any initial datum $U_0 \in \mathcal{D}(\mathcal{A}(0))$, there exists a unique solution $U \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$ of (7).

Proof: To verify (ii), we prove, for a fixed $t > 0$:

- $\mathcal{A}(t)$ is dissipative, ie $\forall U \in D(\mathcal{A}(t)), \langle \mathcal{A}(t)U,\ U\rangle_t \leq 0$,
- $\forall \lambda > 0, \lambda I - \mathcal{A}(t)$ is surjective,
- $\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \forall t,\ s \in [0,\ T]$.

Well-posedness: the general case

We assume now only $0 \leq \tau(t) \leq M, \quad \forall t > 0$ holds. We take

$$\tau_\epsilon(t) = \tau(t) + \epsilon, \quad \forall 0 < \epsilon < \epsilon_0.$$

Then, by the previous theorem, there exists a unique solution $U_\epsilon = (u^\epsilon, v^\epsilon, z^\epsilon)^T \in C([0, +\infty), \mathcal{D}(\mathcal{A}_\epsilon(t))) \cap C^1([0, \infty), \mathcal{H})$, of

$$\left\{ \begin{array}{l} U_\epsilon' = \mathcal{A}_\epsilon(t) U_\epsilon \\ U_\epsilon(0) = (u_0, u_1, f_0(\cdot, - \cdot \tau_\epsilon(0)))^T = U_{\epsilon,0} \in \mathcal{D}(\mathcal{A}_\epsilon(0)), \end{array} \right. \tag{9}$$

where the operator $\mathcal{A}_\epsilon(t)$ is defined by

$$\mathcal{A}_\epsilon(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau_\epsilon'(t)\rho - 1}{\tau_\epsilon(t)} z_\rho \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t) + \epsilon} z_\rho \end{pmatrix},$$ with domain

$\mathcal{D}(\mathcal{A}_\epsilon(t)) = \mathcal{D}(\mathcal{A}(t))$. The aim is then to take the limit of $(u_\epsilon)_{0<\epsilon<\epsilon_0}$ when ϵ tends to 0.

Well-posedness: the general case

To pass at the limit, we need to have more regularity on the solution and, for that purpose, we use a theorem of [Kato,1976]. Introduce $D^2(0) = \{\phi \in \mathcal{D}(\mathcal{A}(0)) \cap X_2 \ : \ -\mathcal{A}(0)\phi \in \mathcal{D}(\mathcal{A}(0))\}$, where

$X_2 = (H^1_{\Gamma_D}(\Omega) \cap H^{3/2}(\Omega)) \times (H^1_{\Gamma_D}(\Omega) \cap H^{3/2}(\Omega)) \times L^2(\Gamma_N; H^2(0,1))$.

Therefore, for all initial data $U_{\epsilon,0} \in D^2(0)$, there exists a unique solution $U_\epsilon \in C^1([0,T], \mathcal{H}) \cap C([0,T], \mathcal{D}(\mathcal{A}_\epsilon(0)))$ of (9) which satisfies, moreover,

$$\frac{d^2}{dt^2} U_\epsilon \in C([0,T], \mathcal{H}).$$

We can give a sense to the derivative of the energy $\tilde{E}_\epsilon$,

$$\tilde{E}_\epsilon(t) = \frac{1}{2} \int_\Omega \left((\Delta u_\epsilon)^2 + (\nabla u_{\epsilon,t})^2 \right) dx + \frac{q' \tau_\epsilon(t)}{2} \int_{\Gamma_N} \int_0^1 u^2_{\epsilon,tt}(x, t - \tau_\epsilon(t)\rho) d\rho d\Gamma.$$

