Finite dimensional issues in PDE control

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Control and Inverse Problems in PDE : Theoretical and Numerical Aspects
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Outline

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1. Introduction
2. Partially dissipative hyperbolic systems
   - Preliminaries in control theory
   - Systems of balance laws
   - Partially dissipative linear hyperbolic systems: (SK) and beyond
   - Future work
3. Connections with hypoellipticity & hypocoercivity
4. Switching and spiky controls
   - Motivation
   - Switching controls
   - Spiky controls

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Basic concepts and results in the theory of control of finite-dimensional systems appear in several relevant issues related to the theory of PDE, its asymptotic behavior and control.

In this lecture we revisit some of them.

We shall focus mainly in the following issues:

- The large time asymptotics for partially dissipative hyperbolic systems.
- Control of hypoelliptic systems.
- Switching and spiky controls for PDEs.

We present some recent results and also point towards some lines of future research.

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The Kalman rank condition

Controllability of finite dimensional linear systems

Let $n, m \in \mathbb{N}^*$ and $T > 0$. Consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0, T), \\ x(0) = x^0. \end{cases} \tag{1}$$

Here A is a real $n \times n$ matrix, B is a real $n \times m$ matrix, $x : [0, T] \longrightarrow \mathbb{R}^n$ represents the *state* and $u : [0, T] \longrightarrow \mathbb{R}^m$ the *control*.

System (1) is **controllable** in time $T > 0$ if given any initial and final one $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0, T, \mathbb{R}^m)$ such that the solution of (1) satisfies $x(T) = x^1$.

Obviously, in practice $m \leq n$. Of particular interest is the case where m is as small as possible.

The control property does not only depend on the dimensions m and n but on how the matrices A and B interact

$$x'(t) = Ax(t) + Bu(t)$$

Theorem

(Kalman, 1958) The system (A, B) is controllable in some time T if and only if

$$rank\,[B, AB, \cdots, A^{n-1}B] = n. \tag{2}$$

Consequently, if system (1) is controllable in some time $T > 0$ it is controllable in any time.

Note that, the so-called controllability matrix $[B, AB, \cdots, A^{n-1}B]$ is of dimension $n \times nm$.

Thus, in the limit case where $m = 1$ (one single control), it is a $n \times n$ matrix. In this case the Kalman rank condition requires it to be invertible.

Proof of Theorem 1:

" ⇒" Suppose that $\mathrm{rank}([B, AB, \cdots, A^{n-1}B]) < n$.
From Cayley-Hamilton Theorem we deduce that $\mathrm{rank}([B, AB, \cdots, A^{n-1}B, ...]) < n$. Thus, the rank of $e^{At}B$ is confined in a vector space of dimension $< n$ for all $t > 0$.
From the variation of constants formula, the solution x of (1) satisfies

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

But $\int_0^t e^{A(t-s)}Bu(s)ds$ is confined to a subspace of dimension $< n$.
Consequently, there are non trivial directions $v \in \mathbb{R}^n$ such that

$$\langle v, x(T)\rangle = \langle v, e^{AT}x^0\rangle + \int_0^T \langle v, e^{A(T-s)}Bu(s)\rangle ds = \langle v, e^{AT}x^0\rangle.$$

Stabilization of finite dimensional linear systems

Assume, to fix ideas, that A is a skew-adjoint matrix, i. e. $A^* = -A$. In this case, $< Ax, x >= 0$. In the absence of control the energy of solutions is conserved.

The following result guarantess the exponential decay of solutions for a suitable *feedback control mechanism:*

Theorem

If A *is skew-adjoint and the pair* (A, B) *is controllable then* $L = -B^*$ *stabilizes the system, i.e. the solution of*

$$x' = Ax - BB^*x, \quad x(0) = x^0 \tag{3}$$

has an uniform exponential decay: $|x(t)| \leq ce^{-\omega t}|x^0|$.

