

Spreading speeds for some reaction-diffusion equations with general initial conditions

François Hamel ^{a, b} and Yannick Sire ^{a*}

^a Aix-Marseille Université, LATP, Faculté des Sciences et Techniques
Avenue Escadrille Normandie-Niemen, F-13397 Marseille Cedex 20, France

^b Institut Universitaire de France

Abstract

This paper is devoted to the study of some qualitative and quantitative aspects of nonlinear propagation phenomena in diffusive media. More precisely, we consider the case a reaction-diffusion equation in a periodic medium with ignition-type nonlinearity, the heterogeneity being on the nonlinearity, the operator and the domain. Contrary to previous works, we study the asymptotic spreading properties of the solutions of the Cauchy problem with general initial conditions which satisfy very mild assumptions at infinity. We introduce several concepts generalizing the notion of spreading speed and we give a complete characterization of it when the initial condition is asymptotically oscillatory at infinity. Furthermore we construct, even in the homogeneous one-dimensional case, a class of initial conditions for which highly nontrivial dynamics can be exhibited.

1 Introduction

We consider reaction-diffusion-advection equations of the type

$$\begin{cases} u_t - \nabla \cdot (A(z)\nabla u) + q(z) \cdot \nabla u = f(z, u), & z \in \bar{\Omega}, \\ \nu A \nabla u = 0, & z \in \partial\Omega \end{cases} \quad (1.1)$$

in an unbounded domain (connected and open) $\Omega \subset \mathbb{R}^N$ which is of class $C^{2,\alpha}$ for some $\alpha > 0$. We denote by ν the outward unit normal on $\partial\Omega$. For any two vectors $\xi = (\xi_1, \dots, \xi_N)$ and $\xi' = (\xi'_1, \dots, \xi'_N)$ in \mathbb{R}^N and any $N \times N$ matrix $B = (B_{ij})_{1 \leq i, j \leq N}$ with real entries, we write

$$\xi B \xi' = \sum_{1 \leq i, j \leq N} \xi_i B_{ij} \xi'_j.$$

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The coefficients of (1.1) are not assumed to be homogeneous in general, as well as the underlying domain Ω . Instead, we just assume that there exist two real numbers $L > 0$ and $R \geq 0$ such that

$$\begin{cases} \forall z = (x, y) \in \Omega, & |y| \leq R, \\ \forall k \in LZ \times \{0\}^{N-1}, & \Omega = \Omega + k, \end{cases} \quad (1.2)$$

where

$$x = x_1, \quad y = (x_2, \dots, x_N), \quad z = (x, y)$$

and $|\cdot|$ denotes the euclidean norm. The domain Ω is then an infinite cylinder which is unbounded in the direction x , its boundary $\partial\Omega$ may be straight or undulating, and Ω may also contain periodic perforations. Let C be the periodicity cell defined by

$$C = \{z = (x, y) \in \Omega, \quad x \in (0, L)\}.$$

Throughout the paper, we assume that the matrix field $z \mapsto A(z) = (A_{ij}(z))_{1 \leq i, j \leq N}$ is of class $C^{1,\alpha}(\overline{\Omega})$, symmetric and uniformly elliptic, that the vector field $z \mapsto q(z) = (q_i(z))_{1 \leq i \leq N}$ is of class $C^{0,\alpha}(\overline{\Omega})$, divergence-free ($\nabla \cdot q = 0$ in $\overline{\Omega}$) and tangent to the boundary of Ω ($q \cdot \nu = 0$ on $\partial\Omega$), and that the nonlinearity $f : (z, u) \in (\overline{\Omega} \times \mathbb{R}) \mapsto f(z, u)$ is continuous, of class $C^{0,\alpha}$ with respect to z locally uniformly in $u \in \mathbb{R}$ and we assume that the restriction of f to $\overline{\Omega} \times [0, 1]$ is of class C^1 with respect to u . All functions A_{ij} , q_i and $f(\cdot, u)$ (for all $u \in \mathbb{R}$) are assumed to be periodic, in the sense that they satisfy

$$w(x + k, y) = w(x, y) \quad \text{for all } (x, y) \in \Omega \text{ and } k \in LZ.$$

The vector field q is normalized in such a way that

$$\int_C q(z) dz = 0.$$

The nonlinearity f is also assumed to be of combustion type: there exists $\theta \in (0, 1)$ such that for every $z \in \overline{\Omega}$,

$$\begin{cases} f(z, \cdot) \equiv 0 & \text{on } (-\infty, \theta] \cup [1, +\infty), \\ f(z, \cdot) > 0 & \text{on } (\theta, 1), \quad \frac{\partial f}{\partial u}(z, 1^-) = -\lim_{s \rightarrow 0^+} \frac{f(z, 1-s)}{s} < 0. \end{cases} \quad (1.3)$$

Under the previous structural assumptions on the domain and the nonlinearity, we study the Cauchy problem

$$\begin{cases} u_t - \nabla \cdot (A(z)\nabla u) + q(z) \cdot \nabla u = f(z, u), & t > 0, \quad z \in \overline{\Omega}, \\ \nu A \nabla u = 0, & t > 0, \quad z \in \partial\Omega, \\ u(0, z) = u_0(z), & z \in \Omega, \end{cases} \quad (1.4)$$

where the initial value $u_0 : \Omega \rightarrow [0, 1]$ is uniformly continuous and satisfies the following mild conditions at infinity

$$\lim_{A \rightarrow +\infty} \left(\sup_{z=(x,y) \in \Omega, x \geq A} u_0(z) \right) < \theta \quad \text{and} \quad \lim_{A \rightarrow -\infty} \left(\inf_{z=(x,y) \in \Omega, x \leq A} u_0(z) \right) > \theta.$$

For sake of conciseness, we will denote the previous limits as

$$\limsup_{x \rightarrow +\infty} u_0(z) < \theta \quad \text{and} \quad \liminf_{x \rightarrow -\infty} u_0(z) > \theta. \quad (1.5)$$

The assumption of uniform continuity for u_0 is just made to ensure the solvability of the Cauchy problem. Notice also that, since u_0 satisfies $0 \leq u_0 \leq 1$ in Ω and is not identically equal to 0 or 1 because of (1.5), the solution u of (1.4) satisfies

$$0 < u(t, z) < 1 \quad \text{for all } t > 0 \text{ and } z \in \overline{\Omega} \quad (1.6)$$

from the strong parabolic maximum principle and Hopf lemma.

The main assumption (1.5) means, roughly speaking, that the initial condition u_0 is front-like in the direction x , uniformly with respect to the orthogonal variables y . But it is important to notice that we do not assume that u_0 converges to some constants as $x \rightarrow \pm\infty$. The goal of this paper is to study propagation phenomena for the solutions u of (1.4) when the initial conditions u_0 just satisfy (1.5). We shall see that these very weak assumptions at initial time give rise to a large variety of asymptotic spreading properties and possibly complex large-time behaviour. To this end, we first define the following two quantities, which shall stand for minimal and maximal asymptotic spreading speeds:

Definition 1.1 *Let u_0 be as before. We define the lower spreading speed $c_*(u_0)$ associated to (1.4) as*

$$c_*(u_0) = \sup \mathcal{E}_*(u_0)$$

where

$$\mathcal{E}_*(u_0) = \left\{ c \in \mathbb{R} \mid \forall c' < c, \lim_{t \rightarrow +\infty} \left(\inf_{z \in \overline{\Omega}, x \leq c't} u(t, z) \right) = 1 \right\}.$$

We also define the upper spreading speed $c^*(u_0)$ associated to (1.4) as

$$c^*(u_0) = \inf \mathcal{E}^*(u_0)$$

where

$$\mathcal{E}^*(u_0) = \left\{ c \in \mathbb{R} \mid \forall c' > c, \limsup_{t \rightarrow +\infty} \left(\sup_{z \in \overline{\Omega}, x \geq c't} u(t, z) \right) < 1 \right\}.$$

Qualitatively, the previous definitions show that an observer who moves at speed c in direction x will see for large times the steady state 1 if $c < c_*(u_0)$ and will be away from 1 if $c > c^*(u_0)$. It follows in particular from Definition 1.1 and (1.6) that, for all $A \in \mathbb{R}$,

$$\lim_{t \rightarrow +\infty} \left(\sup_{(x+s, y) \in \overline{\Omega}, x \leq A, s \leq ct} |u(t, x+s, y) - 1| \right) = 0 \quad \text{if } c < c_*(u_0)$$

and

$$\limsup_{t \rightarrow +\infty} \left(\sup_{(x+s, y) \in \overline{\Omega}, x \geq A, s \geq ct} u(t, x+s, y) \right) < 1 \quad \text{if } c > c^*(u_0).$$

Notice that by definition, there always holds

$$c_*(u_0) \leq c^*(u_0).$$

Furthermore, if $c_*(u_0) \in \mathbb{R}$, resp. $c^*(u_0) \in \mathbb{R}$ –we shall see in Theorem 1.3 that this is automatically true due to (1.5)– then $c_*(u_0) = \max \mathcal{E}_*(u_0)$, resp. $c^*(u_0) = \min \mathcal{E}^*(u_0)$. However, this does not mean in general that

$$\lim_{t \rightarrow +\infty} \left(\inf_{z \in \bar{\Omega}, x \leq c_*(u_0)t} u(t, z) \right) = 1$$

or

$$\limsup_{t \rightarrow +\infty} \left(\sup_{z \in \bar{\Omega}, x \geq c^*(u_0)t} u(t, z) \right) < 1.$$

This paper is devoted to some characterizations of the lower and upper spreading speeds $c_*(u_0)$ and $c^*(u_0)$ given in Definition 1.1, when u_0 satisfies the above conditions (1.5). We will derive some estimates for these spreading speeds and provide an example for which $c^*(u_0) \neq c_*(u_0)$, even in the homogeneous case.

One of the key points to understand propagation phenomena for the Cauchy problem (1.4) is based on the existence of a family of pulsating travelling fronts for (1.1). In particular, we shall relate the spreading speeds $c_*(u_0)$ and $c^*(u_0)$ to various speeds of fronts connecting two stationary states of the equation. Given any real number $\gamma \in (-\infty, \theta)$, a pulsating front connecting γ to 1 and travelling to the right with effective speed $c \neq 0$ is a special time-global solution $u : \mathbb{R} \times \bar{\Omega} \rightarrow (\gamma, 1)$ of (1.1) satisfying the periodicity condition

$$\forall k \in LZ, \forall (t, x, y) \in \mathbb{R} \times \bar{\Omega}, \quad u\left(t - \frac{k}{c}, x, y\right) = u(t, x + k, y) \quad (1.7)$$

and the additional asymptotic conditions

$$\lim_{x \rightarrow +\infty} u(t, z) = \gamma \quad \text{and} \quad \lim_{x \rightarrow -\infty} u(t, z) = 1. \quad (1.8)$$

The previous limits (1.8) are taken locally in time and uniformly in y . Another way to describe a pulsating front is to use a hull function $\varphi : \mathbb{R} \times \bar{\Omega} \mapsto (\gamma, 1)$ and write u as

$$u(t, z) = \varphi(x - ct, z)$$

where the function $z \in \bar{\Omega} \mapsto \varphi(s, z)$ is periodic in $\bar{\Omega}$ for each $s \in \mathbb{R}$, and

$$\varphi(+\infty, \cdot) = \gamma, \quad \varphi(-\infty, \cdot) = 1 \quad \text{uniformly in } \bar{\Omega}.$$

The existence and properties of pulsating travelling fronts have been obtained in [45, 46] for the case of the whole space \mathbb{R}^N and in [3, 4] in the general periodic framework and with general combustion-type nonlinearities, covering the situation of the present paper. We sum up the result in the following theorem

Theorem 1.2 [3, 4] *Let the nonlinearity f be of the combustion type (1.3). For any $\gamma \in (-\infty, \theta)$, there exists a unique speed $c = c_\gamma$, which is positive, such that problem (1.1) has a pulsating travelling front solution u_γ satisfying (1.7) and (1.8). Furthermore, the function u_γ is unique up to shifts in time and the map $\gamma \mapsto c_\gamma$ is continuous and increasing.*

Under assumption (1.3), a great attention has been to be devoted to the properties of fronts in the homogeneous one-dimensional version of (1.1), and then in straight infinite cylinders (see e.g. [11]). Of particular interest are the stability of these fronts and the convergence to the fronts of the solutions of Cauchy problems of the type (1.4) when the initial condition u_0 is in some sense close to a given front and has the same (constant) limit as it when $x \rightarrow +\infty$ [10, 25, 37, 38, 39]. Initial conditions with compact support have also been considered. Under some conditions, that is if they are above and away from θ on a sufficiently large set, then they develop into a pair of diverging fronts [26, 39, 47]. However, in the general periodic setting, the question of the global stability of the travelling fronts still remains open, even for initial conditions having the same constant limit γ as a given front u_γ when $x \rightarrow +\infty$. As a matter of fact, the present paper is at least twofold: firstly, we show the convergence in speed for a more general class of asymptotically periodic (when $x \rightarrow +\infty$) initial conditions, and secondly we prove that such convergence does not hold in general, even in the homogeneous case, when the initial conditions just satisfy (1.5).

Let us mention here that other types of nonlinearities have also been considered in the literature. For instance, some existence and stability results of fronts with bistable reaction terms are known, but they are mainly concerned with homogeneous or close-to-homogeneous media, or with media which are invariant in the direction of propagation [18, 30, 39, 44]. One of the most famous results in this spirit is the following one: in the homogeneous setting with bistable reaction-terms $f : [0, 1] \rightarrow \mathbb{R}$ satisfying

$$f(0) = f(\theta) = f(1) = 0, \quad f < 0 \text{ on } (0, \theta), \quad f > 0 \text{ on } (\theta, 1), \quad f'(0) < 0, \quad f'(1) < 0$$

for some $\theta \in (0, 1)$, front-like initial conditions satisfying (1.5) are known to converge to the unique front connecting the two stable zeroes 0 and 1 of f [18]. This is due to the strong attractivity of these two stable states. As will be seen, in the combustion case (1.3) considered in the present paper, new interesting and more complex phenomena shall occur, due to the existence of a continuum of stationary states (below θ). Lastly, for monostable or particular Kolmogorov-Petrovski-Piskunov [27] type nonlinearities, existence and qualitative properties of pulsating travelling fronts in periodic media have been established in [3, 7, 19, 21, 34, 35, 43]. In this case, the set of possible speeds is a half-line $[w^*, +\infty)$. Estimates of the minimal speeds w^* have been derived in [6, 7, 9, 16, 17, 23, 41, 48]. Since the seminal paper of Aronson and Weinberger [1] in the homogeneous setting in \mathbb{R}^N , much work has also been devoted to asymptotic spreading speeds in KPP-type reaction-diffusion equations with compactly supported initial conditions in periodic or more general media [5, 8, 31, 32, 43], with exponentially decaying initial conditions [2, 12, 20, 21, 24, 29, 33, 39, 42] or with slowly decaying initial conditions [13, 22].

Let us now come back to the Cauchy problem (1.4) under assumption (1.3). As already emphasized, the main goal of this paper is to consider (1.4) with a very large class of front-like initial conditions, satisfying (1.5), which are not required to converge to any constant in the direction of propagation or to be close to any pulsating front. Before stating our main results, we just need to introduce a few more notations. We consider the following linear advection-diffusion equation with the same initial data u_0 as for the nonlinear Cauchy

problem (1.4), but with zero right-hand side:

$$\begin{cases} v_t - \nabla \cdot (A(z)\nabla v) + q(z) \cdot \nabla v = 0, & t > 0, z \in \bar{\Omega}, \\ \nu A \nabla v = 0, & t > 0, z \in \partial\Omega, \\ v(0, z) = u_0(z), & z \in \Omega, \end{cases} \quad (1.9)$$

Then, we introduce the following quantities, which will play an important role in the sequel

$$\alpha_{\min}(u_0) = \lim_{t \rightarrow +\infty} \left(\liminf_{x \rightarrow +\infty} v(t, z) \right)$$

and

$$\alpha_{\max}(u_0) = \lim_{t \rightarrow +\infty} \left(\limsup_{x \rightarrow +\infty} v(t, z) \right).$$

The limits in time in the previous two quantities are well-defined real numbers since the maps $t \mapsto \liminf_{x \rightarrow +\infty} v(t, z)$ and $t \mapsto \limsup_{x \rightarrow +\infty} v(t, z)$ are bounded (in $[0, 1]$) and respectively nondecreasing and nonincreasing in time by the parabolic maximum principle (see Remark 2.3 after Lemma 2.2 for more details). Furthermore, there holds

$$0 \leq \alpha_{\min}(u_0) \leq \alpha_{\max}(u_0) \leq \limsup_{x \rightarrow +\infty} u_0(z) < \theta.$$

In particular, if $u_0(z) \rightarrow \gamma$ as $x \rightarrow +\infty$ for some real number $\gamma \in [0, \theta)$, then $\alpha_{\min}(u_0) = \alpha_{\max}(u_0) = \gamma$.

We can now state the main results of this paper. The first theorem provides lower and upper bounds for the lower and upper spreading speeds $c_*(u_0)$ and $c^*(u_0)$.

Theorem 1.3 *Let u be the solution of (1.4) with any uniformly continuous initial condition $u_0 : \Omega \rightarrow [0, 1]$ satisfying (1.5). Then*

$$\liminf_{t \rightarrow +\infty} \left(\inf_{z \in \bar{\Omega}} u(t, z) \right) \geq \alpha_{\min}(u_0)$$

and

$$c_{\alpha_{\min}(u_0)} \leq c_*(u_0) \leq c^*(u_0) \leq c_{\alpha_{\max}(u_0)}, \quad (1.10)$$

where we recall that $c_{\alpha_{\min}(u_0)}$ and $c_{\alpha_{\max}(u_0)}$ denote the unique speeds of the pulsating fronts of (1.1) connecting $\alpha_{\min}(u_0)$ and $\alpha_{\max}(u_0)$ to 1. Furthermore, for every $c > c_{\alpha_{\max}(u_0)}$, there holds

$$\limsup_{t \rightarrow +\infty} \left(\sup_{z \in \bar{\Omega}, x \geq ct} u(t, z) \right) \leq \alpha_{\max}(u_0).$$

Thus, Theorem 1.3 provides bounds for the asymptotic spreading speeds $c_*(u_0)$ and $c^*(u_0)$. The following theorem states a complete characterization of these spreading speeds when the initial condition is assumed to be asymptotically periodic in the right direction.

