# Symbolic dynamics and relatively hyperbolic groups

François Dahmani[*], AslıYaman[†]

**Abstract :** We study the action of a relatively hyperbolic group on its boundary, by methods of symbolic dynamics. Under a condition on the parabolic subgroups, we show that this dynamical system is finitely presented. We give examples where this condition is satisfied, including geometrically finite kleinian groups.

Associated to any word-hyperbolic group $\Gamma$, there is a dynamical system arising from the action of $\Gamma$ on its Gromov boundary $\partial\Gamma$. Already in [9], M.Gromov uses methods of symbolic dynamics for the study of this action, and in [3] (see also [4]), M.Coornaert, and A.Papadopoulos explain a way to factorize such a dynamical system through a subshift of finite type. They describe a finite alphabet $\mathcal{A}$, and a subshift $\Phi \subset \mathcal{A}^\Gamma$, and they construct a continuous equivariant, surjective map $\Phi \to \partial\Gamma$, which codes the action of $\Gamma$ on its boundary by a subshift of finite type.

The action of a group $\Gamma$ on a compact metric space, $K$ is expansive if there exists $\varepsilon > 0$ such that any pair of distinct points in $K$ can be taken at distance at least $\varepsilon$ from each other by an element of $\Gamma$. It is well known that the action of an hyperbolic group $\Gamma$ on $\partial\Gamma$ is expansive. This property, together with the existence of the coding given in [3], makes the action of a hyperbolic group, $\Gamma$, on its boundary, $\partial\Gamma$, *finitely presented* (see [9], [3]). In [9], M.Gromov describes consequences of such a presentation, like the rationality of some counting functions.

After an idea of Gromov in [9], B.Farb [7] and B.Bowditch [1] developed the theory of relatively hyperbolic groups, as a generalization of geometrically finite Kleinian groups. We will use for this work the definition of relatively hyperbolic groups given by Bowditch in [1]. A group $\Gamma$ is hyperbolic relative to a family, $\mathcal{G}$, of finitely generated subgroups of $\Gamma$ if it acts on an hyperbolic fine graph, with finite stabilizers of edges, finitely many orbits of edges, and such that the stabilizers of infinite valence vertices are exactly the elements of $\mathcal{G}$ (see Definition 2.3). In [7], this definition is known as "relatively hyperbolic with the property BCP".

If one replaces "fine" by "locally finite" in above definition, then $\mathcal{G}$ is empty and the group is hyperbolic. In [7], Farb proves that the fundamental group of a finite volume manifold of pinched negative curvature, with finitely many cusps is hyperbolic relative to the conjugates of the fundamental groups of the cusps, which are virtually nilpotent. Sela's limit groups, or, finitely generated $\omega$-residually-free groups are hyperbolic relative to their maximal abelian non-cyclic subgroups, as shown in [6].

Bowditch describes a boundary for a relatively hyperbolic group in [1]. The group acts on this compactum, and the elements of the family $\mathcal{G}$ are parabolic subgroups for this action. Despite of those parabolic subgroups, the action is expansive (Proposition 3.3). Although the construction of the subshift of finite type given by Coornaert and Papadopoulos will not work properly here (either one would need an infinite alphabet, or the map $\Phi \to \partial\Gamma$ would not be well defined) we found an intrinsic property of the maximal parabolic subgroups that allows successful modifications.

An infinite group has its one-point compactification $G \cup \{\infty\}$ finitely presented with special symbol if there exists an alphabet $\mathcal{A}$, a subshift of finite type $\Phi \subset \mathcal{A}^G$, a continuous surjective

[*]Université Louis Pasteur, Strasbourg, e-mail : dahmani@math.u-strasbg.fr
[†]University of Southampton, e-mail : a.yaman@maths.soton.ac.uk

map $\Pi : \Phi \to (G \cup \{\infty\})$, and a special symbol $\$ \in \mathcal{A}$, such that, for $\sigma \in \Phi$, $\Pi(\sigma) = g \in G$ if and only if $\sigma(g) = \$$.

**Theorem 0.1** *Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and $\partial\Gamma$ be its boundary (in the sense of Bowditch [1]).*

*If each $G \in \mathcal{G}$ has its one-point compactification finitely presented with special symbol, then the action of $\Gamma$ on its boundary $\partial\Gamma$ is finitely presented.*

**Theorem 0.2** *If a group has its one point compactification finitely presented with special symbol, then it is finitely generated.*

*Poly-hyperbolic groups have their one-point compactifications finitely presented with special symbol.*

**Corollary 0.1** *The action of a geometrically finite kleinian group (or more generally, a geometrically finite in the sense of Bowditch, fundamental group of a manifold with pinched negative curvature) on its limit set is finitely presented.*

There is the natural question :
**Problem.** *Which groups have their one-point compactifications finitely presented with special symbol ?*

We give in section 1 definitions related to symbolic dynamics. In section 2 we define relatively hyperbolic groups, their boundaries and introduce some tools such as "angles" and "cones". We prove the Theorem 0.1 in Section 3. The subshift we construct will produce objects which are local Busemann functions on the fine hyperbolic graph associated to the relatively hyperbolic group. To associate a point in the boundary to an element of the subshift, we consider its gradient lines. We prove that they converge to points at infinity, and we make sure that, for a given element of the subshift, all the gradient lines converge to the same point. For this we use the property of special symbol for each stabilizer of infinite valence vertex. In Section 4 we study this property of special symbol, and in particular, prove Theorem 0.2.

We want to thank B.Bowditch, and T.Delzant, for their many advises, and M.Coornaert for his useful explainations.

# 1 Definitions, symbolic dynamics

We borrow the next definitions (1.1 to 1.4) from Gromov [9] 8.4, and Coornaert and Papadopoulos [3], chapter 2. See also Fried [8].

**Definition 1.1** *(Shift, subshift, subshift of finite type)([9], 8.4, [3], Chap. 2)*

*If $\mathcal{A}$ is a finite alphabet and $\Gamma$ is a group, $\mathcal{A}^\Gamma$, with the product topology, is the total shift of $\Gamma$ on $\mathcal{A}$. It admits a natural left $\Gamma$-action given by $(\gamma\sigma)(g) = \sigma(\gamma^{-1}g)$ for all $g \in \Gamma$, $\sigma \in \mathcal{A}^\Gamma$.*

*A closed $\Gamma$-invariant subset of $\mathcal{A}^\Gamma$ is called a subshift.*

*A cylinder $\mathcal{C}$ is a subset of the total shift such that there exists a finite set $F \subset \Gamma$, and a family of maps $M \subset \mathcal{A}^F$ with*

$$\mathcal{C} = \{\sigma \in \mathcal{A}^\Gamma,\ \sigma|_F \in M\}.$$

$\Phi$ *is a* subshift of finite type *if there exists a cylinder $\mathcal{C}$ such that $\Phi = \bigcap_{\gamma\in\Gamma} \gamma^{-1}\mathcal{C}$.*

The subshifts of finite type are subshifts, but the cylinders are not $\Gamma$ invariant.

