CPR-AMG preconditioner for multiphase porous media flows: a few experiments

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Workshop MOMAS
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Two Phase (water - oil) Darcy flow model

\( S^w \) and \( S^o \): volume fractions

\( P^w \) and \( P^o \): phases pressures

\[
\begin{align*}
U^w &= -\frac{k_r^w(S^w)}{\mu^w} K(x) \left( \nabla P^w - \rho^w g \right), \\
U^o &= -\frac{k_r^o(S^o)}{\mu^o} K(x) \left( \nabla P^o - \rho^o g \right),
\end{align*}
\]

\( P^o = P^w + P_c(S^o) \),

\( S^w + S^o = 1 \).
Immiscible two phase (water-oil) Darcy flow

- Conservation equations

\[
\begin{align*}
\partial_t \left( \phi(x) \rho^{o} S^{o} \right) + \text{div} (\rho^{o} U^{o}) &= 0 \\
\partial_t \left( \phi(x) \rho^{w} S^{w} \right) + \text{div} (\rho^{w} U^{w}) &= 0
\end{align*}
\]

- Darcy velocities

\[
\begin{align*}
U^{o} &= -\frac{k^g_r(S^{o})}{\mu^{o}} K(x) \left( \nabla P^{o} - \rho^{o}g \right) \\
U^{w} &= -\frac{k^r_w(S^{w})}{\mu^{w}} K(x) \left( \nabla P^{w} - \rho^{w}g \right)
\end{align*}
\]

- Closure laws

\[
\begin{align*}
S^{w} + S^{o} &= 1, \\
P^{o} &= P^{w} + P_{c}(S^{o}).
\end{align*}
\]
Linear systems arising from Fully Implicit formulations

- Coupling of an elliptic/parabolic unknown (pressure) with hyperbolic/degenerate parabolic unknowns (saturation, compositions)
- Strong heterogeneities, anisotropy of the media
- Usually globally structured but locally unstructured meshes (pinch out, faults, LGR)
Algebraic MultiGrid (AMG)

- Basic ingredients
  - Fixed smoother (Gauss Seidel)
  - Adapted Coarse space built from the graph of the matrix and the coefficients (strong connections)
  - Coefficient dependent prolongation operator

- 1 Vcycle of AMG used as preconditioner combined with a Krylov method

- AMG paradigm: low frequency modes not damped by GS smoother are smooth along strong connections
- Very efficient on 2nd order elliptic problems with strong heterogeneities and anisotropy
- But does not apply on the full $P, S$ system ...
CPR-AMG Preconditioner: multiplicative combination of ILU0 on $A$ and AMG on the elliptic bloc $A_{11}$.

- $X^{(1)} = \text{ILU0}(A)^{-1} R$
- $X^{(2)}_1 = \text{AMG}(A_{11})^{-1} \left( R_1 - A_{11} X^{(1)}_1 - A_{12} X^{(1)}_2 \right)$
- $X = \begin{pmatrix} X^{(1)}_1 + X^{(2)}_1 \\ X^{(1)}_2 \end{pmatrix}$
CPR-AMG preconditioner

When it works?

- The bad conditioning of the system is concentrated in the pressure block and efficiently dealt with AMG.
- Other diagonal blocks and unknowns couplings are efficiently dealt with a cheap preconditioner on the global system like ILU0.
- AMG only apply on a small part of the system: $1/n^2$ of the matrix for $n$ unknowns per cell.
CPR-AMG preconditioner

- Decoupling stage: $A = DJ$ with $D$ block diagonal matrix
  - (i) Define a pressure block adapted to AMG paradigm
  - (ii) Ensure a good decoupling of the $P$ and $S$ unknowns
  - In practice (i) is more important than (ii)
  - (iii) Ensure non-zero diagonal entries for the $S$ block (if pointwise ILU0 is used)

- Examples for the two phase immiscible Darcy flow model:
  - $D_\kappa = \begin{pmatrix} (J_{11})_{\kappa,\kappa} & (J_{12})_{\kappa,\kappa} \\ (J_{21})_{\kappa,\kappa} & (J_{22})_{\kappa,\kappa} \end{pmatrix}^{-1}$ bad choice due to diagonal scaling!
  - $D_\kappa = \begin{pmatrix} 1 & -\frac{(J_{12})_{\kappa,\kappa}}{(J_{22})_{\kappa,\kappa}} \\ 0 & 1 \end{pmatrix}$ can lead to negative diagonal entries!
  - $D_\kappa = \begin{pmatrix} (J_{11} + J_{22})J_{22} \\ J_{11}J_{22} - J_{12}J_{21} \end{pmatrix}_{\kappa,\kappa}^{-1} - \begin{pmatrix} (J_{11} + J_{22})J_{12} \\ J_{11}J_{22} - J_{12}J_{21} \end{pmatrix}_{\kappa,\kappa}$
  - $D_\kappa = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
CPR-AMG preconditioner

