Uniqueness by the two dimensional local Dirichlet-to-Neumann map on an arbitrary subboundary

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Our Achievements

Global Uniqueness by DN map on arbitrary subboundary

Introduction

$\Omega \subset \mathbb{R}^n$: bounded domain

$$\begin{cases} \mathrm{div}(\gamma \nabla u) & = & 0 \text{ in } \Omega \\ u|_{\partial\Omega} & = & f \end{cases}$$

$$\Lambda_\gamma : f \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

Electrical Impedance Tomography:

Find γ from Λ_γ

proposed by Tikhonov, Calderón

Reduction to potential determination

Electrical Impedance Tomography:

$\mathrm{div}(\gamma \nabla u) = 0$

$\Longrightarrow$

Determination of potential in

$$\Delta w + qw = 0, \quad q = -\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}, w = \sqrt{\gamma}u$$

Uniqueness: whole Cauchy data

spatial dimension $n \geq 3$:

- J. Sylvester - G. Uhlmann (1987): $\gamma \in C^2$
- R. Brown - R. Torres (2003): less regular γ ($3/2$ derivatives in $L^p, p > 2n$)
- L. Päivärinta - A. Panchenko - G. Uhlmann (2003): Lipschitz γ

Uniqueness: whole Cauchy data

$n = 2$:

- A. Nachman (1996): $\gamma \in C^2$
- R. Brown - G. Uhlmann (1997): less regular γ
- K. Astala - L. Päivärinta (2006): $\gamma \in L^\infty$
- A. Bukhgeim (2008): L^∞-potential determination

determination of multiple coefficients in scalar equation

- J. Cheng - M. Yamamoto (2004):

  $\Delta u + a_1 \partial_1 u + a_2 \partial_2 u = 0$

- H. Kang - G. Uhlmann (2004)

Partial Cauchy data

$$\begin{cases} \text{div}(\gamma \nabla u) & = & 0 \text{ in } \Omega \\ u|_{\partial\Omega} & = & f \end{cases}$$

$\Gamma, \widetilde{\Gamma} \subset \partial\Omega$: subboundary

$$\widetilde{\Lambda}_\gamma : \{f \in H^{\frac{1}{2}}(\partial\Omega) | \, \text{supp} f \subset \Gamma\} \longrightarrow \gamma \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}$$

uniqueness by partial Cauchy data

Γ: input subboundary, $\widetilde{\Gamma}$: output subboundary

$n \geq 3$:

- V.Isakov (2006): $\Gamma, \widetilde{\Gamma} \subset$ plane or sphere
- A. Bukhgeim - G. Uhlmann (2002):
  $\Gamma = \partial\Omega$, $\widetilde{\Gamma} \supset\supset$ half of $\partial\Omega$, $\gamma \in C^2$
- K. Knusden (2006): $\gamma \in C^{\alpha+\frac{3}{2}}$, $\alpha > 0$
- H. Heck - J.-N. Wang (2006): stability
- C.Dos Santos Ferreira - C.Kenig - J.Sjöstrand - G. Uhlmann (2007): magnetic Schrödinger

- C. Kenig - J. Sjöstrand - G. Uhlmann (2007):
  $\Gamma \supset$ half of $\partial\Omega$, $\widetilde{\Gamma} \supset\supset \partial\Omega \setminus \Gamma$

Open Problem

No Published Results

for arbitrary input boundary and

output boundary

Our Result $\Rightarrow$ One Answer

Main Result I

$\Omega \subset \mathbb{R}^2$, $\widetilde{\Gamma} \subset \partial\Omega$: arbitrary, $j = 1, 2$

$q_j \in C^{1+\alpha}(\overline{\Omega})$ with $\alpha > 0$, complex-valued

$$C_{q_j} = \left\{ \left(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} \right) \middle| (\Delta + q_j)u = 0 \text{ in } \Omega,\ u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0,\ u \in H^1(\Omega) \right\}$$

(if 0 is eigenvalue, C_{q_j} contains the graph)

Theorem 1: $C_{q_1} = C_{q_2} \Longrightarrow q_1 \equiv q_2$

Main Result II

Theorem 2

Assume: $\gamma_j > 0, \in C^{3+\alpha}(\overline{\Omega})$, $j = 1, 2$ with $\alpha > 0$,
$\gamma_1 = \gamma_2$ on $\partial\Omega$.

$$\Lambda_{\gamma_1} u = \Lambda_{\gamma_2} u \text{ on } \widetilde{\Gamma} \text{ for all } u \in H^{\frac{1}{2}}(\Gamma), \ \operatorname{supp} u \subset \widetilde{\Gamma}.$$