**Definition 1.2** *(Dynamical systems of finite type) [9], [3]*

*Let* $\Gamma$ *acts on a compact* $K$*. The dynamical system is of finite type if there exists a finite alphabet* $\mathcal{A}$*, a subshift of finite type* $\Phi \subset \mathcal{A}^\Gamma$ *and a continuous, surjective,* $\Gamma$*-equivariant map* $\pi : \Phi \to K$*.*

*Example :* Set $\Gamma = \mathbb{Z}$, and $\mathcal{A} = \{a, b, \$\}$. The cylinder $\mathcal{C}$ is the set of the maps that agree on $F = \{0, 1\}$ with one of the maps in $M = \{m_i,\ i = 1...4\}$ where $m_1(0, 1) = (a, a)$, $m_2(0, 1) = (a, \$)$, $m_3(0, 1) = (\$, b)$, $m_4(0, 1) = (b, b)\}$.

Let $\Phi$ be the subshift of finite type defined by the cylinder $\mathcal{C}$, i.e $\Phi = \bigcap_{n\in\mathbb{Z}} n + \mathcal{C}$. One can check that the elements of $\Phi$ are the constant word on $a$, the constant word on $b$ and all the words $(...aaa\$bbb...)$ beginning by $a$, until there is a \$ on the $n^{th}$ letter ($n \in \mathbb{Z}$) and then $b$. Although for this example $\Phi$ is countable, in general subshifts of finite type are not countable.

Now consider the compact $K = \mathbb{Z} \cup \{\infty\}$ with the usual topology. There is a natural left action of $\mathbb{Z}$ on $K$, fixing the infinity point. Consider the map $\pi : \Phi \to K$ that send $(...aaaa...)$ and $(...bbbb...)$ on $\infty$, and $(...aaa\$bbb...)$ on $n \in \mathbb{Z}$ where $n$ is the index of the letter \$. The map $\pi$ is surjective, continuous and equivariant, and therefore the action of $\mathbb{Z}$ on $K$ is of finite type.

We now go on with definitions. One can refine the property of being a dynamical system of finite type with the following.

**Definition 1.3** *(Expansivity)*

*The action of a group* $\Gamma$ *on a compactum* $K$ *is* expansive *if there exists* $U$ *a neighborhood of the diagonal* $\Delta \subset K \times K$ *such that* $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$*.*

Note that, if the compactum is metric, this is equivalent to the definition of expansivity given in introduction (see [3], Proposition 2.3).

**Definition 1.4** *(Finitely presented dynamical systems)[9], [3]*

*Let* $\Gamma$ *acts on a compact* $K$*. The dynamical system is finitely presented if it is both of finite type and expansive.*

If one has a subshift of finite type $\Phi \subset \mathcal{A}^\Gamma$ and a surjective continuous equivariant map $\pi : \Phi \to K$, the expansivity of the action of $\Gamma$ on $K$ turns out to be equivalent to the fact that the subshift $\Psi \subset (\mathcal{A} \times \mathcal{A})^\Gamma$ defined by $[(\sigma_1 \times \sigma_2) \in \Psi] \Leftrightarrow [\pi(\sigma_1) = \pi(\sigma_2)]$, is of finite type (cf [3] chapter 2).

If $\Gamma$ is a infinite discrete group, it acts on its alexandrov compactification $\Gamma \cup \{\infty\}$ by multiplication of the left, hence fixing the point at infinity. If $\Gamma$ is finite, its alexandrov compactification is itself.

**Definition 1.5** *(Finite presentation with special symbol)*

*The alexandrov compactification of a discrete group* $\Gamma$ *is said* finitely presented with special symbol *if the* $\Gamma$*-action on* $\Gamma \cup \{\infty\}$ *is finitely presented by a subshift* $\Phi \subset \mathcal{A}^\Gamma$ *and if the presentation map* $\pi : \Phi \to \Gamma \cup \{\infty\}$ *satisfies*

$$\exists \$ \in \mathcal{A} \quad (\pi(\sigma) = \gamma \in \Gamma) \Leftrightarrow (\sigma(\gamma) = \$)$$

Note that in this case, the property of expansivity is always satisfied. The example of dynamical system of finite type described previously is a finite presentation with special symbol of $\mathbb{Z}$. Note also that finite groups which are already compact admit a trivial finite presentation with special symbol.

We give in part 4 several examples of groups with a compactification which is finitely presented with special symbol.

# 2 About Relatively Hyperbolic Groups

## 2.1 Definitions

A *graph* is a set of *vertices* with a set of *edges*. An edge is a pair of vertices. One can equip a graph with a metric where edges have length 1. Thus this geometrical realization allows to consider simplicial, geodesic, quasi-geodesic and locally geodesic paths in a graph.

**Definition 2.1** *(Circuits)*
*A* circuit *in a graph is a simple simplicial loop, i.e without self intersection.*

In [1], B. Bowditch introduces the notion of fineness of a graph.

**Definition 2.2** *(Fineness)[1]*
*A graph $\mathcal{K}$ is* fine *if for all $L > 0$, for all edge $e$, the set of the circuits of length less than $L$, containing $e$ is finite. It is* uniformly fine *if this set has cardinality bounded above by a constant depending only on $L$.*

**Definition 2.3** *(Relatively Hyperbolic Groups)[1]*
*A group $\Gamma$ is* hyperbolic relative *to a family of subgroups $\mathcal{G}$, if it acts on a Gromov-hyperbolic, fine graph $\mathcal{K}$, such that stabilizers of edges are finite, $\Gamma\backslash\mathcal{K}$ is a finite graph, and the stabilizers of the vertices of infinite valence are exactly the elements of $\mathcal{G}$.*

With an abuse of language, we will say that the pair $(\Gamma, \mathcal{G})$ is a relatively hyperbolic group, and that $\mathcal{K}$ is a graph *associated* to it.

We note that as there are finitely many orbits of edges, a graph associated to a relatively hyperbolic group is uniformly fine. Note also that the graph associated to $(\Gamma, \mathcal{G})$, $\mathcal{K}$, can be chosen to be without global cut point, and with positive hyperbolicity constant $\delta$.

## 2.2 Angles

For any graph, one can define a notion of *angle* as follow.