- Constrained pressure residual or Combinative methods
  - A. Behie et al: 1982
  - J.R. Wallis et al: 1983

- CPR-AMG or Combinative AMG
  - Y. Vassilevski, S. Lacroix, M. Wheeler 2001
  - Applications to basin models: R. Scheichl, P. Quandalle, R.M. 2003
  - H. Cao, H. Tchelepi, J. Wallis: 2005
  - State of the art solver in reservoir simulation for large meshes or highly heterogeneous media (Pumaflow 2003, Intersect 2005)
AMG Codes

- AMG1R5: Ruge et Stuben 1985 f77
- HYPRE: Ruge et Stuben based, parallel, connected to PETSC
- TRILINOS: aggregation based AMG, parallel
- DUNE ISTL: aggregation based AMG, parallel
- SAMG: Ruge et Stuben, parallel, systems, commercial
TPFA finite volume discretization of Multiphase Darcy flows

- Discretization
- Cell centred unknowns: $P^\alpha_\kappa$, $S^\alpha_\kappa$, $\alpha = w, o$
- Discrete conservation laws on each cell $\kappa$:

\[
\int_{\kappa} \left( (\phi \rho^o S^o)^n - (\phi \rho^o S^o)^{n-1} \right) dx \\
+ \sum_{\kappa' \in \mathcal{M}_\kappa} \int_{\sigma = \kappa \kappa'} \int_{t^{n-1}}^{t^n} \rho^\alpha \mathbf{U}^\alpha \cdot \mathbf{n}_{\kappa \kappa'} d\sigma =
\]

- Implicit Euler time integration
- Conservative approximation of the fluxes

\[
\int_{\sigma = \kappa \kappa'} \rho^\alpha \mathbf{U}^\alpha \cdot \mathbf{n}_{\kappa \kappa'} d\sigma
\]
TPFA finite volume discretization of two phase Darcy flows

\[
\phi_{\kappa} \left( \frac{\rho_{\kappa}^{\alpha,n} \, S_{\kappa}^{\alpha,n} - \rho_{\kappa}^{\alpha,n-1} \, S_{\kappa}^{\alpha,n-1}}{\Delta t} \right) + \sum_{\kappa' \in \mathcal{M}_{\kappa}} \left( \frac{\rho^{\alpha} \, k_{r\alpha}(S^{\alpha})}{\mu^{\alpha}} \right)^n \, up_{\kappa,\kappa'}^{\alpha} = 0
\]

for all \( \kappa \in \mathcal{M}, \alpha = w, o \), with \( \phi_{\kappa} = \int_{\kappa} \phi(x) \, dx \) and

- the TPFA conservative discretization of the Darcy fluxes

\[
V_{\kappa,\kappa'}^{\alpha} = -V_{\kappa',\kappa}^{\alpha} \sim \int_{\kappa,\kappa'} -K(x) \left( \nabla P^{\alpha} + \rho_{\kappa,\kappa'}^{\alpha} \, g \, \nabla Z \right) \cdot n_{\kappa,\kappa'} \, d\sigma \\
= T_{\kappa,\kappa'} \left( P_{\kappa}^{\alpha} - P_{\kappa'}^{\alpha} + \rho_{\kappa,\kappa'}^{\alpha} \, g(Z_{\kappa} - Z_{\kappa'}) \right)
\]

- the upwinding \( up_{\kappa,\kappa'}^{\alpha} = \begin{cases} 
\kappa & \text{if } V_{\kappa,\kappa'}^{\alpha} \geq 0, \\
\kappa' & \text{if } V_{\kappa,\kappa'}^{\alpha} < 0.
\end{cases} \)
Example of Black Oil model on Cartesian meshes

- 3 phases w,o,g, 3 components water, heavy HC, light HC
- dissolution of the light HC in the oil phase
- 25 injectors, 36 producers in fivespots pattern
- 30 years of production
- Uniform Cartesian meshes of the domain 1kmx1kmx100m: 15x15x8, 30x30x16, 60x60x32
- Log normal permeability field of variance 4,8,12
Example of Black Oil model on Cartesian meshes