Well-posedness: the general case

We verify, under $(1-d)(1-\tau'_{min}) \le 2$, that $\tilde{E}'_\epsilon(t) \le 0$. Consequently

$$\int_\Omega \left((u_{\epsilon,tt})^2 + (\nabla u_{\epsilon,t})^2 \right) dx + \frac{q'\tau_\epsilon(t)}{2} \int_{\Gamma_N} \int_0^1 u^2_{\epsilon,tt}(x, t-\tau_\epsilon(t)\rho) d\rho d\Gamma$$
$$\le \int_\Omega \left((\Delta u_0)^2 + (\nabla u_1)^2 \right) dx + \frac{q'}{2}(\tau(0)+\epsilon) \int_{\Gamma_N} \int_0^1 f^2_{0,t}(x, -(\tau(0)+\epsilon)\rho) d\rho d\Gamma.$$

Assuming that $(f_{0,t}(x, -(\tau(0)+\epsilon)\rho)_{0<\epsilon<\epsilon_0}$ is bounded on $L^2(\Gamma_N \times (0,1))$, there exists $u \in H^1((0,T); H^1_{\Gamma_D}(\Omega)) \cap H^2((0,T); L^2(\Omega))$ such that, up to a subsequence,

$$u_\epsilon \rightharpoonup u \quad \text{in } H^1((0,T); H^1_{\Gamma_D}(\Omega)) \cap H^2((0,T); L^2(\Omega)).$$

The limit u then satisfies (1). Therefore

Theorem

For all initial data $U_0 \in D^2(0)$, there exists a unique solution $u \in H^1((0,T); H^1_{\Gamma_D}(\Omega)) \cap H^2((0,T); L^2(\Omega))$ of (7).

Outline of the talk

Energy

Assume

$$\mu_2 < \sqrt{1-d}\,\mu_1.$$

We define the energy as

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + |\nabla u|^2\right) dx + \frac{q\tau(t)}{2}\int_0^1 \int_{\Gamma_N} u_t^2(x, t-\tau(t)\rho)\, d\rho\, d\Gamma,$$

where $q > 0$ is such that $\frac{\mu_2}{\sqrt{1-d}} < q < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}$.

Proposition

For any regular solution of (7), the energy is decreasing and there exists $D > 0$ such that

$$E'(t) \leq -D \int_{\Gamma_N} \{u_t^2(x,t) + u_t^2(x, t-\tau(t))\}\, d\Gamma.$$

Lyapunov functional

As the system is not invariant by translation in time, we can not use an observability inequality to obtain the stability of the system. We use an other method and introduce the following Lyapounov functional:

$$\mathcal{E}(t) = E(t) + \gamma \left(\int_\Omega [2m \cdot \nabla u + (n-1)u] u_t dx + \mathcal{E}_2(t) \right), \quad (10)$$

where γ is a positive small constant that we will choose later on and $\mathcal{E}_2(t)$ is defined by

$$\mathcal{E}_2(t) := q\tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma. \quad (11)$$

Note that $\mathcal{E}$ is equivalent to the energy E:

$$\exists d_1,\, d_2 > 0, \quad ,\forall t \geq 0, \quad d_1 E(t) \leq \mathcal{E}(t) \leq d_2 E(t).$$

Two lemma

Denoting by $E_0(t) = \frac{1}{2}\int_\Omega (u_t^2(x,t) + |\nabla u(x,t)|^2)dx$:

Lemma

$$\exists C_0,\ C_1 > 0,\ \frac{d}{dt}\int_\Omega [2m\cdot\nabla u + (n-1)u]u_t dx$$
$$\leq -C_0 E_0(t) + C_1 \int_{\Gamma_N} (u_t^2(x,t) + u_t^2(x,t-\tau(t)))d\Gamma,$$

Lemma

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -2\mathcal{E}_2(t) + q\int_{\Gamma_N} u_t^2 d\Gamma.$$

Stability result

We can deduce the exponential stability estimate for system (1).