Theorem 1.4 *Let $u_0 : \Omega \rightarrow [0, 1]$ be a uniformly continuous function such that there exists a uniformly continuous periodic function $w_0 : \Omega \rightarrow [0, \theta)$ satisfying*

$$\lim_{x \rightarrow +\infty} |u_0(z) - w_0(z)| = 0.$$

Then $\alpha_{\min}(u_0) = \alpha_{\max}(u_0) = \langle w_0 \rangle$, and consequently

$$c_*(u_0) = c^*(u_0) = c_{\langle w_0 \rangle},$$

where

$$\langle w_0 \rangle = \int_C w_0 \in [0, \theta)$$

denotes the average of the periodic function w_0 and $c_{\langle w_0 \rangle}$ is the unique speed of the pulsating travelling front of (1.1) connecting $\langle w_0 \rangle$ to 1.

An immediate corollary follows from Theorem 1.4 and the monotonicity of the speeds c_γ with respect to γ . It shows the monotonicity of the spreading speeds with respect to the asymptotic average (when it exists) of the tail of u_0 as $x \rightarrow +\infty$.

Corollary 1.5 *Let u_0 and $\tilde{u}_0 : \Omega \rightarrow [0, 1]$ fulfill the assumptions of Theorem 1.4 with two periodic functions w_0 and $\tilde{w}_0 : \Omega \rightarrow [0, \theta)$. If $\langle w_0 \rangle < \langle \tilde{w}_0 \rangle$, then the spreading speeds satisfy $c_*(u_0) = c^*(u_0) = c_{\langle w_0 \rangle} < c_{\langle \tilde{w}_0 \rangle} = c_*(\tilde{u}_0) = c^*(\tilde{u}_0)$.*

As a consequence of Theorem 1.4, we have convergence in speed in the following sense: for any given value $\lambda \in (\langle w_0 \rangle, 1)$, the set

$$E_{t,\lambda} = \{(x, y) \in \bar{\Omega}, u(t, x, y) = \lambda\}$$

is not empty for large t and

$$\lim_{t \rightarrow +\infty} \frac{\min\{x \mid \exists y, (x, y) \in E_{t,\lambda}\}}{t} = \lim_{t \rightarrow +\infty} \frac{\max\{x \mid \exists y, (x, y) \in E_{t,\lambda}\}}{t} = c_{\langle w_0 \rangle}.$$

In particular, for any family $(x(t), y(t))$ in $\bar{\Omega}$ such that $u(t, x(t), y(t)) = \lambda$, then $x(t)/t$ converges to $c_{\langle w_0 \rangle}$ as $t \rightarrow +\infty$. This corresponds exactly to the notion of convergence in speed. However, it does not mean that $x(t) - c_{\langle w_0 \rangle}t$ converges as $t \rightarrow +\infty$ or is even bounded. But we conjecture that $x(t) - c_{\langle w_0 \rangle}t$ converges as $t \rightarrow +\infty$ provided that u_0 converges to w_0 sufficiently fast (exponentially) as $x \rightarrow +\infty$. A remaining open question is the convergence in profile of $u(t, \cdot)$ to the one-parameter family of time shifts of the pulsating front $u_{\langle w_0 \rangle}$.

In the previous two theorems, we established some general properties and bounds of the lower and upper spreading speeds $c_*(u_0)$ and $c^*(u_0)$, and we considered an important class of initial conditions for which these two quantities are equal. In what follows, we exhibit a class of initial conditions u_0 for which $c_*(u_0) < c^*(u_0)$ and, among other things, we will see that the behaviour of the solution u along the rays with speeds c between $c_*(u_0)$ and $c^*(u_0)$ is rather complex. For the sake of clarity of the presentation, we only consider

here a simple one-dimensional and homogeneous framework –more general heterogeneous equations with the same type of initial conditions and the same type of long-time behaviour could be dealt with. Consider the following Cauchy problem

$$\begin{cases} u_t - u_{xx} = f(u), & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.11)$$

where the nonlinearity f is of the combustion type (1.3), as above, but it depends on u only. For each $\gamma \in (-\infty, \theta)$, let φ_γ denote the unique (up to shifts) travelling front of (1.11) connecting γ to 1, with unique speed c_γ , that is

$$\begin{cases} \varphi_\gamma'' + c_\gamma \varphi_\gamma' + f(\varphi_\gamma) = 0 & \text{in } \mathbb{R}, \\ \varphi_\gamma(-\infty) = 1 > \varphi_\gamma(x) > \varphi_\gamma(+\infty) = \gamma & \text{for all } x \in \mathbb{R}. \end{cases} \quad (1.12)$$

Theorem 1.6 *Let $\alpha < \beta$ be any given real numbers in $[0, \theta)$. There are initial conditions $u_0 : \mathbb{R} \rightarrow [\alpha, 1]$ such that*

$$\begin{cases} \liminf_{x \rightarrow -\infty} u_0(x) > \theta, \\ \exists A \in \mathbb{R}, \forall x \geq A, \quad \alpha \leq u_0(x) \leq \beta \end{cases}$$

and such that, under the general previous notations,

$$\alpha_{\min}(u_0) = \alpha, \quad \alpha_{\max}(u_0) = \beta, \quad c_*(u_0) = c_\alpha < c_\beta = c^*(u_0).$$

Furthermore,

$$\forall t \geq 0, \quad \alpha \leq u(t, \cdot) \leq 1, \quad u(t, -\infty) = 1, \quad \liminf_{x \rightarrow +\infty} u(t, x) = \alpha < \beta = \limsup_{x \rightarrow +\infty} u(t, x)$$

and

$$\left\{ \begin{array}{ll} \forall c < c_*(u_0), \forall A \in \mathbb{R}, & u(t, ct + \cdot) \xrightarrow[t \rightarrow +\infty]{} 1 \text{ uniformly in } (-\infty, A], \\ \forall x \in \mathbb{R}, \exists \alpha_x \in (\alpha, 1], & \left\{ \lim_{t_k \rightarrow +\infty} u(t_k, c_*(u_0)t_k + x) \right\} = [\alpha_x, 1], \\ \forall c \in (c_*(u_0), c^*(u_0)), \forall x \in \mathbb{R}, & \left\{ \lim_{t_k \rightarrow +\infty} u(t_k, ct_k + x) \right\} = [\alpha, 1], \\ \forall x \in \mathbb{R}, \exists \beta_x \in [\beta, 1), & \left\{ \lim_{t_k \rightarrow +\infty} u(t_k, c^*(u_0)t_k + x) \right\} = [\alpha, \beta_x], \\ \forall c > c^*(u_0), \forall A \in \mathbb{R}, & \lim_{t \rightarrow +\infty} \left(\sup_{x \in [A, +\infty)} u(t, ct + x) \right) = \beta, \\ \forall c > c^*(u_0), \forall x \in \mathbb{R}, & \left\{ \lim_{t_k \rightarrow +\infty} u(t_k, ct_k + x) \right\} = [\alpha, \beta]. \end{array} \right.$$

Let us now comment the construction and the long-time behaviour of the solutions u given in Theorem 1.6. The initial conditions u_0 are constructed so that $u_0(x)$ oscillates between α and β as $x \rightarrow +\infty$, on larger and larger intervals. This way, the solution u will somehow oscillate at large times between two approximated fronts whose speeds are approximately equal to $c_*(u_0) = c_\alpha$ and $c^*(u_0) = c_\beta$. In other words, the “location” $\xi(t)$ of the solution, that is $\xi(t) \in \mathbb{R}$ such that $u(t, \xi(t)) = \theta$, oscillates between $c_\alpha t$ and $c_\beta t$, which means in particular no convergence in speed. We nevertheless provide quantitative estimates on $\xi(t)$ over some reasonably large time intervals (precise statements will be given in Section 3, see in particular the proof of Lemma 3.1 and Remark 3.2 below). Thus, the values of $u(t, ct + x)$ along any ray with a given speed $c \in (c_*(u_0), c^*(u_0))$ describe at the limit the whole interval $[\alpha, 1]$, in the sense that the set of limit values of the function $t \mapsto u(t, ct + x)$ as $t \rightarrow +\infty$ is equal to the whole interval $[\alpha, 1]$. In the moving frame with speed $c_*(u_0)$ (resp. $c^*(u_0)$), as we shall see in Section 3, the solution u is actually separated from α (resp. 1) uniformly in $(-\infty, A]$ (resp. $[A, +\infty)$) for any $A \in \mathbb{R}$, in the sense that

$$\begin{cases} \alpha < \liminf_{t \rightarrow +\infty} \left(\inf_{x \in (-\infty, A]} u(t, c_*(u_0)t + x) \right), \\ \limsup_{t \rightarrow +\infty} \left(\sup_{x \in [A, +\infty)} u(t, c^*(u_0)t + x) \right) < 1. \end{cases} \quad (1.13)$$

However, these limits are never uniform in space, since $\inf_{\mathbb{R}} u(t, \cdot) = \alpha$ and $\sup_{\mathbb{R}} u(t, \cdot) = 1$ for all $t \geq 0$.

Under the general notations and assumptions of this paper, the speed $c_*(u_0)$ is by definition the largest speed for which the solution u converges to 1 in any right-moving frame with a speed *smaller* than $c_*(u_0)$ (and even uniformly in any given set $\{x \leq A\}$). However, one of the main interests of Theorem 1.6 is to show that, even for the homogeneous equation (1.11), the solution u may not in general be separated from 1 in all moving frames with speeds *larger* than $c_*(u_0)$: indeed, in Theorem 1.6, there holds $\limsup_{t \rightarrow +\infty} u(t, ct + x) = 1$ for all $c \in [c_*(u_0), c^*(u_0))$ and $x \in \mathbb{R}$, with $c_*(u_0) < c^*(u_0)$. On the other hand, again by virtue of our general definitions, the solution u is always separated from 1 in any right-moving frame with a speed *larger* than $c^*(u_0)$. However, in any such moving frame, the solution u may still have a complex behaviour and it may not converge locally to a constant in general, as seen as a byproduct of the last assertion in Theorem 1.6. Lastly, the solution u of (1.4) may not in general be separated from its infimum value in all moving frames with speeds *smaller* than $c^*(u_0)$, since, in the example given in Theorem 1.6, there holds $\liminf_{t \rightarrow +\infty} u(t, ct + x) = \alpha$ for all $c \in (c_*(u_0), c^*(u_0)]$ and $x \in \mathbb{R}$, with $c_*(u_0) < c^*(u_0)$.

Lastly, we mention that non-convergence results similar to the ones described in Theorem 1.6 for $c \in (c_*(u_0), c^*(u_0))$ are also known to hold for the heat equation, see [14] and Remark 3.3 below. Other complex behavior may also occur for the nonlinear equation $u_t = \Delta u + f(u)$ in \mathbb{R}^2 with bistable-type nonlinearity f and some appropriate initial conditions which are trapped between two shifts of a given conical front (the solutions may not in general converge to a unique shift of the given front, see [40]), as well as for supercritical semilinear heat equations with some initial conditions which are trapped between two ordered stationary states (the solutions may not in general converge to a unique stationary state, see [36]).

Remark 1.7 In the previous results, the initial conditions u_0 were front-like in the sense of (1.5). When

$$\limsup_{|x| \rightarrow +\infty} u_0(z) < \theta$$

and when $u_0(z) \geq \theta + \varepsilon$ for all $z \in \Omega \cap \{x \in [-A, A]\}$ with $\varepsilon > 0$ fixed and A large enough, then $u(t, z)$ converges to 1 as $t \rightarrow +\infty$ locally in $z \in \overline{\Omega}$, see [43]. In that case, the solution u will then spread to the left (as $x \rightarrow -\infty$) and to the right (as $x \rightarrow +\infty$). One could then define left spreading speeds in a similar way as in Definition 1.1 (which was concerned with the right spreading speeds). Actually, when $u(t, z)$ converges to 1 as $t \rightarrow +\infty$ locally in $z \in \overline{\Omega}$, the left spreading speeds are completely independent of the right ones: the left (resp. right) ones only depend on the behavior of u_0 as $x \rightarrow -\infty$ (resp. $x \rightarrow +\infty$). Therefore, the same qualitative and quantitative results about the left and right spreading speeds as in the previous theorems hold immediately if the above conditions on u_0 are fulfilled.

2 General properties

This section is concerned with the proof of the general properties of the lower and upper spreading speeds $c_*(u_0)$ and $c^*(u_0)$. We begin in Subsection 2.1 with the proof of Theorem 1.4, since it follows straightforwardly from Theorem 1.3. Then Subsection 2.2 is devoted to the proof of Theorem 1.3.

2.1 Asymptotically periodic initial conditions

In this subsection, we prove Theorem 1.4, assuming the conclusion of Theorem 1.3. To this end, we shall use the following theorem, providing Gaussian estimates (see Theorem 6.1 in [15]) for the fundamental solution of the linear equation (1.9).

Theorem 2.1 [15] *Let $p(t, z, z')$ be the kernel of the operator $\partial_t - \nabla \cdot (A \nabla) + q \cdot \nabla$ in Ω with no-flux boundary conditions $\nu A \nabla p = 0$ on $\partial\Omega$. Then there exist some constants $C_0 > 0$, $\omega_1 \geq 0$ and $\omega_2 > 0$ such that for all $0 < t < +\infty$ and $(z, z') \in \overline{\Omega}$*

$$|p(t, z, z')| \leq C_0 t^{-\frac{N}{2}} e^{\omega_1 t - \frac{|z-z'|^2}{\omega_2 t}} \quad (2.14)$$

In order to prove Theorem 1.4, we will need the following lemma, which is a consequence of the above Gaussian estimates.

Lemma 2.2 *Let v (resp. w) be the unique solution of the linear equation (1.9) in Ω with a uniformly continuous and bounded initial condition $v_0 : \Omega \rightarrow \mathbb{R}$ (resp. $w_0 : \Omega \rightarrow \mathbb{R}$). Assume furthermore that*

$$\lim_{x \rightarrow +\infty} |v_0(z) - w_0(z)| = 0.$$

Then, for all $t \geq 0$,

$$\lim_{x \rightarrow +\infty} |v(t, z) - w(t, z)| = 0.$$

Proof. The proof uses standard arguments. We just do it here for the sake of completeness. By uniqueness of the solution of the Cauchy problem, the function $\varphi(t, z) = v(t, z) - w(t, z)$ satisfies

$$\varphi(t, z) = \int_{\Omega} p(t, z, z') \varphi(0, z') dz'$$

for all $t > 0$ and $z \in \bar{\Omega}$. Let $t > 0$ and $\varepsilon > 0$ be any two arbitrary positive real numbers. From the assumption of the lemma, there is $A \in \mathbb{R}$ such that $|\varphi(0, z)| \leq \varepsilon$ for all $z = (x, y) \in \Omega$ such that $x \geq A$. Set

$$\Omega^+ = \{z' = (x', y') \in \Omega, x' \geq A\} \quad \text{and} \quad \Omega^- = \{z' = (x', y') \in \Omega, x' \leq A\}.$$

For all $z = (x, y) \in \bar{\Omega}$, there holds

$$\begin{aligned} |\varphi(t, z)| &\leq \varepsilon \int_{\Omega^+} p(t, z, z') dz' + \int_{\Omega^-} p(t, z, z') |\varphi(0, z')| dz' \\ &\leq \varepsilon + C_0 \|v_0 - w_0\|_{\infty} t^{-\frac{N}{2}} e^{\omega_1 t} \int_{\Omega^-} e^{-\frac{|z-z'|^2}{\omega_2 t}} dz' \end{aligned}$$

from Theorem 2.1. From (1.2), it follows that

$$|\varphi(t, z)| \leq \varepsilon + C_0 \|v_0 - w_0\|_{\infty} C_{N,R} t^{-\frac{N}{2}} e^{\omega_1 t} \int_{-\infty}^A e^{-\frac{|x-x'|^2}{\omega_2 t}} dx', \quad (2.15)$$

where $C_{N,R} > 0$ denotes the Lebesgue measure of any euclidean ball of radius R in \mathbb{R}^{N-1} . Since the last integral does not depend on y and converges to 0 as $x \rightarrow +\infty$, one concludes that there exists $B \in \mathbb{R}$ such that $|\varphi(t, z)| \leq 2\varepsilon$ for all $z = (x, y) \in \bar{\Omega}$ such that $x \geq B$, which gives the desired conclusion. \square

Remark 2.3 Under the notations of Lemma 2.2, it follows that the quantities $m(t) = \liminf_{x \rightarrow +\infty} v(t, z)$ and $M(t) = \limsup_{x \rightarrow +\infty} v(t, z)$ are respectively nondecreasing and nonincreasing with respect to time t . It is obviously sufficient to deal with $m(t)$. To do so, let $0 \leq t_1 < t_2 < +\infty$ be fixed, and let $\varepsilon > 0$ be arbitrary. There exist then $A \in \mathbb{R}$ and a uniformly continuous and bounded function $w_0 : \Omega \rightarrow \mathbb{R}$ such that $v(t_1, \cdot) \geq w_0$ in Ω and $w_0(z) = m(t_1) - \varepsilon$ for all $z = (x, y) \in \Omega$ such that $x \geq A$. The maximum principle yields $v(t + t_1, \cdot) \geq w(t, \cdot)$ in $\bar{\Omega}$ for all $t > 0$, where w denotes the solution of (1.9) with initial condition w_0 . But Lemma 2.2 implies that $\lim_{x \rightarrow +\infty} w(t, z) = m(t_1) - \varepsilon$ for all $t \geq 0$. Consequently, $m(t_2) \geq m(t_1) - \varepsilon$. The conclusion follows.