**Definition 2.4** *(Angles)*
*Let $\mathcal{K}$ be a graph, and let $e_1 = (v, v_1)$ and $e_2 = (v, v_2)$ be edges with one common vertex $v$. The angle $\mathrm{Ang}_v(e_1, e_2)$, is the length of the shortest path from $v_1$ to $v_2$, in $\mathcal{K} \setminus \{v\}$ ($+\infty$ if there is none).*

The angle between two simple simplicial (oriented) paths having a common vertex is the angle between their first edges after this vertex.

If $p$ is a simple simplicial path, and $v$ one of its vertices, $\mathrm{Ang}_v(p)$ is the angle between the consecutive edges of $p$ at $v$, and its maximal angle $\mathrm{MaxAng}(p)$ is the maximal angle between consecutive edges of $p$.

In the notation $\mathrm{Ang}_v(p, p')$, we will sometimes omit the subscript if there is no ambiguity.

**Proposition 2.1** *(Some useful remarks)*
1. $\mathrm{Ang}_v(e_1, e_3) \leq \mathrm{Ang}_v(e_1, e_2) + \mathrm{Ang}_v(e_2, e_3)$ *when $e_i$ are edges issued from a vertex $v$.*
2. *If $\gamma$ is an isometry,* $\mathrm{Ang}_v(e_1, e_2) = \mathrm{Ang}_{\gamma v}(\gamma.e_1, \gamma.e_2)$.
3. *Any circuit of length $L \geq 2$ has a maximal angle less than $L - 2$.*

First remark is the triangular inequality for the length distance of $\mathcal{K}\setminus\{v\}$. Second statement is obvious. Finally, if $e_1 = (v_1, v)$ and $e_2 = (v, v_2)$ are two consecutive edges in the circuit, the circuit itself gives a path of length $L-2$ from $v_1$ and $v_2$ avoiding $v$. $\square$

Here is an important property of angles.

**Lemma 2.1** *(Large angles in triangles)*

*Let $[x, y]$ and $[x, z]$ be geodesic segments in a $\delta$-hyperbolic graph, and assume that $\mathrm{Ang}_x([x, y], [x, z]) = \theta \geq 50\delta$. Then the concatenation of the two segments is still a geodesic. Moreover any geodesic segment $[y, z]$ will contain $x$ and $\mathrm{Ang}_x([y, z]) \geq \theta - 50\delta$.*

Let $[y, z]$ be a geodesic, defining a triangle $(x, y, z)$, which is $\delta$-thin. We consider the vertices $y'$ and $z'$ on $[x, y]$ and $[x, z]$ located at distance $10\delta$ from $x$. Because of the angle at $x$, they are not $30\delta$-close to each other. Therefore, they are $\delta$-close to the segment $[y, z]$, and we set $y''$ and $z''$ the corresponding points on $[y, z]$. This gives a loop of length less than $(2 * 10\delta + 2\delta) * 2 \leq 50\delta$, containing $x$, consisting of $[x, y']$, $[y', y'']$, $[y'', z'']$, $[z'', z']$, and $[z', x]$. As the small transitions $[y', y'']$ and $[z'', z']$ are $10\delta$ far away from $x$, they do not contain $x$. Thus the third property of Proposition 2.1 proves that $x \in [y'', z'']$, and $\mathrm{Ang}_x([y'', z'']) \geq \theta - 50\delta$, and therefore $\mathrm{Ang}_x([y, z]) \geq \theta - 50\delta$ . $\square$

## 2.3 Cones

**Definition 2.5** *(Cones)*

*Let $\mathcal{K}$ be a graph, let $d$ and $\theta$ be positive numbers. The* cone *centered at an edge $e = (v, v')$, of radius $d$ and angle $\theta$ is the set of vertices $w$ at distance less than $d$ from $v$ and such hat there exists a geodesic segment $[v, w]$ the maximal angle and the angle with $e$ of which are less than $\theta$:*

$$Cone_{d,\theta}(e, v) = \{w,\ \mathrm{dist}(w, v) \leq d,\ \mathrm{MaxAng}[v, w] \leq \theta, \mathrm{Ang}_v(e, [v, w]) \leq \theta\}$$

**Proposition 2.2** *(Bounded angles imply local finiteness)*

*Let $\mathcal{K}$ be a fine graph. Given an edge $e$ and $\theta > 0$, there exists only finitely many edges $e'$ such that $e$ and $e'$ have a common vertex, and $\mathrm{Ang}(e, e') \leq \theta$.*

Only finitely many circuits shorter than $\theta$ contain $e$. $\square$

**Corollary 2.1** *(Cones are finite)*

*In a fine graph, the cones are finite sets of vertices. If the graph is uniformly fine, the cardinality of $Cone_{d,\theta}(e, v)$ can be bounded above by a function of $d$ and $\theta$.*

Given a cone of radius $d$ and angle $\theta$, one can construct a tree of diameter $d+1$, locally finite according to the previous proposition, that maps onto the cone. As the tree is finite, so is the cone. If the fineness is uniform, the tree constructed has bounded degree. $\square$

We will use the following theorem, which is a reformulation of a result in [5].

**Theorem 2.1** *Let $\Gamma$ be a relatively hyperbolic group and $\mathcal{K}$ be an associated graph, which is $\delta$-hyperbolic. There exists an aspherical (in particular simply connected) simplicial complex such that its vertex set is the one of $\mathcal{K}$, and such that each simplex has all its vertices in a same cone of $\mathcal{K}$, of radius $10\delta + 10$, and angle $100\delta + 30$.*

In [5], the first author defines the *relative Rips complex* $P_{d,r}(\mathcal{K})$ for a relatively hyperbolic group. It is the maximal complex on the set of vertices of $\mathcal{K}$ such that an edge is between two vertices if, in $\mathcal{K}$, a geodesic of length less than $d$ and maximal angle less than $r$ links them. Although in [5], the notion of angle is replaced by "length of traveling in cosets", the proof of Theorem 6.2 remains the same, and gives the asphericity of $P_{d,r}(\mathcal{K})$ for large $d$ and $r$. Theorem 2.1 follows. □

### 2.4 Boundary of a relatively hyperbolic group

Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and let $\mathcal{K}$ be an associated graph. In [1], Bowditch defines the (dynamical) boundary $\partial\Gamma$ of $\Gamma$ by $\partial\Gamma = \partial\mathcal{K} \cup \mathcal{V}_\infty$ where $\partial\mathcal{K}$ is the Gromov boundary of the hyperbolic graph $\mathcal{K}$, and $\mathcal{V}_\infty$ is the set of vertices of infinite valence in $\mathcal{K}$. This boundary admits a natural topology of metrisable compactum (see [1], [2], [10], and also [5]).

## 3 Finite presentation of the boundaries of a relatively hyperbolic group.

We will prove the following theorem.