Proof: With $L = -B^*$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = - < BB^*x(t), x(t) >= - \mid B^*x(t) \mid^2 \leq 0.$$

An example: The harmonic oscillator

Consider the damped harmonic oscillator:

$$mx'' + Rx + kx' = 0, \tag{4}$$

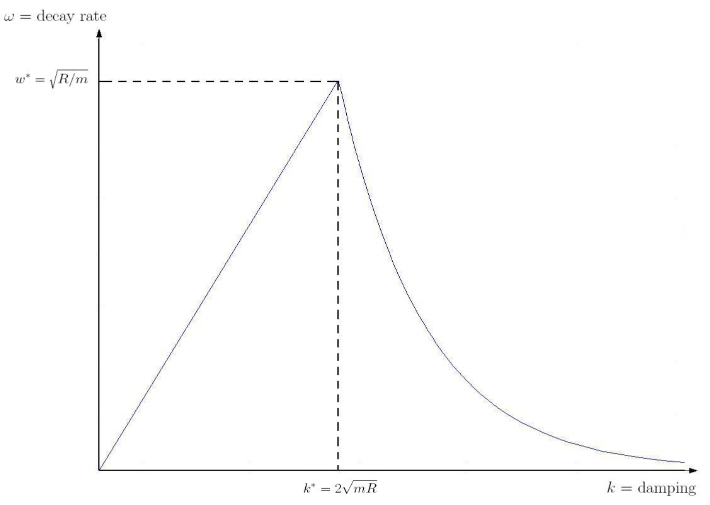
where m, k and R are positive constants.
It is easy to see that the solutions of this equation have an exponential decay property. Indeed, it is sufficient to remark that the two characteristic roots have negative real part. Indeed,

$$mr^2 + R + kr = 0 \Leftrightarrow r_\pm = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\mathrm{Re}\, r_\pm = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \leq 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \geq 4mR. \end{cases}$$

We observe here the classical overdamping phenomenon. Contradicting a first intuition it is not true that the decay rate increases when the value of the damping parameter k increases.



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Systems of balance laws

We consider nonlinear hyperbolic systems of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} \frac{\partial F_j(w)}{\partial x_j} = Q(w), \quad x \in \mathbb{R}^m,\ t > 0,$$

$$\begin{array}{cccccc} w: & \mathbb{R} & \times & \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ & (t & , & x) & \longmapsto & w(t,x) \end{array}$$

arising in so many applications: fluid mechanics, gas dynamics, traffic flow, ralaxation, ...

Local (in time) smooth solutions are known to exist (C. Dafermos, L. Hsiao-T.-P. Liu, A. Majda, D. Serre,...)
But possible singularities (i.e. shock waves) may arise in finite time.
The nonlinear term Q may play the role of a partial dissipation and help to the existence of global smooth solutions.

Global smooth solutions in a neighborhood of a constant equilibrium: $Q(w^*) = 0$.

THE METHOD = LINEARIZATION + FIXED POINT.

If the linearized dynamics exhibits solutions that decay as $t \to \infty$, a perturbation argument may hopefully allow showing that, locally around the constant equilibrium, solutions are global and decay as well.

One has to distinguish:

- Total dissipation : $Q(w) = -Bw$, $B > 0$
- Partial dissipation: $Q(w) = \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix} w$.

  Y. Shizuta & S. Kawashima (85), Y. Zeng (99), W. A. Yong (04), S. Bianchini, B. Hanouzet & R. Natalini (03, $m = 1$),...

Example : Isentropic Euler equations with damping

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial f(u)}{\partial x} = -v$$

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Partially dissipative linear hyperbolic systems

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j} = -Bw \ , \ x \in \mathbb{R}^m \ , \ w \in \mathbb{R}^n \qquad (5)$$

$$\begin{array}{c} A_1, \ldots, A_m \\ \text{symmetric} \end{array} \ \Bigg| \ B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \ \begin{array}{c} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \qquad \begin{array}{c} X^t D X > 0 \\ \forall X \in \mathbb{R}^{n_2} - \{0\} \end{array}$$

Goal: Understand the asymptotic behavior as $t \to \infty$.

Apply Fourier transform:

$$\frac{\partial \hat{w}}{\partial t} = (-B - iA(\xi))\hat{w} \qquad \text{where} \qquad A(\xi) := \sum_{j=1}^{m} \xi_j A_j$$

Lack coercivity :

$\langle [B + iA(\xi)]X, X \rangle = \langle BX, X \rangle = \langle DX_2, X_2 \rangle \not\geqslant c|X|^2$

But possible decay depending on ξ:

$$\exp[(-B - iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}$$

PARTIALLY DISSIPATIVE LINEAR HYPERBOLIC SYSTEM

$\equiv$

m-PARAMETER (ξ) FAMILY OF FINITE-DIMENSIONAL PARTIALLY DISSIPATIVE n-DIMENSIONAL SYSTEMS.