Proof of Theorem 1.4. Let w be the solution of (1.9) with an initial datum w_0 as in the theorem. It is then classical to check that

$$\lim_{t \rightarrow +\infty} w(t, z) = \langle w_0 \rangle \quad \text{uniformly in } \bar{\Omega}, \quad (2.16)$$

where $\langle w_0 \rangle$ denotes the average of the periodic function w_0 . Indeed, $w(t, \cdot)$ remains periodic for each $t > 0$ by uniqueness of the Cauchy problem. Furthermore, $\min_{\bar{\Omega}} w(t, \cdot) = \min_{\bar{\mathcal{C}}} w(t, \cdot)$ and $\max_{\bar{\Omega}} w(t, \cdot) = \max_{\bar{\mathcal{C}}} w(t, \cdot)$ are bounded, and respectively nondecreasing

and nonincreasing in $t > 0$. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of positive times converging to $+\infty$ and $(z_n)_{n \in \mathbb{N}}$ be a sequence of points in \overline{C} such that

$$\lim_{n \rightarrow +\infty} w(t_n, z_n) = \lim_{t \rightarrow +\infty} \left(\min_{\overline{\Omega}} w(t, \cdot) \right) =: m.$$

From standard parabolic estimates, up to extraction of subsequence, the x -periodic functions

$$w_n(t, z) = w(t + t_n, z)$$

converge locally in t and uniformly in $\overline{\Omega}$ as $n \rightarrow +\infty$ to a classical solution w_∞ of the same equation (1.9) in $\mathbb{R} \times \overline{\Omega}$. Furthermore, $w_\infty \geq m$ in $\mathbb{R} \times \overline{\Omega}$ and $\min_{\overline{\Omega}} w_\infty(0, \cdot) = m$. Thus, $w_\infty \equiv m$ in $\mathbb{R} \times \overline{\Omega}$ from the strong parabolic maximum principle. This implies that, given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|w(t_N, \cdot) - m| \leq \varepsilon$ in $\overline{\Omega}$, whence

$$|w(t, z) - m| \leq \varepsilon \quad \text{for all } (t, z) \in [t_N, +\infty) \times \overline{\Omega}$$

from the maximum principle. As a consequence, $w(t, z) \rightarrow m$ as $t \rightarrow +\infty$ uniformly in $\overline{\Omega}$. On the other hand, integrating the equation (1.9) in C at any time $t > 0$ implies that the function $t \mapsto h(t) = \int_C w(t, z) dz$ is constant in $t > 0$, because q is divergence-free in Ω and tangential on $\partial\Omega$. Since $w(t, z) \rightarrow w_0(z)$ as $t \rightarrow 0^+$ for all $z \in C$ and the function w is globally bounded, Lebesgue's dominated convergence theorem implies that $h(t) = \langle w_0 \rangle |C|$ for all $t > 0$, where $|C|$ denotes the Lebesgue measure of the periodicity cell C . Eventually, this yields (2.16).

Therefore, by the uniformity of the limit (2.16) and by Lemma 2.2, we deduce that

$$\alpha_{\min}(u_0) = \alpha_{\max}(u_0) = \langle w_0 \rangle.$$

But $c_{\alpha_{\min}(u_0)} \leq c_*(u_0) \leq c^*(u_0) \leq c_{\alpha_{\max}(u_0)}$ from Theorem 1.3. Hence,

$$c_*(u_0) = c^*(u_0) = c_{\langle w_0 \rangle},$$

and the proof of Theorem 1.4 is complete. \square

2.2 Lower and upper bounds for $c_*(u_0)$ and $c^*(u_0)$

The following section is devoted to the proof of Theorem 1.3. We start with a general result ensuring that any solution of (1.9) with a compactly supported initial datum converges uniformly to 0 as $t \rightarrow +\infty$.

Lemma 2.4 *Let w be the solution of the linear equation*

$$\begin{cases} w_t - \nabla \cdot (A(z)\nabla w) + q(z) \cdot \nabla w = 0, & t > 0, z \in \overline{\Omega}, \\ \nu A \nabla w = 0, & t > 0, z \in \partial\Omega, \\ w(0, z) = w_0(z), & z \in \Omega, \end{cases} \quad (2.17)$$

where $w_0 : \overline{\Omega} \rightarrow \mathbb{R}$ is continuous and compactly supported in $\overline{\Omega}$. Then, for all $t > 0$,

$$\int_{\Omega} w(t, z) dz = \int_{\Omega} w_0(z) dz.$$

Furthermore, $w(t, z) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly in $\overline{\Omega}$.

Proof. First of all, since w_0 has a compact support, denoted by $K = \text{supp}(w_0)$, the following pointwise estimate follows from Theorem 2.1:

$$\forall t > 0, \forall z \in \bar{\Omega}, \quad |w(t, z)| \leq C_0 \|w_0\|_\infty t^{-\frac{N}{2}} e^{\omega_1 t} \int_K e^{-\frac{|z-z'|^2}{\omega_2 t}} dz'. \quad (2.18)$$

By pointwise gradient bounds (see [28]), we get that, for every $t > 0$ and every $z \in \bar{\Omega}$,

$$|\nabla_z w(t, z)| \leq C(t) \max_{t' \in [t/2, t], z' \in \bar{\Omega}, |z'-z| \leq 1} |w(t', z')|,$$

where $C(t)$ depends on t but not on z . As a consequence, for any $0 < a \leq b < +\infty$, there are positive constants $C'_{a,b}$ and $\omega_{a,b}$ which depend on a and b , such that

$$\forall t \in [a, b], \forall z \in \bar{\Omega}, \quad |w(t, z)| + |\nabla_z w(t, z)| \leq C'_{a,b} e^{-\omega_{a,b}|z|^2}.$$

Notice in particular that the integrals of w and $|\nabla_z w|$ over Ω converge at any time $t > 0$.

Fix now any two times $0 < t < t'$. Integrate the equation

$$w_t = \nabla \cdot (A(z)\nabla w) - q \cdot \nabla w$$

over $[t, t'] \times (\Omega \cap B_R)$, where B_R denotes the euclidean ball of \mathbb{R}^N centered at the origin with radius R , and pass to the limit as $R \rightarrow +\infty$. It follows from the previous estimates that

$$\int_\Omega w(t, z) dz = \int_\Omega w(t', z) dz,$$

using once again the assumptions that q is divergence-free in Ω and tangential on the boundary $\partial\Omega$.

Lastly, we know that $w(t, z) \rightarrow w_0(z)$ as $t \rightarrow 0^+$ for all $z \in \Omega$. Moreover, $|w| \leq \|w_0\|_\infty$ in $[0, +\infty) \times \Omega$. On the other hand, there is $\eta > 0$ such that

$$|z - z'| \geq \eta |z| \quad \text{for all } z' \in K \text{ and } z \in \bar{\Omega} \setminus (K + B_1).$$

Therefore, it follows from (2.18) that, for all $0 < t \leq \min(1, 1/\omega_2)$ and $z \in \bar{\Omega} \setminus (K + B_1)$,

$$|w(t, z)| \leq C_0 \|w_0\|_\infty \omega_2^{\frac{N}{2}} e^{\omega_1 t} \int_{\frac{K-z}{\sqrt{\omega_2 t}}} e^{-|z''|^2} dz'' \leq C_0 \|w_0\|_\infty \omega_2^{\frac{N}{2}} e^{\omega_1 t} \int_{\mathbb{R}^N \setminus B_{\eta|z|}} e^{-|z''|^2} dz''.$$

Since the right-hand side does not depend on t and is integrable (with respect to z) over $\bar{\Omega} \setminus (K + B_1)$, Lebesgue's dominated convergence theorem finally yields that

$$\int_\Omega w(t, z) dz \rightarrow \int_\Omega w_0(z) dz \quad \text{as } t \rightarrow 0^+.$$

Hence, for every $t > 0$, the integral of $w(t, \cdot)$ over Ω is the same as that of w_0 .

Let us now prove that w converges to 0 as $t \rightarrow +\infty$ uniformly in $\bar{\Omega}$. Consider first the case when w_0 is nonnegative, whence $w(t, \cdot) \geq 0$ in $\bar{\Omega}$ for all $t > 0$. The quantity

$$\ell(t) = \sup_\Omega w(t, \cdot)$$

belongs to $[0, \|w_0\|_\infty]$ and it is nonincreasing with respect to $t \geq 0$, from the maximum principle. As a consequence, it has a limit in $[0, \|w_0\|_\infty]$ when $t \rightarrow +\infty$, denoted ℓ_∞ . Assume that $\ell_\infty > 0$. Then there exist a sequence $(t_n)_{n \in \mathbb{N}}$ and a sequence of points $(z_n)_{n \in \mathbb{N}}$ in $\bar{\Omega}$ such that $t_n \rightarrow +\infty$ and $w(t_n, z_n) \rightarrow \ell_\infty$ as $n \rightarrow +\infty$. Denote $z_n = (x_n, y_n) = (k_n L + x'_n, y_n)$, where $k_n \in \mathbb{Z}$ and $(x'_n, y_n) \in \bar{C}$. Up to extraction of a subsequence, the points (x'_n, y_n) converge to $z_\infty \in \bar{C}$ and the functions

$$w_n(t, z) = w_n(t, x, y) = w(t + t_n, x + k_n L, y)$$

converge locally uniformly in $\mathbb{R} \times \bar{\Omega}$ to a classical bounded solution w_∞ of the same equation as w , such that $w_\infty \leq \ell_\infty$ in $\mathbb{R} \times \bar{\Omega}$ and $w_\infty(0, z_\infty) = \ell_\infty$. Therefore, $w_\infty \equiv \ell_\infty$ in $\mathbb{R} \times \bar{\Omega}$ from the strong parabolic maximum principle. In other words, the functions $w_n(t, z) = w_n(t, x, y) = w(t + t_n, x + k_n L, y)$ converge locally uniformly in $\mathbb{R} \times \bar{\Omega}$ to the positive constant ℓ_∞ , which implies that the integrals of the nonnegative functions $w(t_n, \cdot)$ over Ω cannot stay bounded. This leads to a contradiction. Thus $w(t, \cdot) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly in $\bar{\Omega}$.

In the general case when w_0 has no sign, one can write $w_0 = w_0^+ - w_0^-$, where $w_0^+(x) = \max(w_0(x), 0)$ and $w_0^-(x) = \max(-w_0(x), 0)$ for all $x \in \bar{\Omega}$. By uniqueness and linearity of the Cauchy problem (2.17), it follows that $w(t, z) = w_1(t, z) - w_2(t, z)$ for all $t > 0$ and $z \in \bar{\Omega}$, where w_1 and w_2 solve (2.17) with initial conditions w_0^+ and w_0^- respectively. But the previous paragraph implies that w_1 and w_2 converge to 0 as $t \rightarrow +\infty$ uniformly in $\bar{\Omega}$, whence $\lim_{t \rightarrow +\infty} \|w(t, \cdot)\|_\infty = 0$ and the proof of Lemma 2.4 is now complete. \square

The next lemma provides the proof of the first statement of Theorem 1.3.

Lemma 2.5 *Let u be a solution of (1.4) satisfying the assumptions of Theorem 1.3. Then u satisfies*

$$\liminf_{t \rightarrow +\infty} \left(\inf_{z \in \bar{\Omega}} u(t, z) \right) \geq \alpha_{\min}(u_0).$$

Proof. Fix any arbitrary $\varepsilon > 0$. By definition of $\alpha_{\min}(u_0)$, there exist $T > 0$ and $A_1 > 0$ such that for every $z = (x, y) \in \bar{\Omega}$ with $x \geq A_1$, there holds

$$v(T, z) \geq \alpha_{\min}(u_0) - \varepsilon.$$

Since $\liminf_{x \rightarrow -\infty} u_0(z) > \theta > \alpha_{\min}(u_0)$ and since the map $t \mapsto \liminf_{x \rightarrow -\infty} v(t, z)$ is nondecreasing (with the same kind of arguments as in Lemma 2.2 and Remark 2.3), it follows that there exists $A_2 < 0$ such that for every $z = (x, y) \in \bar{\Omega}$ with $x \leq A_2$, there holds

$$v(T, z) \geq \theta > \alpha_{\min}(u_0) - \varepsilon.$$

Denote K the compact set

$$K = \{z = (x, y) \in \bar{\Omega}, A_1 \leq x \leq A_2\}.$$

Consequently,

$$\forall z \in \bar{\Omega} \setminus K, \quad v(T, z) \geq \alpha_{\min}(u_0) - \varepsilon.$$

In particular, since v is also globally bounded, there exists a continuous and compactly supported function $w_0 : \overline{\Omega} \rightarrow \mathbb{R}$ such that

$$\forall z \in \overline{\Omega}, \quad v(T, z) \geq \alpha_{\min}(u_0) - \varepsilon - w_0(z).$$

By linearity and uniqueness of the Cauchy problem for (2.17), it follows that

$$\forall t > 0, \forall z \in \overline{\Omega}, \quad v(T + t, z) \geq \alpha_{\min}(u_0) - \varepsilon - w(t, z),$$

where w is the solution of (2.17) with initial condition w_0 . On the other hand, using Lemma 2.4, we know that $\|w(t, \cdot)\|_{\infty} \rightarrow 0$ as $t \rightarrow +\infty$. Hence,

$$\inf_{\overline{\Omega}} v(t, \cdot) \geq \alpha_{\min}(u_0) - 2\varepsilon$$

for t large enough. This gives directly the desired result, letting ε going to zero and using the fact that $u \geq v$ for every $t > 0$ and $z \in \overline{\Omega}$ by the maximum principle (because f is nonnegative). \square

We now come to the proof of the inequalities (1.10) of Theorem 1.3. We first prove the following lemma, which gives directly the first inequality, namely $c_{\alpha_{\min}(u_0)} \leq c_*(u_0)$.

Lemma 2.6 *Let u be a solution of (1.4) satisfying the assumptions of Theorem 1.3. Then*

$$\forall c < c_{\alpha_{\min}(u_0)}, \forall A \in \mathbb{R}, \quad \lim_{t \rightarrow +\infty} \left(\inf_{x \leq A, (x+ct, y) \in \overline{\Omega}} u(t, x + ct, y) \right) = 1.$$

Proof. First, observe that there exist $\beta > 0$, $A < 0$ and a uniformly continuous function $U_0 : \Omega \rightarrow [0, 1]$ such that

$$U_0 \leq u_0 \text{ in } \Omega \text{ and } U_0(x, y) = \theta + \beta \text{ if } x \leq A.$$

Let $\underline{f} : \mathbb{R} \rightarrow \mathbb{R}_+$ be the function defined by $\underline{f}(s) = \min_{z \in \overline{\Omega}} f(z, s)$ for all $s \in \mathbb{R}$. Let U be the solution of the Cauchy problem (1.4) with the function f being replaced by \underline{f} , and with initial condition U_0 . The maximum principle yields $0 \leq U(t, z) \leq u(t, z) \leq 1$ for all $t > 0$ and $z \in \overline{\Omega}$. Set $\xi(t) = \lim_{x \rightarrow -\infty} U(t, z)$ for all $t \geq 0$. The function $\xi : \mathbb{R}_+ \rightarrow [0, 1]$ satisfies $\xi(0) = \theta + \beta \in (\theta, 1]$ and $\xi'(t) = \underline{f}(\xi(t))$ for all $t \geq 0$. From the assumption on f made in (1.3), one concludes that $\xi(t) \rightarrow 1$ as $t \rightarrow +\infty$. Therefore,

$$\liminf_{x \rightarrow -\infty} u(t, z) \rightarrow 1 \text{ as } t \rightarrow +\infty. \quad (2.19)$$

Then, let us consider a family of continuous functions $(\underline{f}_{-\eta})_{0 \leq \eta < 1 - \theta} : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$, with $\underline{f}_0 = f$, such that each function $\underline{f}_{-\eta}$ is periodic with respect to z , is of class $C^{0, \alpha}$ with respect to z locally uniformly in u , has a restriction to $\overline{\Omega} \times [0, 1 - \eta]$ which is of class C^1 with respect to u , and satisfies

$$\forall z \in \overline{\Omega}, \left\{ \begin{array}{l} \underline{f}_{-\eta}(z, \cdot) \equiv 0 \text{ on } (-\infty, \theta] \cup [1 - \eta, +\infty), \\ \underline{f}_{-\eta}(z, \cdot) > 0 \text{ on } (\theta, 1 - \eta), \quad \frac{\partial \underline{f}_{-\eta}}{\partial u}(z, (1 - \eta)^-) = - \lim_{s \rightarrow 0^+} \frac{\underline{f}_{-\eta}(z, 1 - \eta - s)}{s} < 0. \end{array} \right.$$

Furthermore, the functions (\underline{f}_η) are chosen in such a way that $\eta \mapsto \underline{f}_\eta(z, u)$ is nonincreasing in $[0, 1 - \theta]$ for each $(z, u) \in \bar{\Omega} \times \mathbb{R}$, and $\underline{f}_\eta \rightarrow f$ as $\eta \rightarrow 0$ uniformly in $\bar{\Omega} \times \mathbb{R}$. For each $\gamma \in (-\infty, \theta)$ and $\eta \in [0, 1 - \theta]$, it is known [3] that there exists a unique speed $\underline{c}_{\gamma, \eta} > 0$ and a unique (up to time shifts) front

$$\mathbb{R} \times \bar{\Omega} \ni (t, z) \mapsto \underline{u}_{\gamma, \eta}(t, z) = \underline{\varphi}_{\gamma, \eta}(x - \underline{c}_{\gamma, \eta}t, z) \in (\gamma, 1 - \eta)$$

solving (1.1) with \underline{f}_η instead of f , and satisfying (1.7) and (1.8) with $\underline{c}_{\gamma, \eta}$ and $1 - \eta$ instead of c and 1, respectively. From Theorem 1.2, the speeds $c_\gamma = \underline{c}_{\gamma, 0}$ are continuous with respect to $\gamma < \theta$. It also follows from [3, 4] that $\underline{c}_{\gamma, \eta} \rightarrow \underline{c}_{\gamma, 0} = c_\gamma$ as $\eta \rightarrow 0$, for each $\gamma \in (-\infty, \theta)$.