**Theorem 3.1** *Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group. If, for each $G \in \mathcal{G}$, the action of $G$ on its Alexandrov one-point compactification $G \cup \{\infty\}$ is finitely presented with special symbol, then the action of $\Gamma$ on its boundary $\partial\Gamma$ is finitely presented.*

### 3.1 Busemann and radial cocycles

**Definition 3.1** *(Busemann function)(see [9] 7.5.C, and [3] chap. 3, section 3)*

*Let $\rho : [0, \infty) \to \mathcal{K}$ be a geodesic ray starting at $v_0$. The Busemann function of $\rho$ is $h_\rho : \mathcal{V} \to \mathbb{Z}$ defined by the limit (which always exists and is finite) $h_\rho(v) = \lim_{n\to\infty}(\mathrm{dist}(v, \rho(n)) - n)$.*

**Definition 3.2** *(Busemann cocycles)([9] 7.5.E, [3] chap. 3)*

*Let $h_\rho$ be a Busemann function. The cocycle associated to $h_\rho$ is $\varphi_\rho : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$ defined by $\varphi_\rho(w, v) = h_\rho(v) - h_\rho(w)$. A gradient line of $\varphi_\rho$ is a sequence of vertices $(v_n)_n$ such that $\varphi_\rho(v_{i+1}, v_i) = 1$ for all $i$.*

The proof of the next lemma can be found in [3] (Proposition 4.2).

**Lemma 3.1** *If $\varphi$ is a Busemann cocycle associated to $\rho$, then a gradient line is a sequence of vertices of a geodesic ray asymptotic to $\rho$. Moreover there is a gradient line starting from each vertex.*

**Definition 3.3** *(Radial cocycles)*

*Let $p$ a vertex of infinite valence in $\mathcal{K}$. The radial cocycle associated to $p$ is $\varphi_p : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$ defined by $\varphi_h(w, v) = \mathrm{dist}(v, p) - \mathrm{dist}(w - p)$. A gradient line of $\varphi_p$ is a finite family of vertices $(v_n)_{0\leq n\leq m}$ such that $\varphi_p(v_{i+1}, v_i) = 1$ for all $i$, and $v_m = p$.*

The next lemma is direct by definition.

**Lemma 3.2** *If $\varphi$ is a radial cocycle associated to $p$, then its different gradient lines are exactly the sequences of vertices of geodesic segment ending at $p$.*

We set $\theta = 2000\delta$, where $\delta$ is a positive hyperbolicity constant of $\mathcal{K}$.

**Proposition 3.1** *(Properties of Busemann and radialcocycles)*
*Let $\varphi$ be a Busemann or a radial cocycle. Then :*

1. *(Integral values) For $x, y$ adjacent vertices, $\varphi(x, y)$ is 0, 1 or $-1$.*
2. *(Cocycle) For all $x, y, z$ $\varphi(x, y) + \varphi(y, z) + \varphi(z, x) = 0$.*
3. *(Geodesic extension) Let $l = [v, \xi)$ be a gradient line and $[x, v]$ a geodesic segment of length and maximal angle less than $\theta$, such that $\mathrm{Ang}_v([x, v] \cup [v, \xi)) \geq \theta$, then $[x, v] \cup [v, \xi)$ is a gradient line.*
4. *(Exits) If $v$ is a vertex of finite valence, then there exits $w$ adjacent to $v$ with $\varphi(w, v) = 1$.*

Properties 1 and 2 are obvious. Property 4 is consequence of the Lemmas 3.1 and 3.2. Property 3 deserves its proof here. By Lemma 3.1 (if $\xi \in \partial\mathcal{K}$) and Lemma 3.2 (if $\xi$ is a vertex of infinite valence), any gradient line from $x$ is a ray $[x, \xi)$ and produces a triangle $(x, v, \xi)$, which, by assumption, has a large angle at $v$. Hence, by Lemma 2.1, any ray $[x, \xi)$ contains $v$. □

## 3.2 Shift and subshift

Let $(G_i)_{i=1..m}$ be a finite family of representatives of conjugacy classes of parabolic subgroups in $\Gamma$, each stabilizing an infinite valence vertex $p_i$. We also choose, for each $i$, an arbitrary edge $e_i$, adjacent to $p_i$. Recall that we assume that each $G_i$ acts on $G_i \cup \{\infty\}$ as a finitely presented system with a special symbol. That means that we have an alphabet $\mathcal{A}_i$, a finite subset $F_i \subset G_i$ and a set $M_i$ of maps from $F_i$ to $\mathcal{A}_i$ which define a cylinder, hence a subshift of finite type. We note \$ the special symbol in $\mathcal{A}_i$, without distinguishing the indices $i$.

We also set $F_i' = \{\gamma \in G_i, \mathrm{Ang}_{p_i}(e_i, \gamma e_i) \leq \theta/2\}$.

We fix a vertex $v_0$ and an edge $e_0 = (v_0, v)$. We choose $R$ and $\Theta$ such that $Cone_{R,\Theta}(e_0, v_0)$ satisfies $Cone_{10\theta,10\theta}(e_i, p_i) \subset Cone_{R,\Theta}(e_0, v_0)$ for all $i$, no pair of edges at a vertex of finite valence have an angle more than $\frac{\Theta}{2}$, and finally, for all $i$, for all $\gamma$ in $F_i \cup F_i'$, the vertices of $\gamma e_i$ are in $Cone_{R,\Theta}(e_0, v_0)$.

$\mathcal{A}'$ denotes the set of all possible restrictions of Busemann and radial cocycles on $Cone_{R,\Theta}(e_0, v_0) \times Cone_{R,\Theta}(e_0, v_0)$. We set $\mathcal{A}'' = \mathcal{A}_1 \times \ldots \times \mathcal{A}_r$. We choose our alphabet to be $\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$

**Lemma 3.3** *$\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$ is finite.*

Cones are finite, and cocycles have integral values bounded by the diameter. □

An element $\psi$ of $\mathcal{A}^\Gamma$ is a map from $\Gamma$ to $\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$. Thus it has coordinates $\psi_0 : \Gamma \to \mathcal{A}'$ and $\psi_i : \Gamma \to \mathcal{A}_i$ for all $i$. Hence, $\psi_0(\gamma)$ is a map from $Cone_{R,\Theta}(e_0, v_0) \times Cone_{R,\Theta}(e_0, v_0)$ to $\mathbb{Z}$, whereas $\psi_i(\gamma)$ is in $\mathcal{A}_i$ for $i \geq 1$.

Let $F$ be the set of elements in $\Gamma$ such that the vertices of $\gamma.e_0$ are both in $Cone_{R,\Theta}(e_0, v_0)$. As stabilizers of edges are finite, $F$ is a finite set.