The asymptotic behavior of solutions is determined by the behavior of the function $\xi \rightarrow \lambda(\xi)$ giving the decay rate as a function of ξ.

Example: The dissipative wave equation

$$u_{tt} - u_{xx} + u_t = 0.$$

$$u_t = v_x - u; \quad v_t = u_x.$$

Solutions may be decomposed as

- A high frequency component tending to zero exponentially fast as $t \rightarrow 0$;
- A low frequency component with the same decay rate as the heat kernel $t^{-1/2}$ decay in L^∞ for L^1 data.

This corresponds to a function $\lambda(\xi) \sim \min(|\xi|^2, 1)$.

(SK) for partially dissipative linear hyperbolic systems

The pioneering and well-known result by Shizuta-Kawashima (85) may be viewed as a generalization of the example above on the dissipative wave equation.

Under the condition (SK) below
(SK) : $\forall \xi \in \mathbb{R}^m$, **Ker**$(B) \cap \{$**eigenvectors of** $A(\xi)\} = \{0\}$

$$\Rightarrow \exp[(-B - iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}, \qquad \lambda(\xi) = c\min(1, |\xi|^2)$$

Their proof is based on a (long) linear algebra computation.

Consequences :
(1) $\forall w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n), \qquad w = w_l + w_h$

$$\left|w_h(t)\right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ce^{-\gamma t}\left|w^0\right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)}$$

$$\left|w_l(t)\right|_{L^\infty(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ct^{-\frac{m}{2}}\left|w^0\right|_{L^1(\mathbb{R}^m,\mathbb{R}^n)}$$

(2) Global smooth solutions for the NL system for initial data close to a constant equilibrium.

Our contributions (K. Beauchard & E. Z., ARMA, to appear)

1st step :
(SK) and Kalman rank condition

2nd step :
Measure of the decay rate for $B + iA(\xi)$

3rd step :
Classification of the asymptotic behavior for linear hyperbolic systems (with/without (SK))

4th step :
Nonlinear systems of balance laws :
global smooth solutions without (SK)

(SK) and the Kalman rank condition

(SK) and Kalman rank conditions

Lemma

The (SK) condition is equivalent to imposing the Kalman rank condition to the pairs $(A(\xi), B)$ for all values of $\xi \in \mathbb{R}^m$.

$$A \text{ symmetric} \;\Big|\; B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix} \qquad \begin{matrix} X^t D X > 0 \\ \forall X \in \mathbb{R}^{n_2} - \{0\} \end{matrix}$$

(SK) : $\operatorname{Ker}(B) \cap \{\text{eigenvectors of } A\} = \{0\}$

$\Leftrightarrow$ The pair (A, B) satisfies the Kalman rank condition

$\Leftrightarrow Be^{At}X \equiv 0$ for all $t > 0 \implies X \equiv 0$

$\Leftrightarrow \sum_{k=0}^{n-1} |BA^k X|^2$ is a norm on $\mathbb{R}^n$ (the Kalman quadratic form)

- The (SK) condition requires the conditions above to be satisfied for all $\xi \in \mathbb{R}^m$.
- In $1-d$ $(m=1)$ the (SK) condition is sharp. Whenever it fails, travelling wave solutions with L^2-profiles exist, thus making the decay of solutions impossible. This is so because, in $1-d$, the Kalman condition holds for all $\xi \in \mathbb{R}$ if and only if the pair (A, B) sastifies the Kalman condition.
- This is not true in the multi-dimensional case since $A(\xi) = A(\xi) := \sum_{j=1}^{m} \xi_j A_j$ depends on ξ in a non-trivial way.
- In view of the analysis above it is more natural to analyse the positive definiteness of the quadratic form $\sum_{k=0}^{n-1} |BA(\xi)^k X|^2$, as function of ξ.
- This will illustrate the existence of many other scenarios that (SK) excludes, except in one space dimension $(m=1)$. In the multi-dimensional case (SK) is a sufficient condition for decay, but is far from being necessary.