Fix now any real number c such that $c < c_{\alpha_{\min}(u_0)}$, any real number A and any positive real number $\varepsilon > 0$. From the previous paragraph, one can choose $\kappa > 0$ small enough and then $\eta \in (0, \varepsilon)$ small enough so that

$$c < \underline{c}_{\alpha_{\min}(u_0) - \kappa, \eta} =: c'.$$

In order to conclude, we will put below the solution u of (1.4) a pulsating front subsolution which will travel at speed $\underline{c}_{\alpha_{\min}(u_0) - \kappa, \eta}$ and will be larger than $1 - \varepsilon$ on the left. Indeed, from Lemma 2.5 and (2.19), there exists a time $T > 0$ such that

$$\liminf_{x \rightarrow -\infty} u(T, z) > 1 - \eta \quad \text{and} \quad \inf_{z \in \bar{\Omega}} u(T, z) > \alpha_{\min}(u_0) - \kappa.$$

Since $\underline{\varphi}_{\alpha_{\min}(u_0) - \kappa, \eta}(-\infty, \cdot) = 1 - \eta$ and $\underline{\varphi}_{\alpha_{\min}(u_0) - \kappa, \eta}(+\infty, \cdot) = \alpha_{\min}(u_0) - \kappa$, there exists then a time-shift $T_0 \in \mathbb{R}$ such that

$$u(T, z) \geq \underline{u}_{\alpha_{\min}(u_0) - \kappa, \eta}(T + T_0, z) = \underline{\varphi}_{\alpha_{\min}(u_0) - \kappa, \eta}(x - c'(T + T_0), z) \quad \text{for all } z \in \bar{\Omega}.$$

Since $\underline{f}_\eta \leq f$, the function $\underline{u}_{\alpha_{\min}(u_0) - \kappa, \eta}$ is a subsolution of the equation (1.1), whence

$$u(t, z) \geq \underline{u}_{\alpha_{\min}(u_0) - \kappa, \eta}(t + T_0, z) = \underline{\varphi}_{\alpha_{\min}(u_0) - \kappa, \eta}(x - c'(t + T_0), z) \quad \text{for all } t \geq T \text{ and } z \in \bar{\Omega}$$

from the maximum principle. In particular, for all $t \geq T$,

$$\inf_{x \leq A, (x+ct, y) \in \bar{\Omega}} u(t, x + ct, y) \geq \inf_{x \leq A, (x+ct, y) \in \bar{\Omega}} \underline{\varphi}_{\alpha_{\min}(u_0) - \kappa, \eta}(x + (c - c')t - c'T_0, x + ct, y).$$

But since $c < c'$ and $\underline{\varphi}_{\alpha_{\min}(u_0) - \kappa, \eta}(-\infty, \cdot) = 1 - \eta > 1 - \varepsilon$ uniformly in $\bar{\Omega}$, one concludes that

$$\inf_{x \leq A, (x+ct, y) \in \bar{\Omega}} u(t, x + ct, y) \geq 1 - \varepsilon$$

for t large enough. That completes the proof of Lemma 2.6. \square

The next lemma is a key step which will lead to the end of the proof of Theorem 1.3.

Lemma 2.7 *Let u be a solution of (1.4) and v a solution of (1.9), with the same initial condition u_0 satisfying the assumptions of Theorem 1.3. Then, for all $t \geq 0$,*

$$\lim_{x \rightarrow +\infty} \left(u(t, z) - v(t, z) \right) = 0.$$

We postpone the proof of this technical lemma to the end of this subsection and we finish the proof of Theorem 1.3.

End of the proof of Theorem 1.3. We shall prove

$$\limsup_{t \rightarrow +\infty} \left(\sup_{x \geq A, (x+ct, y) \in \bar{\Omega}} u(t, x+ct, y) \right) \leq \alpha_{\max}(u_0) \quad (2.20)$$

for any $A \in \mathbb{R}$ and for any speed c such that $c > c_{\alpha_{\max}(u_0)}$. This will give the last assertion of Theorem 1.3 and will also imply that $c^*(u_0) \leq c_{\alpha_{\max}(u_0)}$.

Observe first that

$$\lim_{t \rightarrow +\infty} \left(\limsup_{x \rightarrow +\infty} u(t, z) \right) \leq \alpha_{\max}(u_0). \quad (2.21)$$

Indeed, the same property holds for v by definition of $\alpha_{\max}(u_0)$, where v is the solution of (1.9) with the same initial condition u_0 as u . Therefore, (2.21) follows from Lemma 2.7.

We shall then construct a pulsating travelling front which will be a supersolution for u and which will travel to the right at a speed larger than but close to $c_{\alpha_{\max}(u_0)}$. The proof proceeds in a similar way as in Lemma 2.6. Consider a family of continuous functions $(\bar{f}_\eta)_{\eta \geq 0} : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$, with $\bar{f}_0 = f$, such that each function \bar{f}_η is periodic with respect to z , is of class $C^{0,\alpha}$ with respect to z locally uniformly in u , has a restriction to $\bar{\Omega} \times [0, 1 + \eta]$ which is of class C^1 with respect to u , and satisfies

$$\begin{cases} \bar{f}_\eta(z, \cdot) \equiv 0 & \text{on } (-\infty, \theta] \cup [1 + \eta, +\infty), \\ \bar{f}_\eta(z, \cdot) > 0 & \text{on } (\theta, 1 + \eta), \quad \frac{\partial \bar{f}_\eta}{\partial u}(z, (1 + \eta)^-) = - \lim_{s \rightarrow 0^+} \frac{\bar{f}_\eta(z, 1 + \eta - s)}{s} < 0. \end{cases}$$

Furthermore, the functions (\bar{f}_η) are chosen in such a way that $\eta \mapsto \bar{f}_\eta(z, u)$ is nondecreasing in $[0, +\infty)$ for each $(z, u) \in \bar{\Omega} \times \mathbb{R}$, and $\bar{f}_\eta \rightarrow f$ as $\eta \rightarrow 0$ uniformly in $\bar{\Omega} \times [0, 1]$. For each $\gamma \in (-\infty, \theta)$ and $\eta \in [0, +\infty)$, there exists a unique speed $\bar{c}_{\gamma, \eta} > 0$ and a unique (up to time shifts) front

$$\mathbb{R} \times \bar{\Omega} \ni (t, z) \mapsto \bar{u}_{\gamma, \eta}(t, z) = \bar{\varphi}_{\gamma, \eta}(x - \bar{c}_{\gamma, \eta} t, z) \in (\gamma, 1 + \eta)$$

solving (1.1) with \bar{f}_η instead of f , and satisfying (1.7) and (1.8) with $\bar{c}_{\gamma, \eta}$ and $1 + \eta$ instead of c and 1 , respectively. Furthermore, $\gamma \mapsto \bar{c}_{\gamma, 0} = c_\gamma$ is continuous and $\bar{c}_{\gamma, \eta} \rightarrow \bar{c}_{\gamma, 0} = c_\gamma$ as $\eta \rightarrow 0$, for each $\gamma \in (-\infty, \theta)$. Fix now any real number c such that $c > c_{\alpha_{\max}(u_0)}$, any real number A and any positive real number $\varepsilon > 0$. One can then choose $\kappa \in (0, \varepsilon)$ small enough and then $\eta > 0$ small enough so that

$$c > \bar{c}_{\alpha_{\max}(u_0) + \kappa, \eta} =: c'.$$

From (2.21), there exists a time $T > 0$ such that

$$\limsup_{x \rightarrow +\infty} u(T, z) < \alpha_{\max}(u_0) + \kappa.$$

Since u is also such that $u(T, \cdot) \leq 1$ in $\bar{\Omega}$, there exists then a time-shift $T_0 \in \mathbb{R}$ such that

$$u(T, z) \leq \bar{u}_{\alpha_{\max}(u_0) + \kappa, \eta}(T + T_0, z) = \bar{\varphi}_{\alpha_{\max}(u_0) + \kappa, \eta}(x - c'(T + T_0), z) \quad \text{for all } z \in \bar{\Omega}.$$

Since $\bar{f}_\eta \geq f$, the function $\bar{u}_{\alpha_{\max}(u_0) + \kappa, \eta}$ is a supersolution of the equation (1.1), whence

$$u(t, z) \leq \bar{u}_{\alpha_{\max}(u_0) + \kappa, \eta}(t + T_0, z) = \bar{\varphi}_{\alpha_{\max}(u_0) + \kappa, \eta}(x - c'(t + T_0), z) \quad \text{for all } t \geq T \text{ and } z \in \bar{\Omega}$$

from the maximum principle. In particular, for all $t \geq T$,

$$\sup_{x \geq A, (x+ct, y) \in \bar{\Omega}} u(t, x + ct, y) \leq \sup_{x \geq A, (x+ct, y) \in \bar{\Omega}} \bar{\varphi}_{\alpha_{\max}(u_0) + \kappa, \eta}(x + (c - c')t - c'T_0, x + ct, y).$$

But since $c > c'$ and $\bar{\varphi}_{\alpha_{\max}(u_0) + \kappa, \eta}(+\infty, \cdot) = \alpha_{\max}(u_0) + \kappa < \alpha_{\max}(u_0) + \varepsilon$ uniformly in $\bar{\Omega}$, one concludes that

$$\sup_{x \geq A, (x+ct, y) \in \bar{\Omega}} u(t, x + ct, y) \leq \alpha_{\max}(u_0) + \varepsilon$$

for t large enough. That completes the proof of (2.20) and Theorem 1.3. \square

Proof of Lemma 2.7. From the maximum principle, we have

$$0 \leq u(t, z) - v(t, z) \quad \text{for all } t > 0 \text{ and } z \in \bar{\Omega}$$

because $f \geq 0$. We denote $w(t, z) = u(t, z) - v(t, z)$. Notice that $w(0, z) = 0$ and the conclusion of Lemma 2.7 holds immediately at time $t = 0$. We are now going to construct a suitable supersolution for w . Let $\gamma < \theta$ be such that

$$\limsup_{x \rightarrow +\infty} u_0(z) < \gamma.$$

Pick any positive real number $\eta > 0$ and consider the front $\bar{u}_{\gamma, \eta}$ connecting $1 + \eta$ to γ , for the nonlinearity \bar{f}_η , under the above notations in the proof of Theorem 1.3. Denote $c = \bar{c}_{\gamma, \eta} > 0$ its speed. Since u_0 is also not larger than 1, there exists $T_0 \in \mathbb{R}$ such that $u_0 \leq \bar{u}_{\gamma, \eta}(T_0, \cdot)$ in Ω . Therefore,

$$u(t, z) \leq \bar{u}_{\gamma, \eta}(t + T_0, z) = \bar{\varphi}_{\gamma, \eta}(x - c(t + T_0), z)$$

for all $t > 0$ and $z \in \bar{\Omega}$ from the maximum principle. Since $\bar{\varphi}_{\gamma, \eta}(+\infty, \cdot) = \gamma < \theta$, it follows that there exists a constant $D > 0$ such that for all $t > 0$ and all $z = (x, y) \in \bar{\Omega}$ such that $x \geq ct + D$, there holds $u(t, z) \leq \theta$.

We now use Duhamel's formula to express the solution w of the problem

$$w_t - \nabla \cdot (A(z)\nabla w) + q(z) \cdot \nabla w = f(u).$$

Denoting $\mathcal{S}(t) = e^{-t\mathcal{L}}$ the strongly continuous semi-group generated by the operator $\mathcal{L} = -\nabla \cdot (A(z)\nabla) + q \cdot \nabla$ with Neumann boundary conditions $\nu A(z)\nabla = 0$ on $\partial\Omega$, we have

$$w(t, z) = \int_0^t \mathcal{S}(t-s)[f(u(s, \cdot))](z) ds.$$

for all $t > 0$ and $z \in \bar{\Omega}$. Therefore, with the notations of Theorem 2.1, we get

$$\forall t > 0, \forall z \in \bar{\Omega}, \quad w(t, z) = \int_0^t \int_{\Omega} p(t-s, z, z') f(u(s, z')) dz' ds.$$

Fix now any $t > 0$ and $\varepsilon > 0$. Choose any δ such that $0 < \delta < \min(t, \varepsilon/\|f\|_{\infty})$ and write

$$w(t, z) = \underbrace{\int_0^{t-\delta} \int_{\Omega} p(t-s, z, z') f(u(s, z')) dz' ds}_{=: I(t, z)} + \underbrace{\int_{t-\delta}^t \int_{\Omega} p(t-s, z, z') f(u(s, z')) dz' ds}_{=: II(t, z)}$$

for all $z \in \bar{\Omega}$. Notice that $0 \leq II(t, z) \leq \|f\|_{\infty} \delta \leq \varepsilon$. Let us now estimate the integral $I(t, z)$. By the Gaussian estimates in Theorem 2.1, we get that

$$0 \leq I(t, z) \leq C_0 \int_0^{t-\delta} \int_{\Omega} (t-s)^{-\frac{N}{2}} e^{\omega_1(t-s) - \frac{|z-z'|^2}{\omega_2(t-s)}} f(u(s, z')) dz' ds$$

for all $z \in \bar{\Omega}$. Remember that, for $z' = (x', y') \in \bar{\Omega}$, there holds $u(s, z') \leq \theta$, whence $f(u(s, z')) = 0$, as soon as $x' \geq cs + D$. Consequently, for all $z \in \bar{\Omega}$,

$$\begin{aligned} 0 \leq I(t, z) &\leq C_0 \|f\|_{\infty} \int_0^{t-\delta} \int_{\{z'=(x', y'), x' < cs+D\}} (t-s)^{-\frac{N}{2}} e^{\omega_1(t-s) - \frac{|z-z'|^2}{\omega_2(t-s)}} dz' ds \\ &\leq C_0 \|f\|_{\infty} \delta^{-\frac{N}{2}} e^{\omega_1 t} C_{N,R} \int_0^{t-\delta} \int_{-\infty}^{cs+D} e^{-\frac{|x-x'|^2}{\omega_2 t}} dx' ds, \end{aligned}$$

where $C_{N,R} > 0$ is given as in (2.15). But the right-hand side of the last inequality does not depend on y and goes to 0 as $x \rightarrow +\infty$. It follows that $\lim_{x \rightarrow +\infty} I(t, z) = 0$, whence $0 \leq w(t, z) \leq 2\varepsilon$ for all $z = (x, y) \in \bar{\Omega}$ such that $x \geq B$, for some large enough B . Since ε is arbitrary small, this gives the desired result. \square

3 Examples for which $c_*(u_0) < c^*(u_0)$

This section is devoted to the proof of Theorem 1.6. Let $f = f(u)$ be a nonlinearity satisfying (1.3). Let α and β be given throughout the section, such that

$$0 \leq \alpha < \beta < \theta.$$

3.1 Proof of Theorem 1.6

In this subsection, we first define some useful notations and we derive rough estimates. Then, we state a key-lemma which enables us to complete the proof of the theorem.