$\Longrightarrow \gamma_1 = \gamma_2.$

Corollary I

Γ: arbitrary subboundary of $\partial\Omega$ such that

$\Gamma_1 \supset \overline{\partial\Omega \setminus \Gamma}$.

$$\widehat{C}_{q_j} = \{(u|_{\Gamma}, \partial_\nu u|_{\Gamma_1});\ (\Delta + q_j)u = 0, \quad u|_{\partial\Omega\setminus\Gamma} = 0\}.$$

Then: $\widehat{C}_{q_1} = \widehat{C}_{q_2}$ implies $q_1 \equiv q_2$ in Ω.

Remark

- C. Kenig, J. Sjöstrand and G. Uhlmann, Ann. of Math. (2007): in dimension ≥ 3
  Γ: larger than the half of $\partial\Omega$
- For Conductivitiy equation: Same uniqueness

Corollary II: in domain with hole

$\Omega, D \in \mathbb{R}^2$: smooth boundary domain such that
$\overline{D} \subset \Omega$.

$V \subset \partial\Omega$: open set.

Let $q_j \in C^{1+\alpha}(\overline{\Omega \setminus D})$ for some $\alpha > 0$, $j = 1, 2$.

$$\widetilde{C}_{q_j} := \{(u|_V, \partial_\nu u|_V); (\Delta + q_j)u = 0 \text{ in } \Omega \setminus \overline{D},$$
$$\text{supp } f \in V,\ u \in H^1(\Omega \setminus \overline{D})\}$$

Then $\widetilde{C}_{q_1} = \widetilde{C}_{q_2} \Longrightarrow q_1 \equiv q_2$.

Remark

For Conductivitiy equation: Same uniqueness

Anisotropic conductivity

$\sigma = \{\sigma_{ij}\}_{1\leq i,j\leq 2}$: positive definite symmetric matrix

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = g.$$

Dirichlet-to-Neumann map:

$$\Lambda_\sigma(g) = \sum_{i,j=1}^{2} \sigma_{ij}\nu_i \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}.$$

Theorem 3 $\sigma_k = \{\sigma_{ij}^{(k)}\}_{1\le i,j\le 2} \in C^{3+\alpha}(\overline{\Omega})$ for $k = 1, 2$ with some constant $\alpha > 0$.

$\Gamma \subset \partial\Omega$: arbitrary relatively open subset.

$\Lambda_{\sigma_1}(g)|_\Gamma = \Lambda_{\sigma_2}(g)|_\Gamma$ for all $g \in H^{\frac{1}{2}}(\partial\Omega)$, supp $g \subset \Gamma$.

Then $\exists F : \overline{\Omega} \to \overline{\Omega}$: diffeomorphism, $F|_{\partial\Omega} = I$,
$F \in C^{4+\alpha}(\overline{\Omega})$, $F_*\sigma_1 = \sigma_2$.
Here $F_*\sigma = \frac{(DF)\cdot\sigma\cdot(DF)^T\cdot F^{-1}}{|detDF|}$, DF: the differential of F.

Traditional key idea for proof

- $u_1 = u_1(\tau)$: parameter τ - depending solution to

  $\Delta u_1 + q_1 u_1 = 0, \quad u_1|_{\partial\Omega\backslash\widetilde{\Gamma}} = 0$

- $u_2 : \Delta u_2 + q_2 u_2 = 0, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$

  DN maps are equal $\Rightarrow \nabla u_1 = \nabla u_2$ on $\widetilde{\Gamma}$

$$u = u_1 - u_2 \Longrightarrow \Delta u + q_2 u = -(q_1 - q_2)u_1$$

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = 0$$

- $v = v(\tau)$: parameter τ-depending solution to

  $\Delta v + q_2 v = 0, \quad v|_{\partial\Omega\backslash\widetilde{\Gamma}} = 0$

$$0 = \int_\Omega v(\Delta u + q_2 u)dx = -\int_\Omega (q_1 - q_2) v u_1 dx$$

$\Longrightarrow \int_\Omega (q_1 - q_2)(x) v(\tau)(x) u_1(\tau)(x) dx = 0$ for all $\tau > 0$

$\Longrightarrow q_1 = q_2$?

How to choose $u_1(\tau)$, $v(\tau)$? $\Leftarrow$

complex geometrical optics solutions $\to$

which choices?