Decay rate for $B + iA(\xi)$ as a function of ξ

A measure of the decay rate as a function of ξ

$A_1, \dots, A_m$ symmetric | $B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix}$ | $A(\xi) := \sum_{j=1}^{m} \xi_j A_j$

$\xi = \rho\omega \in \mathbb{R}^m$ $\quad \rho > 0 \quad \omega \in S^{m-1} \quad (m_k) \uparrow$ well chosen

$$N_{*,\epsilon}(\omega) := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\}.$$

Theorem

$\exists \epsilon_* > 0, c > 0$ *such that* $\forall \epsilon \in (0, \epsilon_*)$,

$$\exp[(-B - i\rho A(\omega))t] \leqslant 2e^{-cN_{*,\epsilon}(\omega)\min\{1,\rho^2\}t}.$$

Remark : **(SK)** $\Leftrightarrow N_{*,\epsilon}(\omega) \geqslant N_{*,\epsilon} > 0, \forall \omega \in S^{m-1}$.
In general, $N_{*,\epsilon}(\omega)$ may vanish for some values of $\omega \in S^{m-1}$, in which case the decomposition of solutions and its asymptotic form is more complex.

Our proof:

- Is based on energy arguments and the construction of a suitable Lyapunov functional that allows exhibiting the exponential decay rate. In that sense it is similar to the techniques employed for proving decay rates for dissipative wave equations, and the works by C. Villani et al. on the decay for kinetic equations and hypocoercivity.
- Yields the result by Shizuta-Kawashima in a simpler way, but shows that it only covers one of the many possible behaviors one may encounter for $m \geq 2$.
- Provides quantitative estimates on the decay rate as a function of ξ that can be used for a better understanding of the nonlinear problem.

An example: the dissipative wave equation.

$$u_{tt} - u_{xx} + u_t = 0.$$

$$w_{tt} + \xi^2 w + w_t = 0.$$

$$e_\xi(t) = \frac{1}{2}[|w_t|^2 + \xi^2|w|^2].$$

$$\frac{de_\xi(t)}{dt} = -|w_t|^2.$$

The exponential decay rate does not come directly out of this. But it is easy to check that

$$f_\xi(t) = e_\xi(t) + \varepsilon w w_t,$$

is, for ε small enough, such that

$$f_\xi \sim e_\xi$$

and, on the other hand,

$$\frac{df_\xi(t)}{dt} \leq -c(\xi, \varepsilon) f_\xi(t).$$

Indeed,

$$\frac{df_\xi(t)}{dt} = -|w_t|^2 + \varepsilon|w_t|^2 + \varepsilon w w_{tt} = -(1-\varepsilon)|w_t|^2 - \varepsilon\xi^2 w^2 - \varepsilon w w_t$$

$$\leq -(1 - \varepsilon - \tfrac{\varepsilon}{2\xi^2})|w_t|^2 - \tfrac{\varepsilon\xi^2}{2} w^2.$$

Then, it suffices to take:

$$\varepsilon = \min\left(\xi^2, \frac{1}{4}\right).$$

Note that there is an extensive literature on the extensions of this result to nonlinear problems.

Decay rate for $B+iA(\xi)$: $\rho<1$

$$\omega \in S^{m-1}:\ \sum_{k=0}^{n-1} \epsilon^{m_k}|BA(\omega)^k x|^2 \geq N_{*,\epsilon} > 0, \forall x \in S^{n-1}.$$

$\dot{x} = (-B - i\rho A(\omega))x,$ **Goal :**$|x(t)| \leqslant 2|x_0|e^{-cN_{*,\epsilon}\rho^2 t}$

Strategy : find $\mathcal{L}(x) \sim |x|^2$ such that $\frac{d\mathcal{L}}{dt} \leqslant -c\rho^2 N_{*,\epsilon}\mathcal{L}$

$$\mathcal{L}(x) = |x|^2 + \rho\sum_{k=1}^{n-1}\epsilon^{m_k}\mathrm{Im}\left(\langle A(\omega)^k BBA(\omega)^{k-1}x, x\rangle\right)$$