Theorem

There exist positive constants C, ν such that for any solution of system (1)

$$E(t) \leq CE(0)e^{-\nu t}, \quad \forall t \geq 0. \tag{12}$$

Remark

The decay rate is given by

$$\nu = \frac{\gamma}{d_2} \min\left(C_0,\, 4e^{-2M}\right).$$

Sketch of the proof (1)

By two previous lemma and the decay of the energy,

$$\frac{d}{dt}\mathcal{E}(t) \leq -D\int_{\Gamma_N}\{u_t^2(x,t)+u_t^2(x,t-\tau(t))\}d\Gamma + \gamma\Big(-C_0E_0(t)$$

$$+C_1\int_{\Gamma_N}(u_t^2(x,t)+u_t^2(x,t-\tau(t)))d\Gamma - 2\mathcal{E}_2(t)+q\int_{\Gamma_N}u_t^2\,d\Gamma\Big).$$

For γ sufficiently small, we obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma C_0E_0(t)-2\gamma\mathcal{E}_2(t).$$

Moreover, as $\tau(t)\leq M$,

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma C_0E_0(t)-2\gamma e^{-2M}q\tau(t)\int_{\Gamma_N}\int_0^1 u_t^2(x,t-\tau(t)\rho)\,d\rho\,d\Gamma.$$

Sketch of the proof (2)

Then for $0 < \gamma' < \gamma \min\left(C_0,\, 4e^{-2M}\right)$,

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma' E(t) \leq -\frac{\gamma'}{d_2}\mathcal{E}(t)$$

and thus

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\frac{\gamma'}{d_2}t}.$$

Therefore

$$E(t) \leq \frac{1}{d_1}\mathcal{E}(t) \leq \frac{1}{d_1}\mathcal{E}(0)e^{-\frac{\gamma'}{d_2}t} \leq \frac{d_2}{d_1}E(0)e^{-\frac{\gamma'}{d_2}t}.$$

Outline of the talk

The nonlinear problem

$$\left\{\begin{array}{l} u_{tt}(x,t)-\Delta u(x,t)=0 \quad \text{in} \quad \Omega\times(0,+\infty)\\ u(x,t)=0 \quad \text{on} \quad \Gamma_D\times(0,+\infty)\\ \frac{\partial u}{\partial \nu}(x,t)=-\beta_1(u_t(x,t))-\beta_2(u_t(x,t-\tau(t))) \quad \text{on} \quad \Gamma_N\times(0,+\infty)\\ u(x,0)=u_0(x) \quad \text{and} \quad u_t(x,0)=u_1(x) \quad \text{in} \quad \Omega\\ u_t(x,t-\tau(0))=g_0(x,t-\tau(0)) \quad \text{in} \quad \Gamma_N\times(0,\tau), \end{array}\right. \tag{13}$$

where $\beta_j:\mathbb{R}\to\mathbb{R}, j=1,2$, satisfy:

$$|\beta_j(s)|\leq c_j|s|, \quad \forall s\in\mathbb{R}, \quad j=1,2,$$

$$\beta_1(s)\cdot s\geq cs^2, \quad \forall s\in\mathbb{R},$$

$$\beta_2(s)\cdot s\geq 0, \quad \forall s\in\mathbb{R},$$

$$\exists\gamma_1>0,\ \forall x,\ y\in\mathbb{R},\ (\beta_1(x)-\beta_1(y))(x-y)\geq\gamma_1(x-y)^2,$$

and

$$\exists\gamma_2>0,\ \forall x,\ y\in\mathbb{R},\ |\beta_2(x)-\beta_2(y)|^2\leq\gamma_2|x-y|^2.$$

Well-posedness

As for the linear case, we rewrite the system as a first order evolution equation:

$$\left\{\begin{array}{l} U' = \mathcal{A}(t)U \\ U(0) = (u_0, u_1, g_0(\cdot, -\cdot\,\tau(0)))^T \end{array}\right. \tag{14}$$

where the operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

$\mathcal{D}(\mathcal{A}(t)) =$
$\Big\{(u, v, z)^T \in \left(E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_D}(\Omega)\right) \times H^1_{\Gamma_D}(\Omega) \times L^2(\Gamma_N; H^1(0,1)) :$
$\frac{\partial u}{\partial \nu} = -\beta_1(v) - \beta_2(z(\cdot, 1)) \,\mathrm{on}\, \Gamma_N;\; v = z(\cdot, 0) \,\mathrm{on}\, \Gamma_N \Big\}.$

We observe that $\mathcal{A}(t)$ is nonlinear (due to the domain of $\mathcal{A}(t)$).