Key to proof

complex geometric optics solution by

Carleman estimate with suitable weight

- A. Bukhgeim - G. Uhlmann (2002)
- C. Kenig - J. Sjöstrand - G. Uhlmann (2007)
- O. Imanuvilov - G. Uhlmann - M. Yamamoto (2008)

Proof: Preliminaries

$i = \sqrt{-1}$, $z = x_1 + ix_2$, $z \leftrightarrow x = (x_1, x_2)$,

$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$,

$\Phi(z) = \varphi(z) + i\psi(z) \in C^2(\overline{\Omega})$: holomorphic in Ω,

Im $\Phi|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0$

($\Rightarrow \nabla\varphi \cdot \nu = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$)

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}$$

Assume: $\mathcal{H} \cap \partial\Omega = \emptyset$, $\partial_z^2 \Phi(z) \neq 0$ $(z \in \mathcal{H})$

$(\Longrightarrow \mathcal{H} = \{\widetilde{x}_1, \dots, \widetilde{x}_\ell\})$

First Step: Carleman estimate

Proposition 1

$u \in H_0^1(\Omega)$: real valued. Then for all large $\tau > 0$:

$$\tau \|ue^{\tau\varphi}\|^2_{L^2(\Omega)} + \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial \nu} e^{\tau\varphi}\right\|^2_{L^2(\partial\Omega \setminus \widetilde{\Gamma})}$$

$$+ \tau^2 \left\| \left|\frac{\partial \Phi}{\partial z}\right| ue^{\tau\varphi} \right\|^2_{L^2(\Omega)}$$

$$\leq C\left(\|(\Delta u)e^{s\varphi}\|^2_{L^2(\Omega)} + \tau \int_{\widetilde{\Gamma}} \left|\frac{\partial u}{\partial \nu}\right|^2 e^{2\tau\varphi} d\sigma \right)$$

application of Carleman estimate

Carleman estimate $\Longrightarrow$ existence of τ-depending solutions to Schrödinger equation with bounds:

$$\Delta u + qu = f \text{ in } \Omega, \quad u|_{\partial\Omega\setminus\widetilde{\Gamma}} = g$$

Proposition 2

Let $\partial\Omega \setminus \widetilde{\Gamma} = \{x \in \partial\Omega; (\nu \cdot \nabla\varphi) = 0\}$. There exists $\tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution such that

$$||ue^{-\tau\varphi}||_{L^2(\Omega)} \leq C(|\tau|^{-1/2}||fe^{-\tau\varphi}||_{L^2(\Omega)} + ||ge^{-\tau\varphi}||_{L^2(\partial\Omega\setminus\widetilde{\Gamma})})$$

Second Step: construction of novel complex geometrical optics solutions

$$\partial_{\bar{z}}^{-1} g(z) := -\frac{1}{\pi}\int_{\Omega} \frac{g(\xi_1,\xi_2)}{\xi_1 + i\xi_2 - z} d\xi_1 d\xi_2, \quad \partial_z^{-1} g := \overline{\partial_{\bar{z}}^{-1}\overline{g}}$$

$$R_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_{\bar{z}}^{-1}(g e^{\tau(\Phi(z)-\overline{\Phi(z)})})$$

$$\widetilde{R}_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_{z}^{-1}(g e^{\tau(\Phi(z)-\overline{\Phi(z)})})$$

Definition of complex geometrical optics solutions

$$\Delta u_1 + q_1 u_1 = 0 \quad \text{in } \Omega$$

$$u_1|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$$

$\Longrightarrow$ Find

$$u_1(x) = e^{\tau\Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau\overline{\Phi(z)}}\overline{(a(z) + a_1(z)/\tau)} + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}$$

Let polynomials $M_1(z)$ and $M_3(z)$ satisfy

$$\partial_z^j(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)) = 0, \quad z \in \mathcal{H}, \ j = 0, 1, 2$$

$$\partial_{\bar{z}}^j(\partial_z^{-1}(\bar{a}q_1)(z) - M_3(\bar{z})) = 0, \quad z \in \mathcal{H}, \ j = 0, 1, 2$$

$e_1, e_2 \in C^\infty(\Omega)$: $e_1 + e_2 \equiv 1$ on $\bar{\Omega}$, e_2 vanishes in a neighborhood of $\mathcal{H}$, e_1 vanishes in a neighborhood of $\partial\Omega$.