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= -2\mathrm{Re}\left(\langle (B+i\rho A_\omega)x, x\rangle\right)\\ &\quad -\rho\sum_{k=1}^{n-1}\epsilon^{m_k}\mathrm{Im}\left(\langle (A_\omega^t)^k B^t BA_\omega^{k-1}(B+i\rho A_\omega)x, x\rangle\right)\\ &\quad -\rho\sum_{k=1}^{n-1}\epsilon^{m_k}\mathrm{Im}\left(\langle (A_\omega^t)^k B^t BA_\omega^{k-1}x, (B+i\rho A_\omega)x\rangle\right)\\ &\leqslant -2C_1|Bx|^2 - \rho^2\sum_{k=1}^{n-1}\epsilon^{m_k}|BA_\omega^k x|^2\\ &\quad +\rho\sum_{k=1}^{n-1}\epsilon^{m_k}|Bx|\left[\left|BA_\omega^{k-1}\right||BA_\omega^k x| + \left|BA_\omega^k\right||BA_\omega^{k-1}x|\right]\\ &\quad +\rho^2\sum_{k=1}^{n-1}\epsilon^{m_k}|BA_\omega^{k-1}x||BA_\omega^{k+1}x|\end{aligned}$$

Asymptotic behavior for linear hyperbolic systems (with or without (SK))

The set of degeneracy

The minimum of Kalman's quadratic form
$N_{*,\epsilon}(\omega) := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\}$
measures the decay rate for $B + i\rho A(\omega)$

$N_{*,\epsilon}(\omega) > 0 \;\; \Leftrightarrow \;\; \mathrm{Ker}(B) \cap \{\text{eigenvectors of } A(\omega)\} = \{0\}$
$\Leftrightarrow$ $(A(\omega), B)$ satisfies the Kalman rank condition.

The set of degeneracy :
$\mathcal{D}(B + iA(\xi)) = \{\xi \in \mathbb{R}^m; rank[B|BA(\xi)|...|BA(\xi)^{n-1}] < n\}$
is an algebraic submanifold

$\rightarrow$ either $|\mathcal{D}| = 0 \Leftrightarrow N_{*,\epsilon} > 0$ a.e. $\Rightarrow$ strong L^2 stability
$\rightarrow$ or $\mathcal{D} = \mathbb{R}^m$: $\exists$ non dissipated solutions
Decomposition ?

$|\mathcal{D}| = 0$, $n_1 = 1$ (B is effective in all but one components)

Theorem

When $n_1 = 1$, $\mathcal{D}$ *is a vector subspace of* $\mathbb{R}^m$ *and*

$$N_{*,\epsilon}(\omega) \geqslant c \min\{1, dist(\omega, \mathcal{D})^2\}, \forall \omega \in S^{m-1}.$$

As a consequence we have the following new decomposition

$$w = w_h + w_l + w_{new}$$

with

$$\begin{aligned}
\left|w_h(t)\right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)} &\leqslant Ce^{-t}\left|w^0\right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)}\\
\left|w_l(t)\right|_{L^\infty(\mathbb{R}^m,\mathbb{R}^n)} &\leqslant Ct^{-\frac{m}{2}}\left|w^0\right|_{L^1(\mathbb{R}^m,\mathbb{R}^n)}\\
\left|w_{new}(t)\right|_{L^\infty(\mathbb{R}^m,\mathbb{R}^n)} &\leqslant Ct^{-\frac{1}{2}}\left|w^0\right|_{*}
\end{aligned}$$

Example: $n = m = 2$; $\mathcal{D} = \{(\xi_1, \xi_2) : a^1_{21}\xi_1 + a^2_{21}\xi_2 = 0\}$.

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Future work

Our key result is the description of how the Kalman rank condition for the pairs $(B + iA(\omega), B)$ varies when ω covers the unit sphere S^{m-1} and in particular the result:

$$N_{*,\epsilon}(\omega) \geqslant c \min\{1, \mathrm{dist}(\omega, \mathcal{D})^2\}, \forall \omega \in S^{m-1}.$$

But these was proved under the condition $n_1 = 1$.
A systematic analysis of this issue is still to be done.
This topic is related also to the uniformity of the controllability properties of pairs of the form $(B + iA(\omega), B)$. Note that

$$A(\omega) := \sum_{j=1}^{m} \omega_j A_j.$$

This issue is somehow related to:

- The existing literature on control of systems of PDE (navier-Stokes, elasticity, reaction-diffusion,...) with a restricted number of controls.
- The cost of controlling as $T \to 0$.