Let $\mathcal{C}$ be the cylinder (in the sense of Definition 1.1) defined on $F$ so that $\psi \in \mathcal{C}$ if the three next conditions, which concern only finitely many elements of $\Gamma$, are fulfilled :

- $[\psi_0(\gamma)](v_1, v_2) = [\psi_0(1_\Gamma)](\gamma^{-1}v_1, \gamma^{-1}v_2)$ whenever $v_1, v_2, \gamma^{-1}v_1, \gamma^{-1}v_2$ are all in $Cone_{R,\Theta}(e_0, v_0)$.
- $(\psi_i)_{F_i}$ is in $M_i$.

- for $\gamma \in F_i$, for $v$ such that $\gamma.e_i = (p_i, v)$, and for $w$ such that $[w, p_i]$ is a geodesic segment of length, and maximal angle less than $\theta$, containing $v$, one has $[\psi_0(\gamma)](\gamma^{-1}w, p_i) \geq (1 - \mathrm{dist}(w, p_i))$ only if there exists $\gamma' \in F_i'$ such that $\psi_i(\gamma\gamma') = \$$.

Let $\Phi$ be the subshift of finite type $\Phi = \bigcap_{\gamma \in \Gamma} \gamma\mathcal{C}$. The next lemmas explain the role of the properties stated above.

**Lemma 3.4** *(About the $\psi_i$, $i \geq 1$)*
*Let $\psi \in \Phi$, and $\gamma \in \Gamma$, for all $i$, $(\psi_i)_{|\gamma.G_i}$ is an element of $\Phi_i$.*

By definition of $\mathcal{C}$, for all $\gamma$, and all $g_i \in G_i$, $(\psi_i)_{\gamma.g_i.F_i}$ is in $M_i$. □

**Lemma 3.5** *(About $\psi_0$)*
*Let $\psi \in \Phi$. If $v$ and $v'$ are vertices in $\gamma Cone_{R,\Theta}(e_0, v_0)$, we set $\varphi_\psi(v, v') = \psi_0(\gamma)(\gamma^{-1}v, \gamma^{-1}v')$. Then the map $\varphi_\psi$ is well defined, and satisfies each property of Proposition 3.1 in every translate of $Cone_{R,\Theta}(e_0, v_0)$.*

Because of the first property of the definition of $\mathcal{C}$, the formula given for $\varphi_\psi(v, v')$ does not depend on the choice of possible $\gamma$, and therefore, the map is well defined. Properties 1 and 4 of Proposition 3.1 are satisfied because each element of our alphabet satisfy them in a cone. Property 2 (cocycle property) *a priori* only makes sense when the three vertices are in a common cone, and in this case, it is satisfied by each element in our alphabet. For property 3, we notice that for $\gamma_0$ arbitrary, \$ can appear at most once in the set of values $\psi_i(\gamma_0\gamma)$ as $\gamma$ ranges over $G_i$. If $[v, \xi)$ is a gradient line, with $v = \gamma_0 p_i$, for some $i$ and some $\gamma_0$, then, for $\gamma$ such that $\psi_i(\gamma_0\gamma) = \$$, there exists $\gamma'$ in $F_i'$ such that $(\gamma_0\gamma\gamma').e_i$ is the first edge of $[v, \xi)$. If $[v, x]$ is a segment such that $\mathrm{Ang}_v([v, x], [v, \xi)) \geq \theta$, then, for all $\gamma'' \in F_i'$, the edge $(\gamma_0\gamma\gamma'').e_i$ is not on $[v, x]$. Therefore, by the third property of the definition of the cylinder, $\varphi_\psi(x, v) \leq -\mathrm{dist}(x, p_i)$. And this, together with $|\varphi_\psi(x, v)| \leq \mathrm{dist}(x, v)$, gives $\varphi_\psi(x, v) = -\mathrm{dist}(x, p_i)$ . In other words, $[x, v] \cup [v, \xi)$ is a gradient line. □

## 3.3 The presentation $\Pi : \Phi \to \partial\Gamma$

Given an element of $\Phi$, we want to associate canonically an element of $\partial\Gamma$.

**Definition 3.4** *(Gradient lines)*
*Let $\psi$ be an element of $\Phi$. A gradient line $l_\psi$ of $\psi$ is a finite or infinite sequence $(v_n)_{n\geq 0}$ of vertices in $\mathcal{K}$ such that $\varphi_\psi(v_{n+1}, v_n) = 1$ for all $n$. Moreover, it is finite only if for the last index $m$, every neighbor $v$ of $v_m$ satisfy $\varphi_\psi(v, v_m) \leq 0$.*

**Lemma 3.6** *Gradient lines are geodesics in $\mathcal{K}$.*

The map $\varphi_\psi$ is defined on pairs of vertices lying in a same translate of $Cone_{R,\Theta}(e_0, v_0)$. Thus it can be seen as a 1-cochain defined on the relative Rips polyhedron given by Theorem 2.1, which is simply connected. As it is a cocycle, it is a coboundary, and there is a map $\tilde{\varphi}$ defined on the set of vertices of $\mathcal{K}$ such that $\varphi_\psi(w, v) = \tilde{\varphi}(w) - \tilde{\varphi}(v)$ for all $v, w$ lying in a translate of $Cone_{R,\Theta}(e_0, v_0)$. This formula allows to extend $\varphi_\psi$ to all pair of vertices (not only those in a same cone of radius $R$ and angle $\Theta$). This gives a cocycle with integral values such that $|\varphi_\psi(w, v)| \leq \mathrm{dist}(w, v)$, for all $v$ and $w$. Now on a gradient line, we have by definition the other inequality : $|\varphi_\psi(w, v)| \geq \mathrm{dist}(w, v)$, and this proves the claim. □

The proof of this lemma involves the globalization of the cocycles (by asphericity of the relative Rips polyhedron). One can see that the cocycle property proved only for vertices in a same cone in Lemma 3.5, holds for arbitrary triple of vertices.

We now state and prove the main property of the elements of $\Phi$.

**Proposition 3.2** *(Coherence of gradient lines)*
*Let $\psi \in \Phi$. All its gradient lines are asymptotic to each other. In other words they all converge to the same element of $\partial\mathcal{K} \cup \mathcal{V}_\infty$.*

We argue by contradiction, and assume that there are two gradient lines with different end points which are in the boundary or in the set of vertices of infinite valence. Such gradient lines are called *divergent*. We need the next lemma, before continuing the proof of Proposition 3.2.