Approximating fronts

As in the proofs of Lemma 2.6 and Theorem 1.3, we consider two families $(\underline{f}_\eta)_{\eta \in [0, 1-\theta]}$ and $(\bar{f}_\eta)_{\eta \in [0, +\infty)}$ of $C^1([0, 1-\eta])$ and $C^1([0, 1+\eta])$ functions such that

$$\begin{cases} \forall \eta \in [0, 1-\theta), & \underline{f}_\eta = 0 \text{ on } [0, \theta] \cup \{1-\eta\}, \underline{f}_\eta > 0 \text{ on } (\theta, 1-\eta), \underline{f}'_\eta(1-\eta) < 0, \\ \forall \eta \in [0, +\infty), & \bar{f}_\eta = 0 \text{ on } [0, \theta] \cup \{1+\eta\}, \bar{f}_\eta > 0 \text{ on } (\theta, 1+\eta), \bar{f}'_\eta(1+\eta) < 0, \end{cases}$$

and \underline{f}_η (resp. \bar{f}_η) is extended by 0 outside the interval $[0, 1-\eta]$ (resp. $[0, 1+\eta]$). Furthermore, these functions are chosen in such a way that $\underline{f}_0 = \bar{f}_0 = f$, that

$$\begin{cases} \underline{f}_{\eta_1} \geq \underline{f}_{\eta_2} \text{ in } \mathbb{R} & \text{if } 0 \leq \eta_1 \leq \eta_2 < 1-\theta, \\ \bar{f}_{\eta_1} \leq \bar{f}_{\eta_2} \text{ in } \mathbb{R} & \text{if } 0 \leq \eta_1 \leq \eta_2, \end{cases}$$

and that $\lim_{\eta \rightarrow 0} \|\underline{f}_\eta - f\|_{L^\infty(\mathbb{R})} = \lim_{\eta \rightarrow 0} \|\bar{f}_\eta - f\|_{L^\infty(\mathbb{R})} = 0$. For each $\gamma < \theta$ and $\eta \in [0, 1-\theta)$, we denote $(\underline{c}_{\gamma, \eta}, \underline{\varphi}_{\gamma, \eta})$ the unique solution of

$$\begin{cases} \underline{\varphi}''_{\gamma, \eta} + \underline{c}_{\gamma, \eta} \underline{\varphi}'_{\gamma, \eta} + \underline{f}_\eta(\underline{\varphi}_{\gamma, \eta}) = 0 & \text{in } \mathbb{R}, \\ \underline{\varphi}_{\gamma, \eta}(-\infty) = 1-\eta > \underline{\varphi}_{\gamma, \eta}(x) > \underline{\varphi}_{\gamma, \eta}(+\infty) = \gamma & \text{for all } x \in \mathbb{R}, \quad \underline{\varphi}_{\gamma, \eta}(0) = \theta. \end{cases} \quad (3.1)$$

Similarly, for each $\gamma < \theta$ and $\eta \in [0, +\infty)$, we denote $(\bar{c}_{\gamma, \eta}, \bar{\varphi}_{\gamma, \eta}) \in \mathbb{R} \times C^2(\mathbb{R})$ the unique solution of

$$\begin{cases} \bar{\varphi}''_{\gamma, \eta} + \bar{c}_{\gamma, \eta} \bar{\varphi}'_{\gamma, \eta} + \bar{f}_\eta(\bar{\varphi}_{\gamma, \eta}) = 0 & \text{in } \mathbb{R}, \\ \bar{\varphi}_{\gamma, \eta}(-\infty) = 1+\eta > \bar{\varphi}_{\gamma, \eta}(x) > \bar{\varphi}_{\gamma, \eta}(+\infty) = \gamma & \text{for all } x \in \mathbb{R}, \quad \bar{\varphi}_{\gamma, \eta}(0) = \theta. \end{cases} \quad (3.2)$$

Notice that with the normalization of $\underline{\varphi}_{\gamma, \eta}$ and $\bar{\varphi}_{\gamma, \eta}$ at 0, these functions are then really unique. With these notations, for each $\gamma < \theta$, there holds $\underline{c}_{\gamma, 0} = \bar{c}_{\gamma, 0} = c_\gamma$ and the functions $\underline{\varphi}_{\gamma, 0}$ and $\bar{\varphi}_{\gamma, 0}$ are equal to φ_γ up to shifts, where $(c_\gamma, \varphi_\gamma)$ solves (1.12). Furthermore, we recall (see [11]) that all functions $\underline{\varphi}_{\gamma, \eta}$ and $\bar{\varphi}_{\gamma, \eta}$ are decreasing in \mathbb{R} , that the speeds $\underline{c}_{\gamma, \eta}$ and $\bar{c}_{\gamma, \eta}$ are positive, that

$$\begin{cases} (\gamma, \eta) \mapsto \underline{c}_{\gamma, \eta} \text{ is increasing w.r.t. } \gamma \text{ and decreasing w.r.t. } \eta & \text{in } (-\infty, \theta) \times [0, 1-\theta), \\ (\gamma, \eta) \mapsto \bar{c}_{\gamma, \eta} \text{ is increasing w.r.t. } \gamma \text{ and increasing w.r.t. } \eta & \text{in } (-\infty, \theta) \times [0, +\infty) \end{cases} \quad (3.3)$$

and that

$$\forall \gamma < \theta, \quad \lim_{(\gamma', \eta) \rightarrow (\gamma, 0)} \underline{c}_{\gamma', \eta} = \lim_{(\gamma, \eta) \rightarrow (\gamma, 0)} \bar{c}_{\gamma, \eta} = c_\gamma > 0. \quad (3.4)$$

Moreover, for each $\gamma < \theta$ and $\eta \in [0, 1 - \theta)$, let $\underline{u}_{\gamma, \eta}$ be the solution of the Cauchy problem

$$\begin{cases} (\underline{u}_{\gamma, \eta})_t - (\underline{u}_{\gamma, \eta})_{xx} = f(\underline{u}_{\gamma, \eta}), & x \in \mathbb{R}, \\ \underline{u}_{\gamma, \eta}(0, x) = \begin{cases} 1 - \eta, & \text{if } x \in (-\infty, 0), \\ 1 - \eta - \frac{(1 - \eta - \gamma)x}{2} & \text{if } x \in [0, 2], \\ \gamma & \text{if } x \in (2, +\infty). \end{cases} \end{cases} \quad (3.5)$$

For each $\gamma < \theta$, let \bar{u}_γ be the solution of the Cauchy problem

$$\begin{cases} (\bar{u}_\gamma)_t - (\bar{u}_\gamma)_{xx} = f(\bar{u}_\gamma), & x \in \mathbb{R}, \\ \bar{u}_\gamma(0, x) = \begin{cases} 1, & \text{if } x \in (-\infty, 0), \\ 1 - \frac{(1 - \gamma)x}{2} & \text{if } x \in [0, 2], \\ \gamma & \text{if } x \in (2, +\infty). \end{cases} \end{cases} \quad (3.6)$$

It is known from [25, 39] that, for each $\eta \in [0, 1 - \theta)$ and each $\gamma < \theta$, there exist two real numbers $\underline{x}_{\gamma, \eta}$ and \bar{x}_γ such that

$$\lim_{t \rightarrow +\infty} \left(\sup_{x \in \mathbb{R}} |u_{\gamma, \eta}(t, x) - \varphi_\gamma(x - c_\gamma t + \underline{x}_{\gamma, \eta})| \right) = 0 \quad (3.7)$$

and

$$\lim_{t \rightarrow +\infty} \left(\sup_{x \in \mathbb{R}} |\bar{u}_\gamma(t, x) - \varphi_\gamma(x - c_\gamma t + \bar{x}_\gamma)| \right) = 0.$$

Definition of a class of initial conditions u_0

Choose any sequence $(x_n)_{n \in \mathbb{N}}$ of positive real numbers such that $x_0 = 1$, $x_{n+1} - x_n \geq 3$ for all $n \in \mathbb{N}$ and

$$\frac{x_{n+1}}{x_n} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty. \quad (3.8)$$

A typical example is when $x_n = n!$ for large n . Let $u_0 : \mathbb{R} \rightarrow [\alpha, 1]$ be the uniformly continuous function defined by

$$u_0(x) = 1 \quad \text{if } x \in (-\infty, 0), \quad u_0(x) = 1 - \frac{(1 - \alpha)x}{2} \quad \text{if } x \in [0, 2] = [0, x_0 + 1] \quad (3.9)$$

and, for all $n \in \mathbb{N}$,

$$u_0(x) = \begin{cases} \alpha & \text{if } x \in (x_{2n} + 1, x_{2n+1} - 1), \\ \alpha + \frac{\beta - \alpha}{2} (x - x_{2n+1} + 1) & \text{if } x \in [x_{2n+1} - 1, x_{2n+1} + 1], \\ \beta & \text{if } x \in (x_{2n+1} + 1, x_{2n+2} - 1), \\ \beta - \frac{\beta - \alpha}{2} (x - x_{2n+2} + 1) & \text{if } x \in [x_{2n+2} - 1, x_{2n+2} + 1]. \end{cases} \quad (3.10)$$

Let u be the solution of the Cauchy problem (1.11) with this initial condition u_0 . Our aim is to prove that the solution u satisfies the conclusion of Theorem 1.6. *In the sequel, u_0 is fixed as above and, for the sake of simplicity, we drop the dependence on u_0 in the quantities $\alpha_{\min}(u_0)$, $\alpha_{\max}(u_0)$, $c_*(u_0)$ and $c^*(u_0)$.*

Values of α_{\min} and α_{\max}

According to the general notations of this paper, set

$$\alpha_{\min} = \liminf_{t \rightarrow +\infty} \left(\liminf_{x \rightarrow +\infty} v(t, x) \right) \quad \text{and} \quad \alpha_{\max} = \limsup_{t \rightarrow +\infty} \left(\limsup_{x \rightarrow +\infty} v(t, x) \right),$$

where

$$v(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy$$

is the solution of the heat equation

$$\begin{cases} v_t = v_{xx}, & x \in \mathbb{R}, \\ v(0, \cdot) = u_0 \end{cases} \quad (3.11)$$

with initial condition u_0 . Notice that $\alpha < v(t, x) < 1$ for all $t > 0$ and $x \in \mathbb{R}$, from the strong parabolic maximum principle. Observe also that, for each $t > 0$,

$$v(t, x) - \beta = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{|x-y|^2}{4t}} (u_0(y) - \beta) dy \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^2 e^{-\frac{|x-y|^2}{4t}} dy,$$

whence $\limsup_{x \rightarrow +\infty} v(t, x) \leq \beta$. For each $n \in \mathbb{N}$, set $y_n = \frac{x_n + x_{n+1}}{2}$. There holds

$$v(t, y_{2n}) - \alpha \leq \frac{1}{\sqrt{4\pi t}} \int_{|y| \geq \frac{x_{2n+1} - x_{2n-2}}{2}} e^{-\frac{|y_{2n} - y|^2}{4t}} dy$$

and

$$v(t, y_{2n+1}) - \beta \geq -\frac{1}{\sqrt{4\pi t}} \int_{|y| \geq \frac{x_{2n+2} - x_{2n+1-2}}{2}} e^{-\frac{|y_{2n+1} - y|^2}{4t}} dy.$$

Since $x_{n+1} - x_n \rightarrow +\infty$ as $n \rightarrow +\infty$, one gets that $\liminf_{x \rightarrow +\infty} v(t, x) = \alpha$ and $\limsup_{x \rightarrow +\infty} v(t, x) = \beta$ for each $t > 0$ (and also for $t = 0$ since $v(0, \cdot) = u_0$). Finally, one concludes that

$$\alpha_{\min} = \alpha \quad \text{and} \quad \alpha_{\max} = \beta.$$

Theorem 1.3 implies then that

$$c_\alpha \leq c_* \leq c^* \leq c_\beta. \quad (3.12)$$

First estimates of $u(t, ct + x)$ when $c \leq c_\alpha$ or $c \geq c_\beta$

Let us come back to the solution u of (1.11) with initial condition u_0 given by (3.9) and (3.10). The function u_0 satisfies $\alpha \leq u_0 \leq 1$ in \mathbb{R} , whence

$$\alpha \leq u(t, \cdot) \leq 1 \quad \text{in } \mathbb{R}$$

for all $t \geq 0$ (since $f(\alpha) = f(1) = 0$). Since $\liminf_{x \rightarrow +\infty} v(t, x) = \alpha$, $\limsup_{x \rightarrow +\infty} v(t, x) = \beta$ and

$$\liminf_{x \rightarrow +\infty} u(t, x) = \liminf_{x \rightarrow +\infty} v(t, x), \quad \limsup_{x \rightarrow +\infty} u(t, x) = \limsup_{x \rightarrow +\infty} v(t, x)$$

from Lemma 2.7, one concludes that

$$\inf_{\mathbb{R}} u(t, \cdot) = \liminf_{x \rightarrow +\infty} u(t, x) = \alpha < \beta = \limsup_{x \rightarrow +\infty} u(t, x) \quad (3.13)$$

for all $t \geq 0$. Furthermore, for each $\eta \in (0, 1 - \theta)$, there is a real number ξ such that $\underline{\varphi}_{\alpha-\eta, \eta}(x + \xi) \leq u_0(x)$ for all $x \in \mathbb{R}$, under the notations (3.1). Since $\underline{f}_{-\eta} \leq f$, the maximum principle implies that

$$\underline{\varphi}_{\alpha-\eta, \eta}(x - \underline{c}_{\alpha-\eta, \eta}t + \xi) \leq u(t, x) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R},$$

whence $\liminf_{x \rightarrow -\infty} u(t, x) \geq \underline{\varphi}_{\alpha-\eta, \eta}(-\infty) = 1 - \eta$ for all $t \geq 0$. Since $\eta > 0$ is arbitrarily small and $u \leq 1$, one gets that

$$\forall t \geq 0, \quad \sup_{\mathbb{R}} u(t, \cdot) = \lim_{x \rightarrow -\infty} u(t, x) = 1.$$

According to the notations (3.5) and (3.6), there holds

$$\alpha \leq \underline{u}_{\alpha, 0}(0, \cdot) \leq u_0 \leq \bar{u}_{\beta}(0, \cdot) \leq 1 \quad \text{in } \mathbb{R}.$$

As a consequence,

$$\alpha \leq \underline{u}_{\alpha, 0}(t, x) \leq u(t, x) \leq \bar{u}_{\beta}(t, x) \leq 1 \quad \text{for all } (t, x) \in [0, +\infty) \times \mathbb{R}. \quad (3.14)$$

Since

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \left(\sup_{x \in \mathbb{R}} |\underline{u}_{\alpha, 0}(t, x) - \varphi_{\alpha}(x - c_{\alpha}t + \underline{x}_{\alpha, 0})| \right) \\ &= \lim_{t \rightarrow +\infty} \left(\sup_{x \in \mathbb{R}} |\bar{u}_{\beta}(t, x) - \varphi_{\beta}(x - c_{\beta}t + \bar{x}_{\beta})| \right) = 0 \end{aligned} \quad (3.15)$$

and since all functions φ_{γ} are continuous and decreasing in \mathbb{R} , it follows that

$$\left\{ \begin{array}{l} \forall c < c_{\alpha}, \forall A \in \mathbb{R}, \quad u(t, ct + \cdot) \xrightarrow[t \rightarrow +\infty]{} 1 \quad \text{uniformly in } (-\infty, A], \\ \forall A \in \mathbb{R}, \quad \alpha < \varphi_{\alpha}(A + \underline{x}_{\alpha, 0}) \leq \liminf_{t \rightarrow +\infty} \left(\inf_{x \in (-\infty, A]} u(t, c_{\alpha}t + x) \right) (\leq 1), \\ \forall A \in \mathbb{R}, \quad (\alpha \leq) \limsup_{t \rightarrow +\infty} \left(\sup_{x \in [A, +\infty)} u(t, c_{\beta}t + x) \right) \leq \varphi_{\beta}(A + \bar{x}_{\beta}) < 1, \\ \forall c > c_{\beta}, \forall A \in \mathbb{R}, \quad (\alpha \leq) \limsup_{t \rightarrow +\infty} \left(\sup_{x \in [A, +\infty)} u(t, ct + x) \right) \leq \beta. \end{array} \right. \quad (3.16)$$

Notice also that, for all $x \in \mathbb{R}$,

$$\alpha < \alpha_x := \liminf_{t \rightarrow +\infty} u(t, c_{\alpha}t + x) \leq \limsup_{t \rightarrow +\infty} u(t, c_{\alpha}t + x) \leq 1 \quad (3.17)$$

and

$$\alpha \leq \liminf_{t \rightarrow +\infty} u(t, c_{\beta}t + x) \leq \limsup_{t \rightarrow +\infty} u(t, c_{\beta}t + x) =: \beta_x < 1. \quad (3.18)$$

Definition of the functions $t \mapsto \xi(t)$ and $x \mapsto \tau(x)$

The function u_0 is Lipschitz-continuous, piecewise C^1 , and the value

$$\xi_0 = \frac{2(1-\theta)}{1-\alpha} \in (0, 2)$$

is the unique real number such that $u_0(\xi_0) = \theta$. Furthermore, $u'_0(\xi_0) = -(1-\alpha)/2 < 0$. Remember also that, for each $t > 0$, the function $u(t, \cdot)$ is continuous and $u(t, -\infty) = 1$, $\limsup_{x \rightarrow +\infty} u(t, x) \leq \beta < \theta$. Since the number of intersection points of the function $u(t, \cdot)$ with the constant θ (which is a solution of the same parabolic equation as u) is nonincreasing in time, one concludes that, for each $t \geq 0$, there is a unique $\xi(t) \in \mathbb{R}$ such that

$$u(t, \xi(t)) = \theta, \quad u(t, \cdot) > \theta \text{ in } (-\infty, \xi(t)), \quad u(t, \cdot) < \theta \text{ in } (\xi(t), +\infty),$$

and $u_x(t, \xi(t)) < 0$ (with these notations, there holds $\xi(0) = \xi_0$). It follows from the implicit function theorem that ξ is a C^1 function of t . Lastly, from (3.14) and (3.15), there exists a constant $M \geq 0$ such that

$$\forall t \geq 0, \quad c_\alpha t - M \leq \xi(t) \leq c_\beta t + M. \quad (3.19)$$

Therefore, for each $x \geq \xi_0$, the real number

$$\tau(x) = \min \{t \geq 0, \xi(t) = x\}$$

is well-defined. Notice that $\tau(x) > 0$ for all $x > \xi_0$. For all $x \geq \xi_0$, there holds $\xi(\tau(x)) = x$ and $u(\tau(x), x) = \theta$. Furthermore, for all $t \in [0, \tau(x)]$, one has $\xi(t) \leq x$. As a consequence,

$$\forall x \geq \xi_0, \forall t \in [0, \tau(x)], \quad u(t, \cdot) \leq \theta \text{ in } [x, +\infty). \quad (3.20)$$

For any $\xi_0 \leq x_1 < x_2$, there holds $u(\tau(x_2), x_1) > \theta$ since $x_1 < x_2 = \xi(\tau(x_2))$. But $u(0, x_1) \leq \theta$ since $x_1 \geq \xi_0$. Consequently, $\tau(x_1) < \tau(x_2)$. Thus, the function $\tau : [\xi_0, +\infty) \rightarrow [0, +\infty)$ is increasing.

Lastly, notice from (3.19) (applied at $t = \tau(x)$) implies that

$$\forall x \geq \xi_0, \quad \frac{x - M}{c_\beta} \leq \tau(x) \leq \frac{x + M}{c_\alpha}. \quad (3.21)$$

In particular, $\lim_{n \rightarrow +\infty} \tau(x_n) = +\infty$, since $\lim_{n \rightarrow +\infty} x_n = +\infty$.