Set $\Phi = \varphi + i\psi$

Choice of a, a_0, a_1:

$$a, a_0, a_1 \in C^2(\overline{\Omega}), \quad \partial_{\bar{z}} a = \partial_{\bar{z}} a_0 = \partial_{\bar{z}} a_1 \equiv 0$$

$$\text{Re}\, a|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0$$

$$(a_0(z) + \overline{a_1(z)})|_{\partial\Omega \setminus \widetilde{\Gamma}} = \frac{(\partial_{\bar{z}}^{-1}(a q_1) - M_1(z))}{4\partial_z \Phi}$$

$$+ \frac{(\partial_z^{-1}(\overline{a(z)} q_1) - M_3(\bar{z}))}{4\overline{\partial_z \Phi}}$$

Choice of u_{11}:

$$u_{11} = -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)))$$

$$-\frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})))$$

$$-\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi}$$

$$-\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z}))}{4\overline{\partial_z\Phi}}$$

Apply Proposition 2.

We have to veriry

$$(\nabla\varphi \cdot \nu) = 0 \quad \text{on } \partial\Omega \setminus \widetilde{\Gamma}$$

$\Leftarrow$ Im $\Phi = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$ and C-R equations

Therefore $\widetilde{\Gamma}$ can be arbitrary!

Proposition 2 $\Longrightarrow$ Find u_{12} such that

$$\Delta(u_{12}e^{\tau\varphi}) + q_1 u_{12} e^{\tau\varphi} = -q_1 u_{11} e^{\tau\varphi} + h_1 e^{\tau\varphi} \quad \text{in } \Omega,$$

$$u_{12} = -u_{11} - \frac{1}{\tau}(a_0 + \overline{a_1}) \quad \text{on } \partial\Omega \setminus \widetilde{\Gamma}$$

$$\|u_{12}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty$$

Here:

$$h_1 = e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(a(z)q_1) - M_1(z))}{4\tau\partial_z\Phi}\right)$$

$$+ e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z}))}{4\tau\overline{\partial_z\Phi}}\right)$$

$$- \frac{a_0 q_1}{\tau}e^{i\tau\psi} - \frac{\overline{a_1} q_1}{\tau}e^{-i\tau\psi}$$

Similarly:

$$\Delta v + q_2 v = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$$

Construct solution v of the form

$$v(x) = e^{-\tau\Phi(z)}(a(z) + b_0(z)/\tau) + e^{-\tau\overline{\Phi(z)}}\overline{(a(z) + b_1(z)/\tau)}$$
$$+ e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}$$

$$v_{11} = -\frac{1}{4}e^{-i\tau\psi}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\bar{z}}^{-1}(q_2 a(z)) - M_2(z)))$$

$$
-\frac{1}{4}e^{i\tau\psi}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline{a(z)})-M_4(\overline{z})))
$$

$$
+\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z))}{4\partial_z\Phi}
$$

$$
+\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}
$$

Here

$$
\partial_z^j(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z))=0,\quad z\in\mathcal{H},\ \ j=0,1,2,
$$

$$
\partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_2)(z)-M_4(\overline{z}))=0,\quad z\in\mathcal{H},\ \ j=0,1,2
$$

b_0, b_1: holomorphic functions such that

$$(b_0 + \overline{b}_1)|_{\partial\Omega\setminus\widetilde{\Gamma}} = -\frac{(\partial_{\overline{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z\Phi} - \frac{(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\overline{z}))}{4\overline{\partial_z\Phi}}$$

v_{12}: solution to

$$\Delta(v_{12}e^{-\tau\varphi}) + q_2 v_{12} e^{-\tau\varphi} = -q_2 v_{11} e^{-\tau\varphi} - h_2 e^{-\tau\varphi} \quad \text{in } \Omega,$$

$$v_{12}|_{\partial\Omega\setminus\widetilde{\Gamma}} = -v_{11}|_{\partial\Omega\setminus\widetilde{\Gamma}} - \frac{1}{\tau}(b_0 + \overline{b_1})$$

$$\|v_{12}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right):$$

$$h_2 = e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(a(z)q_2) - M_2(z))}{4\tau\partial_z\Phi}\right)$$

$$+ e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\bar{z}))}{4\tau\overline{\partial_z\Phi}}\right)$$

$$- \frac{b_0 q_2}{\tau}e^{-i\tau\psi} - \frac{\overline{b_1}q_2}{\tau}e^{i\tau\psi}$$