Connections with hypoellipticity & hypocoercivity

Hypoellipticity $\equiv$ The fundamental solution is C^∞ away form the diagonal (....L. Hörmander, 1968....)

$$\partial_t u - \sum_{j,k=1}^{n} a_{jk}\partial_{jk}^2 u + \sum_{j,k=1}^{n} b_{jk}x_j\partial_k u = 0.$$

Example: Kolmogorov equation: $u_t - u_{xx} - xu_y = 0$.
After applying Fourier transform:

$$U_t - A(\xi,\xi)U - \sum_{j,k=1}^{n} b_{jk}\partial_j(\xi_k U) = 0.$$

Hypoellipticity fails iff $A(e^{Bs\xi}, e^{Bs\xi}) = 0$ for all $s >$ for some $\xi \neq 0$.

Hypocoercivity (....C. Villani, 2006,....)

$$f_t + Lf = 0$$

$$L = A^*A + B, \qquad \text{where } B^* = -B.$$

Example: Similar models as above but in the context of kinetic equations: Fokker-Planck equation:

$$f_t - \Delta_v f + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f - \nabla_v \cdot (vf) = 0.$$

Despite the lack of coercivity of L, under suitable assumptions on the commutators of A and B, one can build a modified energy or suitable Lyapunov functional (similar to our construction of the functional $\mathcal{L}$) in which the time exponential decay can be proved.

Control.

According to recent work in collaboration with K. Beauchard (to appear in Ann. IHP), the Kolmogorov equation $u_t - u_{xx} - xu_y = 0$ is controllable when the control acts on a band covering the whole range of $y's$.

In that case the needed observability inequalities can be proved by using Fourier transform in the y-variable that leads to the one-parameter family of equations $\tilde{u}_t - \tilde{u}_{xx} - i\eta x\tilde{u} = 0$.

One can the apply standard arguments combining the dissipativity properties of the system and Carleman inequalities.

The understanding of the sharp geometric conditions to get observability is a widely open subject. But this needs of a finer use of the interaction of convection and diffusion in the x, y variables at the control level, in the spirit of what is done in the context of hypocoercivity and hypoellipticity.

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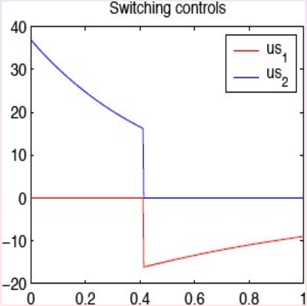
Motivation

To develop systematic strategies allowing to build switching controllers.

The controllers of a system endowed with different actuators are said to be of switching form when only one of them is active in each instant of time.



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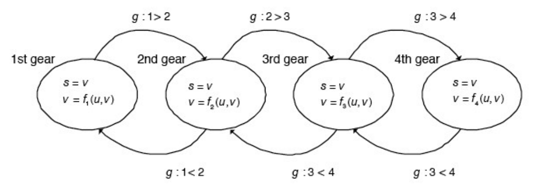
The finite-dimensional case

Consider the finite dimensional linear control system

$$\begin{cases} x'(t) = Ax(t) + u_1(t)b_1 + u_2(t)b_2 \\ x(0) = x^0. \end{cases} \tag{6}$$

$x(t) = \big(x_1(t), \ldots, x_N(t)\big) \in \mathbb{R}^N$ is the state of the system, A is a $N \times N-$matrix, $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are two scalar controls an b_1, b_2 are given control vectors in $\mathbb{R}^N$.

More general and complex systems may also involve switching in the state equation itself:



$$x'(t) = A(t)x(t) + u_1(t)b_1 + u_2(t)b_2, \quad A(t) \in \{A_1, ..., A_M\}.$$

Controllability:

Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for control pairs $\left(u_1,\ u_2\right)$ such that the solution of (16) satisfies

$$x(T) = x^1. \tag{7}$$

In the absence of constraints, controllability holds if and only if the Kalman rank condition is satisfied

$$\left[B,\ AB, \ldots, A^{N-1}B\right] = N \tag{8}$$

with $B = \left(b_1,\ b_2\right)$.

We look for **switching controls:**

$$u_1(t)u_2(t) = 0, \qquad \text{a.e.} \quad t \in (0,\ T). \tag{9}$$

Under the rank condition above, these switching controls always exist.