The key-lemma

The key-point in the proof of Theorem 1.6 is the following lemma, the proof of which is postponed in the next subsection:

Lemma 3.1 Set $z_n = \sqrt{x_n x_{n+1}}$ for each $n \in \mathbb{N}$. For each $0 < \varepsilon < 1$, there exists $\eta_0 = \eta_0(\varepsilon) > 0$ such that the following holds: for all $\eta \in (0, \eta_0)$, there is $N = N(\varepsilon, \eta) \in \mathbb{N}$ such that

$$\forall n \geq N, \quad \left\{ \begin{array}{l} \left| \frac{x_{2n}}{\tau(x_{2n})} - c_\beta \right| + \left| \frac{z_{2n}}{\tau(z_{2n})} - c_\alpha \right| + \left| \frac{x_{2n+1}}{\tau(x_{2n+1})} - c_\alpha \right| + \left| \frac{z_{2n+1}}{\tau(z_{2n+1})} - c_\beta \right| \leq \varepsilon, \\ u(\tau(x_{2n}), \cdot) \geq 1 - \eta \text{ in } (-\infty, x_{2n} - \varepsilon x_{2n}], \\ u(\tau(x_{2n}), \cdot) \leq \alpha + \eta \text{ in } [x_{2n} + \varepsilon x_{2n}, x_{2n+1} - \varepsilon x_{2n+1}], \\ u(\tau(z_{2n}), \cdot) \geq 1 - \eta \text{ in } (-\infty, z_{2n} - \varepsilon z_{2n}], \\ u(\tau(z_{2n}), \cdot) \leq \alpha + \eta \text{ in } [z_{2n} + \varepsilon z_{2n}, x_{2n+1} - \varepsilon x_{2n+1}], \\ u(\tau(x_{2n+1}), \cdot) \geq 1 - \eta \text{ in } (-\infty, x_{2n+1} - \varepsilon x_{2n+1}], \\ |u(\tau(x_{2n+1}), \cdot) - \beta| \leq \eta \text{ in } [x_{2n+1} + \varepsilon x_{2n+1}, x_{2n+2} - \varepsilon x_{2n+2}], \\ u(\tau(z_{2n+1}), \cdot) \geq 1 - \eta \text{ in } (-\infty, z_{2n+1} - \varepsilon z_{2n+1}], \\ |u(\tau(z_{2n+1}), \cdot) - \beta| \leq \eta \text{ in } [z_{2n+1} + \varepsilon z_{2n+1}, x_{2n+2} - \varepsilon x_{2n+2}]. \end{array} \right.$$

End of the proof of Theorem 1.6

First, let c be any given speed such that $c < c_\beta$, let x be any given real number and let us prove that $\limsup_{t \rightarrow +\infty} u(ct, t + x) = 1$. Let $0 < \varepsilon < 1$ be such that

$$c < (1 - \varepsilon) \times (c_\beta - \varepsilon).$$

Let $\eta_0 = \eta_0(\varepsilon)$ be given by Lemma 3.1. Pick any $\eta \in (0, \eta_0)$ and let $N = N(\varepsilon, \eta) \in \mathbb{N}$ be given by Lemma 3.1. Since $\tau(x_{2n}) \rightarrow +\infty$ as $n \rightarrow +\infty$, there is $N_1 \geq N$ such that

$$\forall n \geq N_1, \quad c + \frac{x}{\tau(x_{2n})} \leq (1 - \varepsilon) \times (c_\beta - \varepsilon).$$

For any $n \geq N_1$, it follows then from Lemma 3.1 that

$$c + \frac{x}{\tau(x_{2n})} \leq (1 - \varepsilon) \times (c_\beta - \varepsilon) \leq (1 - \varepsilon) \times \frac{x_{2n}}{\tau(x_{2n})},$$

whence $c\tau(x_{2n}) + x \leq x_{2n} - \varepsilon x_{2n}$. Thus,

$$\forall n \geq N_1, \quad u(\tau(x_{2n}), c\tau(x_{2n}) + x) \geq 1 - \eta$$

from Lemma 3.1. Since η is arbitrary in $(0, \eta_0)$ and since $u \leq 1$, one concludes that

$$\forall c < c_\beta, \forall x \in \mathbb{R}, \quad \limsup_{t \rightarrow +\infty} u(t, ct + x) = 1. \quad (3.22)$$

Let now c be any given speed such that $c > c_\alpha$, let x be any given real number and let us prove that $\liminf_{t \rightarrow +\infty} u(ct, t + x) = \alpha$ and $\limsup_{t \rightarrow +\infty} u(t, ct + x) \geq \beta$. Let $0 < \varepsilon < 1$ be such that

$$(1 + \varepsilon) \times (c_\alpha + \varepsilon) < c,$$

let $\eta_0 = \eta_0(\varepsilon)$ be given by Lemma 3.1, pick any $\eta \in (0, \eta_0)$ and let $N = N(\varepsilon, \eta) \in \mathbb{N}$ be given by Lemma 3.1. Since $\tau(z_{2n}) \rightarrow +\infty$ and $\tau(x_{2n+1}) \rightarrow +\infty$ as $n \rightarrow +\infty$, there is $N_1 \geq N$ such that

$$(1 + \varepsilon) \times (c_\alpha + \varepsilon) \leq \min \left(c + \frac{x}{\tau(z_{2n})}, c + \frac{x}{\tau(x_{2n+1})} \right).$$

Lemma 3.1 also implies that, for any $n \geq N_1$,

$$\begin{cases} (1 + \varepsilon) \times \frac{z_{2n}}{\tau(z_{2n})} \leq (1 + \varepsilon) \times (c_\alpha + \varepsilon) \leq c + \frac{x}{\tau(z_{2n})}, \\ (1 + \varepsilon) \times \frac{x_{2n+1}}{\tau(x_{2n+1})} \leq (1 + \varepsilon) \times (c_\alpha + \varepsilon) \leq c + \frac{x}{\tau(x_{2n+1})}. \end{cases} \quad (3.23)$$

On the other hand,

$$\frac{x_{2n+1}}{\tau(z_{2n})} = \frac{z_{2n}}{\tau(z_{2n})} \times \frac{z_{2n}}{x_{2n}} = \frac{z_{2n}}{\tau(z_{2n})} \times \sqrt{\frac{x_{2n+1}}{x_{2n}}} \rightarrow +\infty \text{ as } n \rightarrow +\infty$$

from (3.8) and (3.21). Moreover,

$$\frac{x_{2n+2}}{\tau(x_{2n+1})} = \frac{x_{2n+1}}{\tau(x_{2n+1})} \times \frac{x_{2n+2}}{x_{2n+1}} \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

In particular, there exists $N_2 \geq N_1$ such that

$$\forall n \geq N_2, \quad \begin{cases} c + \frac{x}{\tau(z_{2n})} \leq (1 - \varepsilon) \times \frac{x_{2n+1}}{\tau(z_{2n})}, \\ c + \frac{x}{\tau(x_{2n+1})} \leq (1 - \varepsilon) \times \frac{x_{2n+2}}{\tau(x_{2n+1})}. \end{cases} \quad (3.24)$$

Eventually, it follows from (3.23) and (3.24) that

$$\forall n \geq N_2, \quad \begin{cases} z_{2n} + \varepsilon z_{2n} \leq c\tau(z_{2n}) + x \leq x_{2n+1} - \varepsilon x_{2n+1}, \\ x_{2n+1} + \varepsilon x_{2n+1} \leq c\tau(x_{2n+1}) + x \leq x_{2n+2} - \varepsilon x_{2n+2}, \end{cases}$$

whence

$$\forall n \geq N_2, \quad \begin{cases} u(\tau(z_{2n}), c\tau(z_{2n}) + x) \leq \alpha + \eta, \\ |u(\tau(x_{2n+1}), c\tau(x_{2n+1}) + x) - \beta| \leq \eta \end{cases}$$

from Lemma 3.1. Since η is arbitrary in $(0, \eta_0)$ and since $u \geq \alpha$, one concludes that

$$\forall c > c_\alpha, \forall x \in \mathbb{R}, \quad \liminf_{t \rightarrow +\infty} u(t, ct + x) = \alpha < \beta \leq \limsup_{t \rightarrow +\infty} u(t, ct + x). \quad (3.25)$$

Since the function u is continuous, properties (3.22) and (3.25) yield:

$$\forall c \in (c_\alpha, c_\beta), \forall x \in \mathbb{R}, \quad \left\{ \lim_{t_k \rightarrow +\infty} u(t_k, ct_k + x) \right\} = [\alpha, 1]. \quad (3.26)$$

Notice that, from (3.12) and the general definitions of c_* and c^* given in the introduction, formula (3.26) implies in particular that

$$c_* = c_\alpha < c_\beta = c^*.$$

The second and third assertions in (3.16) then yield (1.13).

Furthermore, property (3.22) also implies that, for all $x \in \mathbb{R}$,

$$\left\{ \lim_{t_k \rightarrow +\infty} u(t_k, c_\alpha t_k + x) \right\} = [\alpha_x, 1],$$

where $\alpha_x = \liminf_{t \rightarrow +\infty} u(t, c_\alpha t + x) \in (\alpha, 1]$ (see (3.17)). Similarly, property (3.25) implies that, for all $x \in \mathbb{R}$, the real number $\beta_x \in [\alpha, 1)$ given by (3.18), namely $\beta_x = \limsup_{t \rightarrow +\infty} u(t, c_\beta t + x)$, is such that $\beta_x \in [\beta, 1)$ and

$$\left\{ \lim_{t_k \rightarrow +\infty} u(t_k, c_\beta t_k + x) \right\} = [\alpha, \beta_x].$$

Lastly, for any speed $c > c_\beta$ and for any real number x , it follows from the last assertion in (3.16) and from (3.25) that

$$\forall c > c_\beta, \forall x \in \mathbb{R}, \quad \left\{ \lim_{t_k \rightarrow +\infty} u(t_k, ct_k + x) \right\} = [\alpha, \beta].$$

Furthermore, (3.13) and (3.16) imply that

$$\forall c > c_\beta, \forall A \in \mathbb{R}, \quad \lim_{t \rightarrow +\infty} \left(\sup_{x \in [A, +\infty)} u(t, ct + x) \right) = \beta.$$

That completes the proof of Theorem 1.6. □

Remark 3.2 It follows from (3.19), (3.21) and Lemma 3.1 that

$$c_\alpha = \liminf_{t \rightarrow +\infty} \frac{\xi(t)}{t} < \limsup_{t \rightarrow +\infty} \frac{\xi(t)}{t} = c_\beta$$

and

$$\frac{1}{c_\beta} = \liminf_{x \rightarrow +\infty} \frac{\tau(x)}{x} < \limsup_{x \rightarrow +\infty} \frac{\tau(x)}{x} = \frac{1}{c_\alpha}.$$

In particular, there is no speed c such that the function $t \mapsto \xi(t) - ct$ is bounded and there are no $\gamma < \theta$ and $x_0 \in \mathbb{R}$ such that $u(t, \xi(t) + \cdot)$ converges as $t \rightarrow +\infty$ to a front $\varphi_\gamma(\cdot + x_0)$. These properties are very different from the usual results of the literature, which are concerned with initial conditions u_0 converging to a constant as $x \rightarrow +\infty$.

3.2 Proof of Lemma 3.1

Choices of $\eta_0 = \eta_0(\varepsilon)$ and parameters depending on $\eta \in (0, \eta_0)$

Let $0 < \varepsilon < 1$ be given. Let $\rho > 0$ be chosen so that

$$0 < \rho < \frac{1}{2} \quad \text{and} \quad c_\beta - \frac{\varepsilon}{4} < (c_\beta^{-1} + \rho \varepsilon c_\alpha^{-1})^{-1}. \quad (3.27)$$

From (3.3) and (3.4), there exists $\eta_0 = \eta_0(\varepsilon) \in (0, \min(1 - \theta, \theta - \beta))$ such that, for all $\eta \in (0, \eta_0)$,

$$\forall \eta \in (0, \eta_0), \quad \begin{cases} 0 < c_\alpha \leq \bar{c}_{\alpha+\eta, \eta} < c_\alpha + \frac{\varepsilon}{4}, \\ c_\beta - \frac{\varepsilon}{4} < \underline{c}_{\beta-\eta, \eta/2} \leq c_\beta \leq \bar{c}_{\beta+\eta, \eta} < c_\beta + \frac{\varepsilon}{4}, \\ 1 - \frac{c_\alpha}{\bar{c}_{\alpha+\eta, \eta}} \leq \frac{\varepsilon}{8}, \\ 1 - \frac{\underline{c}_{\beta-\eta, \eta/2}}{\bar{c}_{\beta+\eta, \eta}} \leq \frac{\rho \varepsilon}{4} \leq \frac{\varepsilon}{8}, \\ c_\beta - \frac{\varepsilon}{4} < \left(\underline{c}_{\beta-\eta, \eta/2}^{-1} + \rho \varepsilon \underline{c}_{\alpha-\eta, 3\eta/4}^{-1} \right)^{-1}. \end{cases} \quad (3.28)$$

In the sequel, let η be any given real number in the interval $(0, \eta_0)$. Remember that the pairs $(\underline{c}_{\alpha-\eta, \eta/4}, \underline{\varphi}_{\alpha-\eta, \eta/4})$, $(\underline{c}_{\alpha-\eta, 3\eta/4}, \underline{\varphi}_{\alpha-\eta, 3\eta/4})$ and $(\underline{c}_{\beta-\eta, \eta/2}, \underline{\varphi}_{\beta-\eta, \eta/2})$ solve (3.1) with nonlinearities $\underline{f}_{\eta/4}$, $\underline{f}_{3\eta/4}$ and $\underline{f}_{\eta/2}$ respectively, and limit values

$$\begin{cases} \underline{\varphi}_{\alpha-\eta, \eta/4}(-\infty) = 1 - \frac{\eta}{4} > \alpha - \eta = \underline{\varphi}_{\alpha-\eta, \eta/4}(+\infty), \\ \underline{\varphi}_{\alpha-\eta, 3\eta/4}(-\infty) = 1 - \frac{3\eta}{4} > \alpha - \eta = \underline{\varphi}_{\alpha-\eta, 3\eta/4}(+\infty), \\ \underline{\varphi}_{\beta-\eta, \eta/2}(-\infty) = 1 - \frac{\eta}{2} > \beta - \eta = \underline{\varphi}_{\beta-\eta, \eta/2}(+\infty) \end{cases}$$

and that the pairs $(\bar{c}_{\alpha+\eta, \eta}, \bar{\varphi}_{\alpha+\eta, \eta})$ and $(\bar{c}_{\beta+\eta, \eta}, \bar{\varphi}_{\beta+\eta, \eta})$ solve (3.2) with nonlinearity \bar{f}_η and limit values

$$\begin{cases} \bar{\varphi}_{\alpha+\eta, \eta}(-\infty) = 1 + \eta > \alpha + \eta = \bar{\varphi}_{\alpha+\eta, \eta}(+\infty), \\ \bar{\varphi}_{\beta+\eta, \eta}(-\infty) = 1 + \eta > \beta + \eta = \bar{\varphi}_{\beta+\eta, \eta}(+\infty). \end{cases}$$

There exists a real number $A = A(\eta) > 0$, which is fixed in the sequel, such that

$$\left\{ \begin{array}{ll} \bar{\varphi}_{\alpha+\eta,\eta} \geq 1 & \text{in } (-\infty, -A], \\ \varphi_\alpha \geq 1 - \frac{\eta}{8} \geq \theta + \frac{\eta}{8} & \text{in } (-\infty, -A], \\ \bar{\varphi}_{\beta+\eta,\eta} \geq 1 & \text{in } (-\infty, -A], \\ \underline{\varphi}_{\alpha-\eta,\eta/4} \leq \alpha & \text{in } [A, +\infty), \\ \underline{\varphi}_{\alpha-\eta,\eta/4} \geq 1 - \frac{\eta}{2} & \text{in } (-\infty, -A], \\ \underline{\varphi}_{\beta-\eta,\eta/2} \leq \beta - \frac{\eta}{2} & \text{in } [A, +\infty), \\ \underline{\varphi}_{\beta-\eta,\eta/2} \geq 1 - \frac{3\eta}{4} & \text{in } (-\infty, -A], \\ \underline{\varphi}_{\alpha-\eta,3\eta/4} \leq \alpha & \text{in } [A, +\infty), \\ \underline{\varphi}_{\alpha-\eta,3\eta/4} \geq 1 - \eta & \text{in } (-\infty, -A]. \end{array} \right. \quad (3.29)$$

Because of (3.7), there exists also a time $T = T(\eta) \geq 0$ such that

$$\forall t \geq T, \forall x \in \mathbb{R}, \quad |\underline{u}_{\alpha,\eta}(t, x) - \varphi_\alpha(x - c_\alpha t + \underline{x}_{\alpha,\eta})| \leq \frac{\eta}{8}. \quad (3.30)$$

Comparisons with solutions of heat equations

Let v be the solution of the heat equation (3.11) with initial condition u_0 . We know that $\alpha \leq u, v \leq 1$ in $[0, +\infty) \times \mathbb{R}$. Furthermore, since $f \geq 0$, one gets that

$$\forall (t, x) \in [0, +\infty) \times \mathbb{R}, \quad 0 \leq \alpha \leq v(t, x) \leq u(t, x) \leq 1.$$

On the other hand, for any given $x \in (\xi_0, +\infty)$, there holds $u_t(t, y) = u_{yy}(t, y)$ for all $(t, y) \in (0, \tau(x)] \times [x, +\infty)$ and $u(t, x) \leq \theta$ for all $t \in [0, \tau(x)]$ from (3.20). The maximum principle implies that

$$\forall (t, y) \in [0, \tau(x)] \times [x, +\infty), \quad u(t, y) \leq v(t, y) + w(t, y),$$

where w solves the heat equation $w_t = w_{yy}$ in $(0, +\infty) \times (x, +\infty)$, with $w(0, y) = 0$ in $(x, +\infty)$ and $w(t, x) = \theta$ for all $t > 0$. The function w is explicitly given by

$$\forall (t, y) \in (0, +\infty) \times [x, +\infty), \quad w(t, y) = \frac{2\theta}{\sqrt{\pi}} \int_{\frac{y-x}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz.$$

Finally,

$$\forall x > \xi_0, \forall (t, y) \in (0, \tau(x)] \times [x, +\infty), \quad u(t, y) \leq v(t, y) + \frac{2\theta}{\sqrt{\pi}} \int_{\frac{y-x}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz.$$

Let now $B = B(\eta) > 0$ be given so that

$$\frac{1}{\sqrt{\pi}} \int_B^{+\infty} e^{-z^2} dz \leq \frac{\eta}{4}, \quad (3.31)$$

and $\xi_1 = \xi_1(\varepsilon, \eta) > \xi_0 (> 0)$ be such that

$$\forall x \geq \xi_1, \quad \min \left(\frac{x^{3/4} - 1}{2\sqrt{\tau(x + x^{3/4})}}, \frac{x - 2}{2\sqrt{\tau(x)}} \right) \geq B. \quad (3.32)$$

The choice of ξ_1 is possible because of (3.21). In particular, there holds $x^{3/4}/(2\sqrt{\tau(x)}) \geq B$ for all $x \geq \xi_1$, since τ is increasing. Thus,

$$\forall x \geq \xi_1, \quad \forall t \in [0, \tau(x)], \quad u(t, \cdot) \leq v(t, \cdot) + \frac{2\theta\eta}{4} \leq v(t, \cdot) + \frac{\eta}{2} \quad \text{in } [x + x^{3/4}, +\infty). \quad (3.33)$$

Notice indeed that the above inequality is immediate at time $t = 0$.