Third Step: Stationary Phase

Key Theorem: Assume: $F \in C^0(\mathbb{R}^2)$,

$\{y \in \text{supp}\, F |\, \nabla F(x) = 0\} = \{y_1, \ldots, y_\ell\}$,

$\det\, (\partial_i \partial_j \varphi)(y_k) \neq 0$

$$\Longrightarrow \int_{\mathbb{R}^2} e^{i\tau\varphi(y)} F(y) dy$$

$$= \frac{1}{2\pi\tau} \sum_{k=1}^{\ell} \frac{e^{i\tau\varphi(y_k)} F(y_k)}{|\det \partial_i \partial_j \varphi(y_k)|^{\frac{1}{2}}} \times e^{\frac{i\pi}{4} \text{sgn}\, (\partial_i \partial_j \varphi(y_k))}$$

$$+ o\left(\frac{1}{\tau}\right)$$

Here:

$$\mathrm{sgn}\,A := {}^{\#}\,[\text{positive eigenvalues of } A] - {}^{\#}[\text{negative eigenvalues of } A]$$

Result by stationary phase

Proposition 3: Let $\{\widetilde{x}_1, \ldots, \widetilde{x}_\ell\}$ be the set of critical points of the function $Im\Phi$. Then for any potentials $q_1, q_2 \in C^{1+\alpha}(\overline{\Omega})$, $\alpha > 0$ with the same Dirichlet-to-Neumann maps and for any holomorphic function a, we have

$$2\sum_{k=1}^{\ell} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x}_k)\mathrm{Re}\, e^{2i\tau \mathrm{Im}\Phi(\widetilde{x}_k)}}{|(\det \mathrm{Im}\Phi'')(\widetilde{x}_k)|^{\frac{1}{2}}} + R = 0, \quad \tau > 0$$

Here R is independent of τ:

$$R = \int_\Omega (q_1 - q_2)(a(a_0 + b_0) + \overline{a}(\overline{a_1} + \overline{b_1}))dx$$
$$+\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\frac{\partial_{\overline{z}}^{-1}(aq_2) - M_2(z)}{\partial_z \Phi} + \overline{a}\frac{\partial_z^{-1}(q_2\overline{a}) - M_4(\overline{z})}{\overline{\partial_z \Phi}}\right)dx$$
$$-\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\frac{(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{\partial_z \Phi} + \overline{a}\frac{(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))}{\overline{\partial_z \Phi}}\right)dx$$

Sketch of Proof of Proposition 3

- Take geometric optics solution u_1 to
  $\Delta u_1 + q_1 u_1 = 0, \quad u_1|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$
- $u_2 : \Delta u_2 + q_2 u_2 = 0, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$
  DN maps are equal $\Rightarrow \nabla u_1 = \nabla u_2$ on $\widetilde{\Gamma}$

$$u = u_1 - u_2 \Longrightarrow \Delta u + q_2 u = -(q_1 - q_2)u_1$$

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = 0$$

- Take geometric optics solution v to
  $\Delta v + q_2 v = 0, \quad v|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$

$$0 = \int_\Omega v(\Delta u + q_2 u)dx = -\int_\Omega (q_1 - q_2) v u_1 dx :$$

Stationary phase + estimates for $u_{12} \Longrightarrow$

$$2\sum_{k=1}^{\ell} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x_k})\,\mathrm{Re}\, e^{2i\tau \mathrm{Im}\Phi(\widetilde{x_k})}}{|(\det \mathrm{Im}\Phi'')(\widetilde{x_k})|^{\frac{1}{2}}}$$

$$+ R = o(1) \quad \text{as } \tau \to \infty$$

[left side] = almost periodic function in τ

Bohr's theorem implies [left side] = 0 for all τ

Hint to Completion of Theorem 1

We can choose Φ such that

$$\operatorname{Im}\Phi(\widetilde{x_k}) \neq \operatorname{Im}\Phi(\widetilde{x_j}),\ j \neq k$$

Let $a(\widetilde{x_k}) \neq 0$

Then Proposition 3 implies

$$q_1(\widetilde{x_k}) = q_2(\widetilde{x_k})$$

Further Topics

Uniqueness : pointwise uniqueness by choice of Φ

$\Longrightarrow$

- Stability
- Reconstruction
- $C_q = \left\{ \left(u|_\Gamma, \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} \right) \middle| (\Delta + q)u = 0 \text{ in } \Omega,\ u|_{\Omega \setminus \Gamma} = 0,\right.$

  $\left. u \in H^1(\Omega) \right\}$: $\Gamma, \widetilde{\Gamma}$: arbitrary without $\partial\Omega \setminus \Gamma \subset \widetilde{\Gamma}$