**Lemma 3.7** *Under this assumption, there are two divergent gradient lines starting at the same vertex, or at two adjacent vertices.*

Let $l_1$ and $l_2$ be two divergent gradient lines, and $v_1$ and $v_2$ vertices on them. On a geodesic segment $[v_1, v_2]$, consider $v$ the first vertex from which there is a gradient line $l$ divergent from $l_1$. Either $v = v_1$ (and we are in the first case of the lemma), or there is a vertex, $v'$, of $[v_1, v]$ adjacent to $v$. By definition of $v$, all gradient lines starting at $v'$ are asymptotic to $l_1$, and we are in the second case of the lemma. □

Now we can assume that $l_1$ and $l_2$ are two divergent gradient lines starting at the same vertex, or at two adjacent vertices. Thus, there is a geodesic (possibly bi-infinite) $l_3$, such that $(l_1, l_2, l_3)$ define a geodesic triangle with vertices $x_1, x_2, x_3$ (see Figure 1), with $x_1$ and $x_2$ possibly at infinity.

At distance $(x_1 \cdot x_2)_{x_3} - 100\delta$ from $x_3$, we connect $l_1$ and $l_2$ a transition segment of length less than $10\delta$, and we connect $l_1$ to $l_3$, and $l_3$ to $l_2$ at distance $(x_1 \cdot x_2)_{x_3} + 100\delta$ from $x_3$ by two others transition segments of length less than $10\delta$. Thus we have a loop of length less than $1000\delta$ around the center of the triangle, and by Proposition 2.1, no circuit of this length contains an angle more than $1000\delta \leq \Theta$. Hence, if $v$ is a vertex of $l_1$ such that $(x_1 \cdot x_2)_{x_3} - 50\delta < \mathrm{dist}(x_3, v) < (x_1 \cdot x_2)_{x_3} + 50\delta$ and if $\mathrm{Ang}_v(l_1)$ is more than $\Theta$ then either $l_2$ or $l_3$ pass through $v$ as the transition segments connecting $l_1$, $l_2$ and $l_3$ are $10\delta$ short and $50\delta$ far from $v$. The next lemma proves that in fact $l_2$ passes through $v$.

**Lemma 3.8** *The lines $l_1$ and $l_2$ pass through $v$ and $\mathrm{Ang}_v(l_1, l_2) \leq \frac{\Theta}{2}$.*

It is enough to show that $v$ is on $l_2$, because in this case, the third point of Proposition 3.1 (which is satisfied by Lemma 3.5) implies that the two gradient lines make an angle smaller than $\theta \leq 2000\delta \leq \frac{\Theta}{2}$. As $v$ is either on $l_3$ or on $l_2$, we assume that $v$ is on $l_3$. In this case, we have a simple path from $x_3$ to $x_2$ consisting of the concatenation of the piece of $l_1$ between $x_3$ and $v$ and the piece of $l_3$ between $v$ and $x_2$. The hyperbolicity of the space ensures that this path remains at distance less than $60\delta$ from $l_2$. We consider two adjacent vertices on $l_2$, $w$ and $w'$ such that $\mathrm{dist}(v, w) < \mathrm{dist}(v, w') \leq 60\delta + 1$ (two such vertices necessarily exist since $l_2$ is a geodesic going to infinity). On a geodesic segment $[w', v]$ containing $w$, we mark the consecutive vertices where there is an angle greater than $\theta$. If $\mathrm{Ang}_v([v, w'], [v, x_1]) < \theta$, then by triangular inequality for angles, $\mathrm{Ang}_v([v, w'], [v, x_3]) \geq (\Theta - \theta)$, and therefore Lemma 2.1 for the triangle $(x_3, v, w')$ implies that $v \in l_3$. On the other hand, if $\mathrm{Ang}_v([v, w'], [v, x_1]) \geq \theta$, we apply the third point of Proposition 3.1, (which is satisfied by Lemma 3.5) to each subsegment of $[v, w']$ containing no large angle to see that $[w', v]$ is a gradient line. This is a contradiction, since the edge $(w, w')$ would be a gradient line in both directions. This proves Lemma 3.8. □

Let us end the proof of Proposition 3.2. From the previous lemma, and the definition of the Gromov product, we see that a vertex in $l_1$ satisfying the assumption of the previous lemma is in fact located at distance less than $(x_1 \cdot x_2)_{x_3}$ from $x_3$. Let $v$ be the last vertex satisfying the previous lemma (or, if there is none, the vertex on $l_1$ at distance $(x_1 \cdot x_2)_{x_3} - 50\delta$ from $x_3$). Therefore, the two rays do not have large angle until they arrive at distance $50\delta$ from the small transition segments connecting $l_1$, $l_3$, and $l_2$, $l_3$. Thus, there is a cone centered on the first edge of $l_1$ after $v$, of angle and radius $10\theta$, in which $l_1$ and $l_2$ have a subsegment of length at least $20\delta$.

In this cone, the two lines are $\delta$-close at the beginning, and $10\delta$-far later (see Figure 1), or possibly reach $x_1$ and $x_2$, which, in this case, belong to the cone. By definition of our alphabet $\mathcal{A}$, there must be a Busemann or a radial cocycle whose restriction on this cone gives rise to the same segments of gradient lines. This rules out the second case, and in the first case, by hyperbolicity, two geodesic rays with such subsegments would diverge at infinity, and we know that this cannot happen for gradient lines of Busemann or radial cocycles. This is a contradiction, and it proves the proposition. $\square$

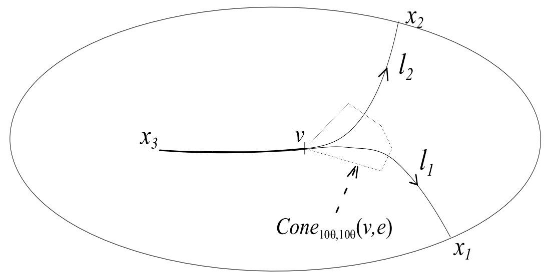


Figure 1: Gradient lines and cone at the center of the triangle

We can now define the map $\Pi : \Phi \to \partial\Gamma$. For an element $\psi$ in $\Phi$, we associate $\Pi(\psi) \in \partial\Gamma$, the point to which any gradient line of $\varphi_\psi$ converge.

### 3.4 End of the proof of Theorem 3.1

In order to complete the proof we need to show that $\Pi : \Phi \to \partial\Gamma$ satisfies the Definition 1.2 (Lemma 3.9), and secondly that the action of $\Gamma$ on $\partial K$ is expansive (Proposition 3.3).