Furthermore, for all $x \geq \max(\xi_1, 2)$ and $(t, y) \in (0, \tau(x)] \times [x, +\infty)$, there holds that

$$\begin{aligned} v(t, y) - \beta &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{|y-z|^2}{4t}} (u_0(z) - \beta) dz \\ &\leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^2 e^{-\frac{|y-z|^2}{4t}} dz = \frac{1}{\sqrt{\pi}} \int_{\frac{y-2}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz \leq \frac{1}{\sqrt{\pi}} \int_{\frac{x-2}{2\sqrt{\tau(x)}}}^{+\infty} e^{-z^2} dz \leq \frac{\eta}{4} \end{aligned}$$

from (3.31) and (3.32). Hence, it follows from (3.33) that

$$\forall x \geq \max(\xi_1, 2), \quad \forall t \in [0, \tau(x)], \quad u(t, \cdot) \leq \beta + \frac{\eta}{4} + \frac{\eta}{2} \leq \beta + \eta \quad \text{in } [x + x^{3/4}, +\infty), \quad (3.34)$$

where the above inequality also holds immediately at time $t = 0$.

Choice of a first iteration point x_{2N_0}

Remember that $z_m = \sqrt{x_m x_{m+1}}$ for each $m \in \mathbb{N}$, and that

$$\lim_{m \rightarrow +\infty} x_m = \lim_{m \rightarrow +\infty} \frac{x_{m+1}}{x_m} = \lim_{m \rightarrow +\infty} \frac{z_m}{x_m} = \lim_{m \rightarrow +\infty} \frac{x_{m+1}}{z_m} = +\infty.$$

Let $N_0 = N_0(\varepsilon, \eta) \in \mathbb{N}$ be such that

$$\forall m \geq 2N_0, \quad \left\{ \begin{array}{l} \xi_1 + 2 + A + M + \bar{c}_{\alpha+\eta, \eta} T + |\underline{x}_{\alpha, \eta}| + \frac{8A}{\varepsilon} + A^{4/3} + \frac{1}{(\rho\varepsilon)^4} + \frac{4A}{(1-2\rho)\varepsilon} \leq x_m, \\ x_m < x_m + x_m^{3/4} \leq x_m + \varepsilon x_m \leq 4x_m \leq z_m \\ z_m + z_m^{3/4} \leq z_m + \varepsilon z_m \leq x_{m+1} - \varepsilon x_{m+1} \leq x_{m+1} - 2x_{m+1}^{3/4} < x_{m+1}, \\ 4x_m \leq \left(6 + \frac{3c_\beta}{c_\alpha}\right) x_m \leq \frac{\varepsilon z_m}{4} \leq \frac{\rho\varepsilon x_{m+1}}{2}, \\ 4x_m + 2x_{m+1}^{3/4} \leq \frac{\varepsilon x_{m+1}}{4}, \\ A \leq \frac{\rho\varepsilon x_m}{4} - 2x_m^{3/4}, \end{array} \right. \quad (3.35)$$

where $\underline{x}_{\alpha,\eta} \in \mathbb{R}$, $M \geq 0$, $\rho \in (0, 1/2)$, $A \geq 0$ and $T \geq 0$ are given in (3.7), (3.19), (3.27), (3.29) and (3.30).

Estimates of v in intervals of the type $[x + x^{3/4}, x_{m+1} - x_{m+1}^{3/4}]$

Choose any integer n such that $n \geq N_0$, any real number $x \in [x_{2n}, x_{2n+1} - x_{2n+1}^{3/4}]$, any real number $t \in (0, \tau(x + x^{3/4})]$ and any real number $y \in [x + x^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}]$ (when this interval is not empty). Since $v(0, \cdot) = u_0 = \alpha$ in the interval $[x_{2n} + 1, x_{2n+1} - 1]$, there holds that

$$\begin{aligned} |v(t, y) - \alpha| &\leq \frac{1}{\sqrt{4\pi t}} \times \left(\int_{-\infty}^{x_{2n+1}} e^{-\frac{|y-z|^2}{4t}} |u_0(z) - \alpha| dz + \int_{x_{2n+1}-1}^{+\infty} e^{-\frac{|y-z|^2}{4t}} |u_0(z) - \alpha| dz \right) \\ &\leq \frac{1}{\sqrt{\pi}} \times \left(\int_{-\infty}^{\frac{x_{2n+1}-y}{2\sqrt{t}}} e^{-z^2} dz + \int_{\frac{x_{2n+1}-1-y}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz \right), \end{aligned}$$

while

$$\frac{x_{2n+1}-y}{2\sqrt{t}} \leq \frac{-x^{3/4}+1}{2\sqrt{\tau(x+x^{3/4})}} \leq -B \leq B \leq \frac{x^{3/4}-1}{2\sqrt{\tau(x+x^{3/4})}} \leq \frac{x_{2n+1}^{3/4}-1}{2\sqrt{t}} \leq \frac{x_{2n+1}-1-y}{2\sqrt{t}}$$

from (3.32) and (3.35). It follows then from (3.31) that

$$\begin{aligned} \forall n \geq N_0, \forall x \in [x_{2n}, x_{2n+1} - x_{2n+1}^{3/4}], \forall t \in [0, \tau(x + x^{3/4})], \\ |v(t, \cdot) - \alpha| \leq \frac{\eta}{2} \quad \text{in } [x + x^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}], \end{aligned} \quad (3.36)$$

provided that the space interval is not empty. Similarly, since $v(0, \cdot) = u_0 = \beta$ in the interval $[x_{2n+1} + 1, x_{2n+2} - 1]$, one gets that

$$\begin{aligned} \forall n \geq N_0, \forall x \in [x_{2n+1}, x_{2n+2} - x_{2n+2}^{3/4}], \forall t \in [0, \tau(x + x^{3/4})], \\ |v(t, \cdot) - \beta| \leq \frac{\eta}{2} \quad \text{in } [x + x^{3/4}, x_{2n+2} - x_{2n+2}^{3/4}], \end{aligned} \quad (3.37)$$

provided that the space interval is not empty.

Refined estimates of $u(\tau(x), \cdot)$ in intervals of the type $[x + x^{3/4}, x_{m+1} - x_{m+1}^{3/4}]$

Let $n \geq N_0$ be given. Let us first show that $\alpha \leq u(t, \cdot) \leq \alpha + \eta$ in $[x_{2n} + x_{2n}^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}]$ for all $t \in (0, \tau(x_{2n}))$. This would then, in particular, yield the same inequality, at time $t = \tau(x_{2n})$, in the smaller interval $[x_{2n} + \varepsilon x_{2n}, x_{2n+1} - \varepsilon x_{2n+1}]$, from the choice of N_0 in (3.35). Remember that the lower bound $u(t, x) \geq \alpha$ always holds. Furthermore, since $x_{2n} \geq \xi_1$ and $\tau(x_{2n}) \leq \tau(x_{2n} + x_{2n}^{3/4})$, properties (3.33) and (3.36) –with $x = x_{2n}$ – imply that

$$\forall t \in [0, \tau(x_{2n})], \forall y \in [x_{2n} + x_{2n}^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}], \quad u(t, y) \leq v(t, y) + \frac{\eta}{2} \leq \alpha + \eta.$$

Eventually,

$$\forall n \geq N_0, \forall t \in [0, \tau(x_{2n})], \quad \alpha \leq u(t, \cdot) \leq \alpha + \eta \quad \text{in } [x_{2n} + x_{2n}^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}]. \quad (3.38)$$

With the same arguments, the following estimates hold:

$$\left\{ \begin{array}{ll} \alpha \leq u \leq \alpha + \eta & \text{in } [0, \tau(z_{2n})] \times [z_{2n} + z_{2n}^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}], \\ \beta - \frac{\eta}{2} \leq v \leq u \leq \beta + \eta & \text{in } [0, \tau(x_{2n+1})] \times [x_{2n+1} + x_{2n+1}^{3/4}, x_{2n+2} - x_{2n+2}^{3/4}], \\ \beta - \frac{\eta}{2} \leq v \leq u \leq \beta + \eta & \text{in } [0, \tau(z_{2n+1})] \times [z_{2n+1} + z_{2n+1}^{3/4}, x_{2n+2} - x_{2n+2}^{3/4}] \end{array} \right. \quad (3.39)$$

for all $n \geq N_0$. The last two properties follow from (3.33) and (3.37) applied with $x = x_{2n+1}$ and $x = z_{2n+1}$ respectively. Notice that these three properties then hold a fortiori in the smaller space intervals $[z_{2n} + \varepsilon z_{2n}, x_{2n+1} - \varepsilon x_{2n+1}]$, $[x_{2n+1} + \varepsilon x_{2n+1}, x_{2n+2} - \varepsilon x_{2n+2}]$ and $[z_{2n+1} + \varepsilon z_{2n+1}, x_{2n+2} - \varepsilon x_{2n+2}]$ respectively. Actually, one gets more generally that

$$\forall x \in [x_{2n}, x_{2n+1} - x_{2n+1}^{3/4}], \quad \alpha \leq u \leq \alpha + \eta \quad \text{in } [0, \tau(x)] \times [x + x^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}] \quad (3.40)$$

and

$$\forall x \in [x_{2n+1}, x_{2n+2} - x_{2n+2}^{3/4}], \quad \beta - \frac{\eta}{2} \leq u \leq \beta + \eta \quad \text{in } [0, \tau(x)] \times [x + x^{3/4}, x_{2n+2} - x_{2n+2}^{3/4}]$$

for all $n \geq N_0$, provided that the space intervals are not empty.

From time $t = \tau(x_{2n})$ to time $t = \tau(x_{2n+1})$

The heart of the proof of Lemma 3.1 consists in estimating from below $u(\tau(x), \cdot)$ on $(-\infty, x - \varepsilon x]$ and estimating some ratios $x/\tau(x)$, for $x = x_{2n}$, z_{2n} , x_{2n+1} and z_{2n+1} . We will do that by induction on n and step by step, from time $\tau(x_{2n})$ to time $\tau(x_{2n+1})$, and from time $\tau(x_{2n+1})$ to time $\tau(x_{2n+2})$.

Step 1: lower bound of $\tau(x)$ for $x \in [x_{2n}, x_{2n+1} - 2x_{2n+1}^{3/4}]$. Choose any integer n such that $n \geq N_0$. There holds

$$u(\tau(x_{2n}), \cdot) \leq \alpha + \eta \quad \text{in } [x_{2n} + x_{2n}^{3/4}, x_{2n+1} - x_{2n+1}^{3/4}]$$

from (3.38). Moreover, $u(\tau(x_{2n}), \cdot) \leq 1$ in \mathbb{R} . It follows then from the first assertion in (3.29) and from the inequality $\bar{\varphi}_{\alpha+\eta, \eta} \geq \alpha + \eta$ in \mathbb{R} that

$$u(\tau(x_{2n}), x) \leq \bar{\varphi}_{\alpha+\eta, \eta}(x - (x_{2n} + x_{2n}^{3/4}) - A) \quad \text{for all } x \in (-\infty, x_{2n+1} - x_{2n+1}^{3/4}].$$

Furthermore, since

$$x_{2n} \leq x_{2n+1} - 2x_{2n+1}^{3/4} \quad \text{and} \quad (x_{2n+1} - 2x_{2n+1}^{3/4}) + (x_{2n+1} - 2x_{2n+1}^{3/4})^{3/4} \leq x_{2n+1} - x_{2n+1}^{3/4},$$

it follows from (3.40), applied at $x = x_{2n+1} - 2x_{2n+1}^{3/4}$, that

$$\forall t \in [0, \tau(x_{2n+1} - 2x_{2n+1}^{3/4})], \quad u(t, x_{2n+1} - x_{2n+1}^{3/4}) \leq \alpha + \eta.$$

Since $\bar{f}_\eta \geq f$, the function $\bar{\varphi}_{\alpha+\eta,\eta}(x - \bar{c}_{\alpha+\eta,\eta}t)$ is a supersolution of the equation satisfied by u . Since $\bar{\varphi}_{\alpha+\eta,\eta} \geq \alpha + \eta$ in \mathbb{R} , the maximum principle applied in the set where $(t, x) \in [\tau(x_{2n}), \tau(x_{2n+1} - 2x_{2n+1}^{3/4})] \times (-\infty, x_{2n+1} - x_{2n+1}^{3/4}]$ then yields

$$u(t, x) \leq \bar{\varphi}_{\alpha+\eta,\eta} \left(x - x_{2n} - x_{2n}^{3/4} - A - \bar{c}_{\alpha+\eta,\eta}(t - \tau(x_{2n})) \right).$$

for all $(t, x) \in [\tau(x_{2n}), \tau(x_{2n+1} - 2x_{2n+1}^{3/4})] \times (-\infty, x_{2n+1} - x_{2n+1}^{3/4}]$. In particular, by choosing $t = \tau(x)$ and $x \in [x_{2n}, x_{2n+1} - 2x_{2n+1}^{3/4}]$, one has

$$\theta = u(\tau(x), x) \leq \bar{\varphi}_{\alpha+\eta,\eta} \left(x - x_{2n} - x_{2n}^{3/4} - A - \bar{c}_{\alpha+\eta,\eta}(\tau(x) - \tau(x_{2n})) \right).$$

But $\bar{\varphi}_{\alpha+\eta,\eta}$ is decreasing and equals θ at 0. Hence,

$$\begin{aligned} \forall x \in [x_{2n}, x_{2n+1} - 2x_{2n+1}^{3/4}], \quad \tau(x) &\geq \frac{x - x_{2n} - x_{2n}^{3/4} - A}{\bar{c}_{\alpha+\eta,\eta}} + \tau(x_{2n}) \\ &\geq \frac{x - 3x_{2n}}{\bar{c}_{\alpha+\eta,\eta}} + \tau(x_{2n}), \end{aligned} \quad (3.41)$$

since $x_{2n} \geq \max(1, A)$ from (3.35).

Step 2: upper bound of $\tau(x)$ for $x \geq 3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T$. Let n be any given integer such that $n \geq N_0$, and let $X \geq 0$ be such that

$$u(\tau(x_{2n}), \cdot) \geq 1 - \eta \quad \text{in } (-\infty, x_{2n} - \varepsilon'x_{2n} - X], \quad (3.42)$$

where we set

$$\varepsilon' = \frac{\varepsilon}{2}.$$

Notice that such a $X \geq 0$ always exists since $u(\tau(x_{2n}), -\infty) = 1$. Owing to the definition of $\underline{u}_{\alpha,\eta}$ in (3.5), and since $u(\tau(x_{2n}), \cdot) \geq \alpha$ in \mathbb{R} , there holds then

$$\forall x \in \mathbb{R}, \quad \underline{u}_{\alpha,\eta}(0, x - (x_{2n} - \varepsilon'x_{2n} - X) + 2) \leq u(\tau(x_{2n}), x).$$

But $\underline{u}_{\alpha,\eta}$ is a subsolution of the equation satisfied by u , since $\underline{f}_\eta \leq f$. Thus,

$$\forall t \geq \tau(x_{2n}), \quad \forall x \in \mathbb{R}, \quad \underline{u}_{\alpha,\eta}(t - \tau(x_{2n}), x - x_{2n} + \varepsilon'x_{2n} + X + 2) \leq u(t, x).$$

Hence, for all $(t, x) \in [T + \tau(x_{2n}), +\infty) \times \mathbb{R}$,

$$\varphi_\alpha \left(x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(t - \tau(x_{2n})) \right) - \frac{\eta}{8} \leq u(t, x) \quad (3.43)$$

from (3.30). Since $3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T \leq 4x_{2n} \leq x_{2n+1} - 2x_{2n+1}^{3/4}$ from (3.35), and since τ is increasing, there holds

$$\forall x \geq 3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T, \quad \tau(x) \geq \tau(3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T) \geq T + \tau(x_{2n}), \quad (3.44)$$

where the last inequality follows from (3.41). In particular, by choosing $x \geq 3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T$ and $t = \tau(x) \geq T + \tau(x_{2n})$ in (3.43), one gets that

$$\varphi_\alpha(x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(\tau(x) - \tau(x_{2n}))) - \frac{\eta}{8} \leq u(\tau(x), x) = \theta,$$

whence

$$x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(\tau(x) - \tau(x_{2n})) \geq -A$$

from the second assertion in (3.29) and since φ_α is decreasing. Thus,

$$\begin{aligned} \forall x \geq 3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T, \quad \tau(x) &\leq \frac{x - x_{2n} + \varepsilon'x_{2n} + X + A + 2}{c_\alpha} + \tau(x_{2n}) \\ &\leq \frac{x + \varepsilon'x_{2n} + X + A + 2 + M}{c_\alpha} \\ &\leq \frac{x + 2x_{2n} + X}{c_\alpha} \end{aligned} \quad (3.45)$$

from (3.21) and (3.35), and since $\varepsilon' = \varepsilon/2 < 1/2 < 1$.