CPU time per Newton step for the log normal field of variance 8

<table>
<thead>
<tr>
<th>variance</th>
<th>CPR-AMG</th>
<th>ILU0</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6.8 it. - 1.25 s</td>
<td>72 it. - 3.0 s</td>
</tr>
<tr>
<td>8</td>
<td>7.8 it. - 1.27 s</td>
<td>128 it. - 5.3 s</td>
</tr>
<tr>
<td>12</td>
<td>9 it. - 1.44 s</td>
<td>235 it. - 9.9 s</td>
</tr>
</tbody>
</table>

CPU and GMRES iterations per Newton step for different variances of the K field on the mesh 30x30x16.
Example of Black Oil model: PAB test case (P. Quandalle, IFPEN)

- 3 phases w,o,g, 3 components water, heavy HC, light HC
- 20 injectors, 30 producers in fivespots pattern
- 4.223.000 cells
Example of gas storage simulation (with J.F. Thebault, GDFSuez-Storengy)

- Two phases immiscible water gas Darcy flow
- 4 injectors (or producers) at centre
- Repeated 6 months of injection followed by 6 months of production
- 5kmx5kmx30m spherical (300m between highest and lowest point)
- Uniform Cartesian meshes: 50x50x6, 100x100x12, 130x130x16
- Homogeneous media, anisotropy ratio of 10
- Perfect gaz for methane, slightly compressible water
- Ratio of viscosity around 40 (strong coupling)
Example of gas storage simulation (with J.F. Thebault, GDFSuez-Storengy)

- **Reduction of the Jacobian:** elimination of $S_g$ in single phase cells
- **No reduction:** both phases considered as present in the full domain
Vertex Approximate Gradient (VAG) discretization of two phase Darcy flows ($\alpha = o, w$).

The Darcy fluxes between $\kappa$ and $s$ are discretized by:

$$V^\alpha_{\kappa,s} = -V^\alpha_{\kappa,s} = F^\alpha_{\kappa,s}(P^\alpha) + \rho^\alpha_{\kappa,s} g F^\alpha_{\kappa,s}(Z).$$

with $F^\alpha_{\kappa,s}(u) = \sum_{s' \in V_{\kappa}} T^s_{\kappa,s} (u_\kappa - u_{s'})$

\[
\begin{align*}
    m_{\kappa} \phi_{\kappa} \frac{S^\alpha_{\kappa,n} - S^\alpha_{\kappa,n-1}}{\Delta t} + \sum_{s \in V_{\kappa}} \frac{k_r^\alpha (S_{up}^{\alpha,n})}{\mu^\alpha} V^\alpha_{\kappa,s} &= 0, \kappa \in M, \\
    m_{s} \phi_{s} \frac{S^\alpha_{s,n} - S^\alpha_{s,n-1}}{\Delta t} - \sum_{\kappa \in M_{s}} \frac{k_r^\alpha (S_{up}^{\alpha,n})}{\mu^\alpha} V^\alpha_{\kappa,s} &= 0, s \in V \setminus V_D, \\
    \end{align*}
\]

$$u^{\alpha} = \begin{cases} 
    \kappa & \text{if } V^\alpha_{\kappa,s} \geq 0, \\
    s & \text{if } V^\alpha_{\kappa,s} < 0.
\end{cases}$$
Example of immiscible two-phase flows

\[
\begin{align*}
\phi \partial_t S + \text{div}(\phi S(f(S)K\nabla \rho) - \lambda(S)K\nabla \rho) + \text{div}(-K\nabla \varphi(S)) &= 0, \\
S|_{x=0} &= 1, \quad -\lambda(S)K\nabla \rho \cdot \mathbf{n}|_{x=0} = V, \\
S|_{t=0} &= 0.
\end{align*}
\]

\[
\lambda(S) = \frac{S^2}{5} + (1-S)^2, \quad f(S) = \frac{S^2}{5} + (1-S)^2, \quad \varphi(S) = \frac{P_{c,1}}{5} \int_0^S \frac{u^2(1-u)}{\frac{u^2}{5} + (1-u)^2} du.
\]

Oil saturation and Global Pressure for $P_{c,1} = 0.1$ and $K$ constant.
Example of immiscible two-phase flows: $P_{c,1} = 0$, Cartesian meshes

Homogeneous case

Heterogeneous case
Example of immiscible two-phase flows: $P_{c,1} = 1$, Cartesian meshes