**Lemma 3.9** *The map* $\Pi : \Phi \to \partial\Gamma$ *is surjective continuous and equivariant.*

Given a point $\xi$ in $\partial\Gamma$, one can find a Busemann or a radial cocycle associated to $\xi$. By Proposition 3.1, this defines an element of $\Phi$ which has a (hence all) gradient line converging to $\xi$. Thus, the map is surjective. If a sequence $\psi_n$ converge to $\psi$, then the gradient lines of $\psi_n$ will coincide with the gradient lines of $\psi$ on large finite subset of $\mathcal{K}$, which ensures the continuity of $\Pi$. Finally, the translate of a ray converge to the corresponding translate of the point at infinity, hence the map is equivariant. $\square$

**Proposition 3.3** *(Expansivity)*
*The action of a relatively hyperbolic group on its boundary is expansive.*

If $\Delta$ is the diagonal of $(\partial\Gamma) \times (\partial\Gamma)$, then we have to find a neighborhood $U$ of $\Delta$ such that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$.

Let $e_1, \dots, e_m$ a set of orbit representatives of the edges in $\mathcal{K}$. Let $X$ be the set of pairs of points $(\xi_1, \xi_2) \in (\partial\mathcal{K})^2$ such that there is a bi-infinite geodesic between $\xi_1$ and $\xi_2$ passing through one of the $e_i$. Let now be $p_1, \dots, p_l$ a set of orbit representatives of the infinite valence points. Because they are bounded parabolic points, the stabilizer $G_i$ of $p_i$ acts on $\partial\Gamma \setminus \{p_i\}$ with compact quotient. Let then $Y$ be the set of pairs of points $(p_i, \zeta)$ where $\zeta$ is in a chosen compact fundamental domain for the action of $G_i$.

We now choose $U = (\partial\Gamma \times \partial\Gamma) \setminus (X \cup Y)$. First we show that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$. The direct inclusion is trivial.

Let $(\xi_1, \xi_2) \in U$ and assume it is not in $\Delta$. We will show that it is not in all translates of $U$. We consider two cases, either they are both in $\partial\mathcal{K}$, or one them, say $\xi_1$, is a vertex of $\mathcal{K}$ of infinite valence. In the first case, there is a bi-infinite geodesic from one point to another, and it can be translated so that its image passes by one of the $e_i$. Therefore, there is $\gamma$ such that $\gamma(\xi_1, \xi_2)$ is in $X$, hence not in $U$. In the second case, there is $\gamma \in \Gamma$ such that $\gamma\xi_1$ is one of the $p_i$. Now there is $\gamma' \in G_i$ such that $\gamma'\gamma(\xi_1, \xi_2)$ is in $Y$, hence not in $U$. This proves that the intersection of the translates of $U$ is equal to the diagonal set.

Now we have to show that $U$ is a neighborhood of $\Delta$. That is to say that a sequence of elements in $X \cup Y$ cannot converge to a point of $\Delta$.

Let $(x_n = (\xi_1^n, \xi_2^n))_n$ be a converging sequence of elements of $X$. After passing on a subsequence, one can assume that, for all $n$, there is a bi-infinite geodesic between $\xi_1^n$ and $\xi_2^n$ passing through a same edge $e_i$. If $\xi_1^n \to \zeta_1$ and $\xi_2^n \to \zeta_2$, we see that $\zeta_1$ and $\zeta_2$ are linked by a geodesic passing through $e_i$, hence non-trivial. Therefore $\zeta_1 \neq \zeta_2$.

Let now $(y_n = (\xi_1^n, \xi_2^n))_n$ be a converging sequence of elements of $Y$. After extraction, and without loss of generality, one can assume that $\xi_1^n = p_i$, for all $n$, and for some $i$. Then, $\xi_2^n$ is in a compact fundamental domain for $G_i$ in $\partial\Gamma \setminus \{p_i\}$, and therefore does not converge to $p_i$. This finally proves that $U$ is a neighborhood of $\Delta$, and ends the proof of Proposition 3.3. □

# 4 Groups admitting a compactification finitely presented with special symbol

In this section we give examples of groups admitting a compactification finitely presented with special symbol, and we introduce a condition for it.

Let us begin with a necessary condition.

**Proposition 4.1** *If $\Gamma$ has a compactification finitely presented with special symbol, then $\Gamma$ is finitely generated.*

Let $\pi : \Phi \to \Gamma \cup K$ be a finite presentation with special symbol. Let $\mathcal{A}$ be the alphabet. Let $\mathcal{C}$ be a cylinder defining $\Phi$, and itself defined by a finite subset, $F$, of $\Gamma$ and a set, $M$, of maps from $F$ onto $\mathcal{A}$. The set of translates of $F$ is a covering of $\Gamma$. Let $P$ be the nerve of the covering. As $F$ is finite, $P$ is a finite dimensional, locally finite polyhedron on which $\Gamma$ acts properly discontinuously cocompactly. The set of vertices of $P$ is naturally identified with $\Gamma$. The claim is that $P$ is connected. If it was not, there would be distinct connected components, $C_i$. Let $\gamma_i \in C_i$, and consider $\sigma_i \in \Phi$ such that $\pi(\sigma_i) = \gamma_i$. Let $\sigma \in \mathcal{A}^\Gamma$ such that $\sigma|_{C_i} \equiv \sigma_i|_{C_i}$. Now, $\sigma$ has several special symbols (one in each $C_i$). On the other hand all the cylinder conditions defining $\Phi$ are satisfied, as by definition they are read on the connected components of $P$. This is a contradiction, and it proves the claim. Therefore, $\Gamma$ is generated by $F$ which is a finite set. □

The next proposition is in fact is a slight variation of a theorem of Gromov, a detailed proof of which can be found in [3] (Corollary 8.2).

**Proposition 4.2** *If $\Gamma$ is an hyperbolic group, then its one-point compactification is finitely presented with special symbol.*

We do again the proof of the main theorem, seeing $\Gamma$ relatively hyperbolic relative to the trivial subgroup $\{1\}$. A Cayley graph plays the role of $\mathcal{K}$, and we consider the same cocycles. They can define either a point at infinity, or a vertex of the graph. Thus, we obtain our presentation choosing the special symbol to be the restriction of a radial cocycle. □

Although it could be seen as a consequence of the proposition above, the example 2 in part 1 already gave the basic examples of $\mathbb{Z}$ and of finite groups. Most of our remaining examples come from the following remarks.

**Proposition 4.3** *If a group $\Gamma$ splits in a short exact sequence $\{1\} \to N \to \Gamma \to H \to \{1\}$, and if both $N$ and $H$ have their alexandrov compactification finitely presented with special symbol, then the alexandrov compactification of $\Gamma$ is finitely presented with special symbol.*

**Proposition 4.4** *Let $G$ be a subgroup of finite index of a group $\Gamma$. The group $G$ has its one-point compactification finitely presented with special symbol if, and only if, the one-point compactification of $\Gamma$ is finitely presented with special symbol.*

Before giving the proofs, we give a consequence. A group $\Gamma$ is said poly-hyperbolic if there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd ... \lhd N_{k-1} \lhd N_k = \Gamma$, with all the quotients $N_{i+1}/N_i$ hyperbolic.