The classical theory guarantees that the standard controls (u_1, u_2) may be built by minimizing the functional

$$J\left(\varphi^0\right) = \frac{1}{2}\int_0^T \left[|b_1 \cdot \varphi(t)|^2 + |b_2 \cdot \varphi(t)|^2\right] dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0),$$

among the solutions of the adjoint system

$$\begin{cases} -\varphi'(t) = A^*\varphi(t), & t \in (0,\ T) \\ \varphi(T) = \varphi^0. \end{cases} \tag{10}$$

The rank condition for the pair (A, B) is equivalent to the following unique continuation property for the adjoint system which suffices to show the coercivity of the functional:

$$b_1 \cdot \varphi(t) = b_2 \cdot \varphi(t) = 0, \quad \forall t \in [0,\ T] \rightarrow \varphi \equiv 0.$$

Under further rank conditions, the following functional, which is a variant of the functional J, with the same coercivity properties, allows building switching controllers, without an a priori partition of the time interval $[0, T]$:

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\left(\left|b_1 \cdot \varphi(t)\right|^2, \left|b_2 \cdot \varphi(t)\right|^2\right) dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0). \tag{11}$$

Theorem

Assume that the pairs $(A, b_2 - b_1)$ and $(A, b_2 + b_1)$ satisfy the rank condition. Then, for all $T > 0$, J_s achieves its minimum at least on a minimizer $\tilde{\varphi}^0$. Furthermore, the switching controllers

$$\begin{cases} u_1(t) = \tilde{\varphi}(t)\cdot b_1 & \text{when} \quad \left|\tilde{\varphi}(t)\cdot b_1\right| > \left|\tilde{\varphi}(t)\cdot b_2\right| \\ u_2(t) = \tilde{\varphi}(t)\cdot b_2 & \text{when} \quad \left|\tilde{\varphi}(t)\cdot b_2\right| > \left|\tilde{\varphi}(t)\cdot b_1\right| \end{cases} \tag{12}$$

where $\tilde{\varphi}$ is the solution of (10) with datum $\tilde{\varphi}^0$ at time $t = T$, control the system.

- The rank conditions on the pairs $\big(A,\ b_2 \pm b_1\big)$ are needed to ensure that the set

$$\big\{ t \in (0,\ T) :\ \big|\varphi(t) \cdot b_1\big| = \big|\varphi(t) \cdot b_2\big| \big\} \tag{13}$$

  is of null measure, which ensures that the controls in (12) are genuinely of switching form.

The Euler-Lagrange equations associated to the minimization of J_s take the form

$$\int_{S_1} \tilde{\varphi}(t)\cdot b_1\, \psi(t)\cdot b_1 dt + \int_{S_2} \tilde{\varphi}(t)\cdot b_2\, \psi(t)\cdot b_2 dt - x^1\cdot\psi^0 + x^0\cdot\psi(0) = 0,$$

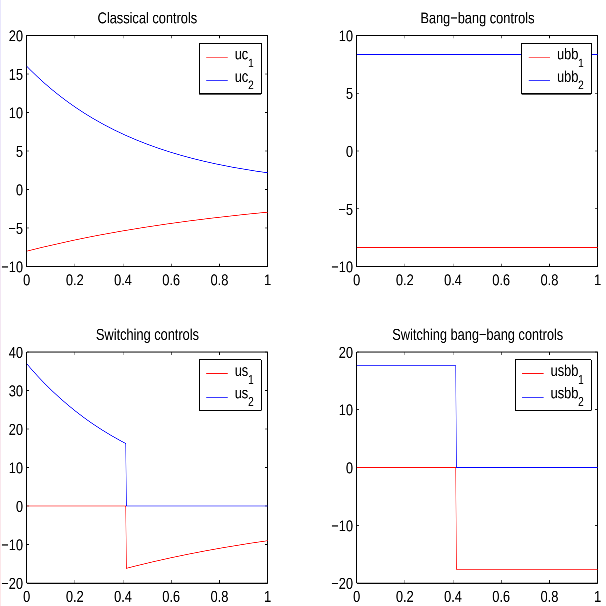
for all $\psi^0 \in \mathbb{R}^N$, where

$$\begin{cases} S_1 = \{t \in (0,\ T) : |\tilde{\varphi}(t)\cdot b_1| > |\tilde{\varphi}(t)\cdot b_2|\}, \\ S_2 = \{t \in (0,\ T) : |\tilde{\varphi}(t)\cdot b_1| < |\tilde{\varphi}(t)\cdot b_2|\}. \end{cases} \tag{14}$$