Homogeneous case

Heterogeneous case
Example of immiscible two-phase flows: tetrahedral meshes

Homogeneous case

Heterogeneous case
Example of density driven compositional single phase flows

\[
\begin{align*}
\frac{\partial_t \rho(c)}{} + \text{div} \left( -\rho(c) \frac{K(x)}{\mu} (\nabla p - \rho(c)g) \right) &= 0, \\
\phi(x) \frac{\partial_t (\rho(c)c)}{} + \text{div} \left( -\rho(c)c \frac{K(x)}{\mu} (\nabla p - \rho(c)g) - \rho(c)D\nabla c \right) &= 0. 
\end{align*}
\]

\[ (1) \]

- In the drain coordinate system:

\[
K_{blue} = \begin{pmatrix}
    k & 0 & 0 \\
    0 & k & 0 \\
    0 & 0 & k/10
\end{pmatrix}
\]

\[ K_{red} = 100K_{blue} \]
Example of density driven compositional single phase flows
Example of density driven compositional single phase flows
Extension of the scheme [Brenner et al 2011] to the VAG discretization on general meshes

- Continuous phase pressures at vertices: $P^o_s$ and $P^w_s$

- Discontinuous saturations at the interfaces between two different rocktypes:
  \[ S^o_{\kappa,s}, \quad \kappa \in \mathcal{M}_s, \]
  with
  \[ S^o_{\kappa,s} = (\tilde{P}_{c,\kappa})^{-1}(p^o_s - p^w_s), \quad \kappa \in \mathcal{M}_s \]

- Fluxes continuity at a given interface $s$: given by the conservation equations at $s$
VAG discretization

\[
\begin{align*}
\left\{ \begin{array}{l}
m_{\kappa} \phi_{\kappa} & \frac{S_{\kappa}^{\alpha,n} - S_{\kappa}^{\alpha,n-1}}{\Delta t} + \sum_{s \in \mathcal{V}_{\kappa}} k_{r,\kappa} (S_{\kappa}^{\alpha,n}) \frac{(V_{\kappa,s}^{\alpha})^+}{\mu^\alpha} + k_{r,\kappa} (S_{\kappa}^{\alpha,n}) \frac{(V_{\kappa,s}^{\alpha})^-}{\mu^\alpha} = 0, \\
& k \in \mathcal{M}, \quad \alpha = w, o,
\end{array} \right.
\end{align*}
\]

\[
\sum_{\kappa \in \mathcal{M}_s} m_{\kappa,s} \phi_{\kappa} \frac{S_{\kappa,s}^{\alpha,n} - S_{\kappa,s}^{\alpha,n-1}}{\Delta t} - \sum_{\kappa \in \mathcal{M}_s} k_{r,\kappa} (S_{\kappa,s}^{\alpha,n}) \frac{(V_{\kappa,s}^{\alpha})^+}{\mu^\alpha} + k_{r,\kappa} (S_{\kappa,s}^{\alpha,n}) \frac{(V_{\kappa,s}^{\alpha})^-}{\mu^\alpha} = 0,
\]

\[
\text{for } s \in \mathcal{V} \setminus \mathcal{V}_D, \quad \alpha = w, o.
\]

\[
\left\{ \begin{array}{l}
S_{\kappa,s}^{o,n} = (\tilde{P}_{c,\kappa})^{-1} (p_{s}^{o,n} - p_{s}^{w,n}), \quad \kappa \in \mathcal{M}_s, \quad s \in \mathcal{V} \setminus \mathcal{V}_D, \\
S_{\kappa}^{o,n} = (\tilde{P}_{c,\kappa})^{-1} (p_{\kappa}^{o,n} - p_{\kappa}^{w,n}), \quad \kappa \in \mathcal{M}.
\end{array} \right.
\]
Oil migration in a 3D basin with barriers
Oil migration in a 3D basin with barriers
Oil migration in a 3D basin with barriers

50 days of simulation
Oil migration in a 3D basin with barriers

7000 days of simulation
Conclusions

- Very efficient with TPFA
- Less efficient for schemes with larger stencils
  - AMG less efficient (not M-matrix) and more expensive
  - ILU0 more efficient due to larger fill-in
- Less efficient with capillary driven flows
- Tests to be done on larger meshes in parallel computing
- Room for new preconditioners
  - More scalable than ILU0 but cheaper than AMG
  - Reduce the cost of AMG set up phase (incorporate some knowledge on the geology? columns?)
  - More adapted to convection diffusion (capillary effect)