**Corollary 4.1** *Every poly-hyperbolic group has its one-point compactification finitely presented with special symbol. In particular, this includes virtually polycyclic (hence, also virtually nilpotent) groups.*

If $\Gamma$ is poly-hyperbolic, there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd ... \lhd N_{k-1} \lhd N_k = \Gamma$, with all the quotients $N_{i+1}/N_i$ hyperbolic. Using the proposition 4.3, and the fact that hyperbolic groups have their one-point compactifications finitely presented with special symbol, an induction on $i$ tells that each $N_i$ has its one-point compactification finitely presented with special symbol, and especially $N_k$ which is $\Gamma$. □

*Proof of Prop. 4.3.*

Let us denote by $\mathcal{A}_N$, $\mathcal{A}_H$, $\$_N$, $\$_H$, $\mathcal{C}_N$, $\mathcal{C}_H$, $\Phi_N$, $\Phi_H$, the alphabets, special symbols, cylinders, and subshifts of finite type for the presentations of $N \cup \{\infty\}$ and $H \cup \{\infty\}$. Let $F_N$, $F_H$, $M_N$ and $M_H$ be the finite subsets of $N$ and $H$, and the sets of maps defining the two given cylinders. From proposition 4.1, $N$ is finitely generated, then up to enlarging $F_N$, we can assume that $F_N$ generates $N$ (in fact, in the proof of proposition 4.1, it is proved that necessarily, $F_N$ generates $N$). Let $\mathcal{A} = \mathcal{A}_H \times \mathcal{A}_N$. Let us choose $\widetilde{H}$ be a set of representative of $H$ in $\Gamma$, and let $F$ be the finite subset of $\Gamma$ as follow, $F = \{\tilde{h}.n,\ \tilde{h} \in \widetilde{H}, h \in F_H, n \in F_N\}$. Let $M$ be the following set of maps. $M = \{(m : F \to \mathcal{A}), \exists m_H \in M_H, \forall n \in F_N, m(\cdot.n)_1 = m_H\ ;\ \forall h \in F_H, m(\tilde{h}.\cdot)_2 \in M_N\}$, where the subscripts 1 and 2 denote the coordinates in the product $\mathcal{A} = \mathcal{A}_H \times \mathcal{A}_N$. Consider the cylinder defined by $F$ and $M$, and the associated subshift of finite type, $\Phi$. We need the following lemma.

**Lemma 4.1** *For any $\sigma \in \Phi$, there is at most one element $\gamma \in \Gamma$ such that $\sigma(\gamma) = (\$_H, \$_N)$.*

We first prove that for any $\sigma \in \Phi$, there is at most one left coset of $N$, $\tilde{h}N$, such that $\forall n \in N$, $\sigma(\tilde{h}.n)_1 = \$_H$. By definition of $M$, if $n \in F_N$, $n_0 \in N$, then $\sigma(\tilde{h}.n_0.n)_1$, the first coordinate of $\sigma(\tilde{h}.n_0.n)$ only depends on $\tilde{h}$ and $n_0$. But $F_N$ was chosen generating $N$, hence $\sigma(\tilde{h}.n_0.n)_1$ only depends on $\sigma(\tilde{h})$. But, by definition of $M$, the map $h \in H \mapsto \sigma(\tilde{h})_1$ is in $\Phi_H$, and therefore, by the special symbol property, there is at most one value of $\tilde{h}$ where it takes the value $\$_H$, this proves the first step of the lemma. We now prove that, if $\tilde{h}$ is such that $\sigma(\tilde{h}.n)_1 = \$_H$, there is at most one $n \in N$ such that $\sigma(\tilde{h}.n)_2 = \$_N$. This is because of the definition of $M$ as the map $n \in n \mapsto \sigma(\tilde{h}.n)_2$ is in $\Phi_N$. This proves the lemma. □

Now, we define the map $\pi$ so that it sends a element $\sigma \in \Phi$ on the point at infinity, if $\sigma$ does not contain the symbol $(\$_H\$_N)$, and on $\gamma \in \Gamma$ if $\sigma(\gamma) = (\$_H\$_N)$. The map $\pi$ is well defined, and gives a finite presentation with special symbol of $\Gamma \cup \{\infty\}$. □

*Proof of Prop. 4.4.*

Assume that $\Gamma$ has its one point compactification finitely presented with special symbol, and let $\mathcal{A}_\Gamma$, $\$_\Gamma$, $\mathcal{C}_\Gamma$, $\Phi_\Gamma$ the alphabet, special symbol, cylinder, and subshift of finite type associated. The cylinder is defined, as before, by two sets : $F_\Gamma \subset \Gamma$ and $M_\Gamma \subset \mathcal{A}^{F_\Gamma}$. We consider $\gamma_1, \ldots, \gamma_n$ a set of orbit representatives of left coset of $G$ in $\Gamma$, and we choose $F = (\bigcup_{i=1}^n \gamma_i^{-1} F_\Gamma) \cap G$, a finite subset of $G$. We set $\mathcal{A} = (\mathcal{A}_\Gamma)^n$ and $M \subset \mathcal{A}^F$ is the set of the maps $m$ from $F$ to $(\mathcal{A}_\Gamma)^n$ such that there exists $m_\Gamma \in M_\Gamma$ whose translates $\gamma_i^{-1} m_\Gamma$ coincide with the $i$-th coordinate of $m$. Those three choices define a subshift of finite type $\Phi \subset \mathcal{A}^G$. By definition of $M$, one sees that there is a natural map $\Phi \to \Phi_\Gamma$ which consists of pushing the $i$-th coordiante of an element $\sigma$ on the coset $\gamma_i G$. This map is continuous $G$-equivariant, and it is a bijection, its inverse being the map that associates to $\varphi \in \Phi_\Gamma$ the element $\sigma \in \Phi$ whose $i$-th coordinate coincide with $\gamma_i^{-1}\varphi$. Therefore, one has a map $\Phi \to \Gamma \cup \{\infty\} \to G \cup \{\infty\}$, the second map being identity on $G$ and sending each $\gamma_i$ to 1. At this point we do not have a special symbol, but, by property of $\Phi_\Gamma$, an element of $\Phi$ can take a value in $\mathcal{A}$ which has $\$_\Gamma$ among its coordinate, only once. Hence, by renaming each of those symbol by a single one \$, we get the expected presentation with special symbol.

Conversely, it suffices to see that the intersection of all the conjugates of $G$ is of finite index in $G$ (hence it has its one point compactification finitely presented with special symbol). It is normal and of finite index in $\Gamma$, and we can apply Proposition 4.3. □

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