Well-posedness

We use this time the following result due to [Kato, 1967]:

Theorem

For $T > 0$ and $t \in [0, T]$ we assume that there exists $\langle .,. \rangle_t$ on X:
$\exists c > 0, \forall u \in X, \forall t, s \in [0, T], \quad \frac{\|u\|_t}{\|u\|_s} \leq e^{c|t-s|}.$
Assume furthermore that:
(i) for all $t \in [0, T]$, $A(t)$ is a maximal monotone operator for $\langle .,. \rangle_t$;
(ii) the domain $\mathcal{D}(A(t))$ of $A(t)$ is independent of t, for all $t \in [0, T]$;
(iii) $\exists K > 0, \forall u \in \mathcal{D}(A(t)), \forall s, t \in [0, T]$,
$\|A(t)u - A(s)u\|_0 \leq K\,|t-s|\left(\|u\|_0 + \|A(s)u\|_0\right).$
Then for all $v \in \mathcal{D}(A(t))$ the evolution equation
$u' + A(t)u = 0$ *for* $0 \leq t \leq T, \quad u(0) = v$ *has a unique solution*
$u \in C([0, T]; X)$ *such that $u(t)$ belongs to $\mathcal{D}(A(t))$ for all $t \in [0, T]$, its strong derivative $u'(t) = -A(t)u(t)$ exists and is continuous except at a countable numbers of values t.*

Well-posedness

We assume that

$$\sqrt{\gamma_2} \leq \gamma_1 \sqrt{1-d}$$

and define the following inner product on $\mathcal{H}$:

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_\Omega \{\nabla u \nabla \tilde{u} + v\tilde{v}\} dx + q\tau(t) \int_{\Gamma_N} \int_0^1 z(x,\rho)\tilde{z}(x,\rho)\, d\rho\, d\Gamma,$$

where

$$\frac{\sqrt{\gamma_2}}{\sqrt{1-d}} \leq q \leq 2\gamma_1 - \frac{\sqrt{\gamma_2}}{\sqrt{1-d}}.$$

Theorem

If β_2 is nondecreasing and $0 < \tau_0 \leq \tau(t) \leq M$, $\forall t > 0$, for any $U_0 \in \mathcal{D}(\mathcal{A}(t))$, then there exists a unique solution $U \in C([0,+\infty), \mathcal{D}(\mathcal{A}(t))) \cap C^1([0,+\infty), \mathcal{H})$ of (14).

Stability

We assume that

$$c > \frac{c_2}{\sqrt{1-d}},$$

where c, c_2 are the constants such that

$$|\beta_2(s)| \leq c_2|s|, \quad \beta_1(s)\cdot s \geq cs^2, \quad \forall s \in \mathbb{R}.$$

We define the energy associated to (13) by

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + |\nabla u|^2\right) dx + \frac{q\tau(t)}{2}\int_0^1\int_{\Gamma_N} u_t^2(x, t-\tau(t)\rho)d\rho d\Gamma,$$

where $q > 0$ is such that $\frac{c_2}{\sqrt{1-d}} < q < 2c - \frac{c_2}{\sqrt{1-d}}$.

Exponential stability

By introducing the same Lyapunov functional as previously with γ sufficiently small, we deduce:

Theorem

There exist positive constants C', ν' such that for any solution of system (13)

$$E(t) \leq C'E(0)e^{-\nu' t}, \quad \forall t \geq 0. \tag{15}$$

Outline of the talk

Conclusion

- Stabilization of the wave equation with boundary time-varying delay
  - under hypothesis on the delay, on the weights and on the domain,
  - by using Lyapunov functional,
  - influence of the delay.
- Extension to nonlinear case.

Open problems

- Stabilization of second order evolution equations with unbounded feedback with time-dependent delay.
- Internal stabilization of the wave equation with time-varying delay, stabilization of the wave equation on networks with time-varying delay.
- Well-posedness of the nonlinear case if the delay vanishes.