In view of this we conclude that

$$u_1(t) = \tilde{\varphi}(t)\cdot b_1\, 1_{S_1}(t), \quad u_2(t) = \tilde{\varphi}(t)\cdot b_2\, 1_{S_2}(t), \tag{15}$$

where 1_{S_1} and 1_{S_2} stand for the characteristic functions of the sets S_1 and S_2, are such that the switching condition holds and the corresponding solution satisfies the final control requirement.


Classical controls
uc1
uc2
20
15
10
5
0
−5
−10
0
0.2
0.4
0.6
0.8
1
Bang−bang controls
ubb1
ubb2
10
5
0
−5
−10
Switching controls
us1
us2
40
30
20
10
0
−10
−20
Switching bang−bang controls
usbb1
usbb2
20
10
0
−10
−20

Optimality:

The switching controls we obtain this way are of minimal $L^2\left(0,\ T;\ \mathbb{R}^2\right)$-norm, the space $\mathbb{R}^2$ being endowed with the ℓ^1 norm, i. e. with respect to the norm

$$||(u_1, u_2)||_{L^2(0,\ T;\,\ell^1)} = \left[\int_0^T (|\tilde{u}_1| + |\tilde{u}_2|)^2 dt\right]^{1/2}.$$

Switching bang-bang controls are of minimal $L^\infty\left(0,\ T;\ \mathbb{R}^2\right)$-norm

Outline

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4. Switching and spiky controls
    - Motivation
    - Switching controls
    - Spiky controls

Spiky controls:

We have seen that, to some extent, "L^∞-minimization", leads to controls of minimal L^1-norm.

The same ideas can be implemented in classical control problems to get "spiky" controls, combination of isolated Diract deltas.

Consider the finite-dimensional system

$$\begin{cases} x'(t) = Ax(t) + u(t)b \\ x(0) = x^0, \end{cases} \tag{16}$$

and the following quadratic functional associated to the solutions of the adjoint system:

$$J\left(\varphi^0\right) = \frac{1}{2}||b\cdot\varphi(t)||^2_{L^\infty(0,T)} - x^1\cdot\varphi^0 + x^0\cdot\varphi(0).$$

The functional is coercive and convex and the minimizer exists under the Kalman rank condition.

What is the Euler-Lagrange equation?

Let us assume that the L^∞-norm of the minimizer $\tilde{\varphi}^0$ is achieved on a single point $t_0 \in [0, T]$. Then, when computing the derivative of J at $\tilde{\varphi}^0$ in the direction of ψ^0 we get

$$b \cdot \tilde{\varphi}(t_0) b \cdot \tilde{\psi}(t_0) - x^1 \cdot \psi^0 + x^0 \cdot \psi(0) = 0.$$

This implies that the control we obtain is of the form

$$u(t) = \lambda \delta_{t_0}, \quad \lambda = b \cdot \tilde{\varphi}(t_0).$$

These controls are of minimal "L^1-norm". They yield, obviously, discontinuous trajectories.

But, what about the assumption on the uniqueness of the time instants t for which the norm $||b \cdot \tilde{\varphi}(t)||_{L^\infty(0,T)}$ is achieved? Obviously it fails in general. The set of time instants t for which this maximum is attained can well be a finite set of isolated points. This would correspond to a finite number of spiky controls.

But can this set contain an infinite number of points?
Then, necessarily, by analiticity, $|b \cdot \tilde{\varphi}(t)|$ has to be a constant.
When the control is scalar, this means that $b \cdot \tilde{\varphi}(t)$ is necessarily a constant. By the Kalman rank condition this means that $\tilde{\varphi}$ is a constant solution which can be easily ruled out assuming that A is invertible, which is generically true.
But what happens when the number of controllers is bigger than one?
What extra condition is needed in the pair (A, B) to guarantee that the controls are spiky?
These questions make sense for PDE's too where the role of spiky controls also arises in what concerns their space distribution.

- E. Z., Switching controls, JEMS, to appear.