Step 3: estimates of $\tau(z_{2n})$ and $\tau(x_{2n+1})$ and lower bound of u on the left of z_{2n} and x_{2n+1} .
Notice that

$$3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T \leq 4x_{2n} \leq z_{2n} \leq x_{2n+1} - 2x_{2n+1}^{3/4} \leq x_{2n+1}$$

because of (3.35). As a consequence, it follows from (3.41), (3.45) and the monotonicity of τ , that

$$\left\{ \begin{array}{l} \frac{z_{2n} - 3x_{2n}}{\bar{c}_{\alpha+\eta,\eta}} \leq \frac{z_{2n} - 3x_{2n}}{\bar{c}_{\alpha+\eta,\eta}} + \tau(x_{2n}) \leq \tau(z_{2n}) \leq \frac{z_{2n} + 2x_{2n} + X}{c_\alpha}, \\ \frac{x_{2n+1} - 2x_{2n+1}^{3/4} - 3x_{2n}}{\bar{c}_{\alpha+\eta,\eta}} \leq \frac{x_{2n+1} - 2x_{2n+1}^{3/4} - 3x_{2n}}{\bar{c}_{\alpha+\eta,\eta}} + \tau(x_{2n}) \leq \dots \\ \dots \leq \tau(x_{2n+1} - 2x_{2n+1}^{3/4}) \leq \tau(x_{2n+1}) \leq \frac{x_{2n+1} + 2x_{2n} + X}{c_\alpha}, \end{array} \right. \quad (3.46)$$

provided that (3.42) holds. Since $x_{2n+1} \geq z_{2n} \geq 3x_{2n} + \bar{c}_{\alpha+\eta,\eta}T$, it follows from (3.43) and (3.44) that

$$\left\{ \begin{array}{l} u(\tau(z_{2n}), x) \geq \varphi_\alpha(x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(\tau(z_{2n}) - \tau(x_{2n}))) - \frac{\eta}{8}, \\ u(\tau(x_{2n+1}), x) \geq \varphi_\alpha(x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(\tau(x_{2n+1}) - \tau(x_{2n}))) - \frac{\eta}{8} \end{array} \right. \quad (3.47)$$

for all $x \in \mathbb{R}$. On the other hand, for all $x \leq z_{2n} - \varepsilon'z_{2n} - X = z_{2n} - \varepsilon z_{2n}/2 - X$, there holds

$$\begin{aligned} x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(\tau(z_{2n}) - \tau(x_{2n})) \\ \leq \left(1 - \frac{c_\alpha}{\bar{c}_{\alpha+\eta,\eta}} - \frac{\varepsilon}{2}\right) z_{2n} + 3x_{2n} + 2 + \underline{x}_{\alpha,\eta} \\ \leq \left(\frac{\varepsilon}{8} - \frac{\varepsilon}{2}\right) z_{2n} + 4x_{2n} \leq -\frac{\varepsilon}{8} z_{2n} \leq -A \end{aligned}$$

from (3.28), (3.35), (3.46) and since $-x_{2n} + \varepsilon'x_{2n} \leq 0$. Thus,

$$u(\tau(z_{2n}), \cdot) \geq \varphi_\alpha(-A) - \frac{\eta}{8} \geq 1 - \frac{\eta}{4} \geq 1 - \eta \quad \text{in } (-\infty, z_{2n} - \varepsilon'z_{2n} - X] \quad (3.48)$$

from (3.47), from the second assertion in (3.29), and since φ_α is decreasing. Similarly, for all $x \leq x_{2n+1} - \varepsilon'x_{2n+1} - X$, there holds

$$\begin{aligned} x - x_{2n} + \varepsilon'x_{2n} + X + 2 + \underline{x}_{\alpha,\eta} - c_\alpha(\tau(x_{2n+1}) - \tau(x_{2n})) \\ \leq \left(1 - \frac{c_\alpha}{\bar{c}_{\alpha+\eta,\eta}} - \frac{\varepsilon}{2}\right) x_{2n+1} + 3x_{2n} + 2 + \underline{x}_{\alpha,\eta} + 2x_{2n+1}^{3/4} \\ \leq \left(\frac{\varepsilon}{8} - \frac{\varepsilon}{2}\right) x_{2n+1} + 4x_{2n} + 2 + \underline{x}_{\alpha,\eta} + 2x_{2n+1}^{3/4} \leq -\frac{\varepsilon}{8} x_{2n+1} \leq -A \end{aligned}$$

from (3.28), (3.35) and (3.46), whence

$$u(\tau(x_{2n+1}), \cdot) \geq \varphi_\alpha(-A) - \frac{\eta}{8} \geq 1 - \frac{\eta}{4} \geq 1 - \eta \quad \text{in } (-\infty, x_{2n+1} - \varepsilon'x_{2n+1} - X] \quad (3.49)$$

from (3.29), (3.47) and the monotonicity of φ_α .

From time $t = \tau(x_{2n+1})$ to time $t = \tau(x_{2n+2})$

Step 1: lower bound of $\tau(x)$ for $x \geq x_{2n+1}$. Choose any integer n such that $n \geq N_0$. There holds

$$u(\tau(x_{2n+1}), \cdot) \leq \beta + \eta \quad \text{in } [x_{2n+1} + x_{2n+1}^{3/4}, +\infty)$$

from (3.34) and (3.35). Furthermore, $u(\tau(x_{2n+1}), \cdot) \leq 1$, $\bar{\varphi}_{\beta+\eta,\eta} \geq \beta + \eta$ in \mathbb{R} , and $\bar{\varphi}_{\beta+\eta,\eta} \geq 1$ in $(-\infty, -A]$ from the third assertion in (3.29). Thus,

$$\forall x \in \mathbb{R}, \quad u(\tau(x_{2n+1}), x) \leq \bar{\varphi}_{\beta+\eta,\eta} \left(x - (x_{2n+1} + x_{2n+1}^{3/4}) - A \right).$$

Since $\bar{\varphi}_{\beta+\eta,\eta}(x - \bar{c}_{\beta+\eta,\eta}t)$ is a supersolution of the equation satisfied by u , the maximum principle implies that

$$\forall (t, x) \in [\tau(x_{2n+1}), +\infty) \times \mathbb{R}, \quad u(t, x) \leq \bar{\varphi}_{\beta+\eta,\eta} \left(x - x_{2n+1} - x_{2n+1}^{3/4} - A - \bar{c}_{\beta+\eta,\eta}(t - \tau(x_{2n+1})) \right).$$

In particular, by choosing any $x \geq x_{2n+1}$ and $t = \tau(x) \geq \tau(x_{2n+1})$, one gets that

$$\theta = u(\tau(x), x) \leq \bar{\varphi}_{\beta+\eta,\eta} \left(x - x_{2n+1} - x_{2n+1}^{3/4} - A - \bar{c}_{\beta+\eta,\eta}(\tau(x) - \tau(x_{2n+1})) \right).$$

Since $\bar{\varphi}_{\beta+\eta,\eta}(0) = \theta$ and the function $\bar{\varphi}_{\beta+\eta,\eta}$ is decreasing, the argument of $\bar{\varphi}_{\beta+\eta,\eta}$ in the above formula is nonpositive, whence

$$\forall x \geq x_{2n+1}, \quad \tau(x) \geq \frac{x - x_{2n+1} - x_{2n+1}^{3/4} - A}{\bar{c}_{\beta+\eta,\eta}} + \tau(x_{2n+1}) \geq \frac{x - 3x_{2n+1}}{\bar{c}_{\beta+\eta,\eta}} + \tau(x_{2n+1}) \quad (3.50)$$

from (3.35).

Step 2: upper bound of $\tau(x)$ for $x \in [x_{2n+1} + x_{2n+1}^{3/4}, x_{2n+2} - 2x_{2n+2}^{3/4}]$. Let n be any given integer such that $n \geq N_0$, and let $Y \geq 0$ be such that

$$u(\tau(x_{2n+1}), \cdot) \geq 1 - \frac{\eta}{4} \text{ in } (-\infty, x_{2n+1} - \varepsilon' x_{2n+1} - Y]. \quad (3.51)$$

We are going to estimate from below, by suitable travelling fronts, the solution u on the time intervals $[\tau(x_{2n+1}), \tau(x_{2n+1} + x_{2n+1}^{3/4})]$ and $[\tau(x_{2n+1} + x_{2n+1}^{3/4}), \tau(x_{2n+2} - 2x_{2n+2}^{3/4})]$.

Remember that $u(\tau(x_{2n+1}), \cdot) \geq \alpha$ and $\underline{\varphi}_{\alpha-\eta, \eta/4} \leq 1 - \eta/4$ in \mathbb{R} and that $\underline{\varphi}_{\alpha-\eta, \eta/4} \leq \alpha$ in $[A, +\infty)$ from the fourth assertion in (3.29). Thus,

$$\forall x \in \mathbb{R}, \quad u(\tau(x_{2n+1}), x) \geq \underline{\varphi}_{\alpha-\eta, \eta/4}(x - (x_{2n+1} - \varepsilon' x_{2n+1} - Y) + A).$$

Since $\underline{\varphi}_{\alpha-\eta, \eta/4}(x - \underline{c}_{\alpha-\eta, \eta/4} t)$ is a subsolution of the equation satisfied by u , the maximum principle implies that, for all $(t, x) \in [\tau(x_{2n+1}), +\infty) \times \mathbb{R}$,

$$u(t, x) \geq \underline{\varphi}_{\alpha-\eta, \eta/4}(x - x_{2n+1} + \varepsilon' x_{2n+1} + Y + A - \underline{c}_{\alpha-\eta, \eta/4}(t - \tau(x_{2n+1}))). \quad (3.52)$$

Let us now find a better subsolution of u for times larger than $\tau(x_{2n+1} + x_{2n+1}^{3/4})$. It follows from (3.37) –applied at $x = x_{2n+1}$ – and the inequality $u \geq v$, that

$$u(\tau(x_{2n+1} + x_{2n+1}^{3/4}), \cdot) \geq v(\tau(x_{2n+1} + x_{2n+1}^{3/4}), \cdot) \geq \beta - \frac{\eta}{2} \text{ in } [x_{2n+1} + x_{2n+1}^{3/4}, x_{2n+2} - x_{2n+2}^{3/4}].$$

Since $u(\tau(x_{2n+1} + x_{2n+1}^{3/4}), \cdot) \geq \theta \geq \beta - \eta/2$ in $(-\infty, x_{2n+1} + x_{2n+1}^{3/4}]$, one gets that

$$u(\tau(x_{2n+1} + x_{2n+1}^{3/4}), \cdot) \geq \beta - \frac{\eta}{2} \text{ in } (-\infty, x_{2n+2} - x_{2n+2}^{3/4}].$$

Furthermore, since τ is increasing and $\underline{\varphi}_{\alpha-\eta, \eta/4}$ is decreasing, it follows from the fifth assertion in (3.29) and from (3.52) that

$$u(\tau(x_{2n+1} + x_{2n+1}^{3/4}), \cdot) \geq \underline{\varphi}_{\alpha-\eta, \eta/4}(-A) \geq 1 - \frac{\eta}{2} \text{ in } (-\infty, x_{2n+1} - \varepsilon' x_{2n+1} - Y - 2A].$$

Since $\underline{\varphi}_{\beta-\eta, \eta/2} \leq \beta - \eta/2$ in $[A, +\infty)$ by virtue of the sixth assertion in (3.29), and since $\underline{\varphi}_{\beta-\eta, \eta/2} \leq 1 - \eta/2$ in \mathbb{R} , it resorts from the last two formulas that

$$u(\tau(x_{2n+1} + x_{2n+1}^{3/4}), \cdot) \geq \underline{\varphi}_{\beta-\eta, \eta/2}(\cdot - (x_{2n+1} - \varepsilon' x_{2n+1} - Y - 2A) + A) \text{ in } (-\infty, x_{2n+2} - x_{2n+2}^{3/4}].$$

On the other hand, there holds $x_{2n+1} \leq x_{2n+2} - 2x_{2n+2}^{3/4} \leq x_{2n+2} - x_{2n+2}^{3/4}$ and

$$(x_{2n+2} - 2x_{2n+2}^{3/4}) + (x_{2n+2} - 2x_{2n+2}^{3/4})^{3/4} \leq x_{2n+2} - x_{2n+2}^{3/4}.$$

It follows then from (3.37) applied at $x = x_{2n+2} - 2x_{2n+2}^{3/4}$ and from the monotonicity of τ that

$$\forall t \in [0, \tau(x_{2n+2} - 2x_{2n+2}^{3/4})], \quad u(t, x_{2n+2} - x_{2n+2}^{3/4}) \geq v(t, x_{2n+2} - x_{2n+2}^{3/4}) \geq \beta - \frac{\eta}{2}.$$

Set

$$T' = \min \left(\tau(x_{2n+2} - 2x_{2n+2}^{3/4}), \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x_{2n+2} - x_{2n+2}^{3/4} - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 2A}{\underline{c}_{\beta-\eta, \eta/2}} \right).$$

Observe that $T' \in [\tau(x_{2n+1} + x_{2n+1}^{3/4}), \tau(x_{2n+2} - 2x_{2n+2}^{3/4})]$ and that

$$\underline{\varphi}_{\beta-\eta, \eta/2} \left(x_{2n+2} - x_{2n+2}^{3/4} - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 3A - \underline{c}_{\beta-\eta, \eta/2} (t - \tau(x_{2n+1} + x_{2n+1}^{3/4})) \right) \leq \beta - \frac{\eta}{2}$$

for all $t \in [\tau(x_{2n+1} + x_{2n+1}^{3/4}), T']$, because of the sixth assertion in (3.29) and the monotonicity of $\underline{\varphi}_{\beta-\eta, \eta/2}$. Eventually, since $\underline{\varphi}_{\beta-\eta, \eta/2}(x - \underline{c}_{\beta-\eta, \eta/2}t)$ is a subsolution of the equation satisfied by u , the maximum principle applied in $[\tau(x_{2n+1} + x_{2n+1}^{3/4}), T'] \times (-\infty, x_{2n+2} - x_{2n+2}^{3/4}]$ yields

$$u(t, x) \geq \underline{\varphi}_{\beta-\eta, \eta/2} \left(x - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 3A - \underline{c}_{\beta-\eta, \eta/2} (t - \tau(x_{2n+1} + x_{2n+1}^{3/4})) \right) \quad (3.53)$$

for all $(t, x) \in [\tau(x_{2n+1} + x_{2n+1}^{3/4}), T'] \times (-\infty, x_{2n+2} - x_{2n+2}^{3/4}]$.

Pick any $x \in [x_{2n+1} + x_{2n+1}^{3/4}, x_{2n+2} - 2x_{2n+2}^{3/4}]$, set

$$t(x) = \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 3A}{\underline{c}_{\beta-\eta, \eta/2}}$$

and assume that $\tau(x) > t(x)$. Then $t(x) \leq \tau(x) \leq \tau(x_{2n+2} - 2x_{2n+2}^{3/4})$ since τ is increasing. On the other hand,

$$\begin{aligned} t(x) &\leq \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x_{2n+2} - 2x_{2n+2}^{3/4} - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 3A}{\underline{c}_{\beta-\eta, \eta/2}} \\ &\leq \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x_{2n+2} - x_{2n+2}^{3/4} - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 2A}{\underline{c}_{\beta-\eta, \eta/2}} \end{aligned} \quad (3.54)$$

since $-x_{2n+2}^{3/4} + A \leq 0$, because of (3.35). Thus, $t(x) \leq T'$. Observe also that

$$t(x) \geq \tau(x_{2n+1} + x_{2n+1}^{3/4})$$

by definition of $t(x)$ and since $x \geq x_{2n+1} + x_{2n+1}^{3/4} \geq x_{2n+1}$ and all parameters Y and A are nonnegative. One can then apply (3.53) at the point $(t(x), x)$ and one gets

$$u(t(x), x) \geq \underline{\varphi}_{\beta-\eta, \eta/2}(0) = \theta,$$

whence $\tau(x) \leq t(x)$, owing to the definition of $\tau(x)$. As a consequence, the assumption $\tau(x) > t(x)$ cannot hold and one concludes that

$$\begin{aligned} \forall x \in [x_{2n+1} + x_{2n+1}^{3/4}, x_{2n+2} - 2x_{2n+2}^{3/4}], \\ \tau(x) &\leq t(x) = \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x - x_{2n+1} + \varepsilon' x_{2n+1} + Y + 3A}{\underline{c}_{\beta-\eta, \eta/2}} \\ &\leq \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x + Y + 3A}{\underline{c}_{\beta-\eta, \eta/2}}. \end{aligned} \quad (3.55)$$

Step 3: estimate of $\tau(z_{2n+1})$ and lower bound of u on the left of z_{2n+1} . It follows from (3.21), (3.50), (3.55) and the inequality

$$x_{2n+1} + x_{2n+1}^{3/4} \leq z_{2n+1} \leq x_{2n+2} - 2x_{2n+2}^{3/4}$$

that

$$\begin{aligned} \frac{z_{2n+1} - 3x_{2n+1}}{\bar{c}_{\beta+\eta,\eta}} &\leq \frac{z_{2n+1} - 3x_{2n+1}}{\bar{c}_{\beta+\eta,\eta}} + \tau(x_{2n+1}) \leq \dots \\ \dots &\leq \tau(z_{2n+1}) \leq \frac{x_{2n+1} + x_{2n+1}^{3/4} + M}{c_\alpha} + \frac{z_{2n+1} + Y + 3A}{\underline{c}_{\beta-\eta,\eta/2}}, \end{aligned} \quad (3.56)$$

provided that (3.51) holds. Furthermore,

$$\begin{aligned} \tau(x_{2n+1} + x_{2n+1}^{3/4}) &\leq \tau(z_{2n+1}) \leq \tau(x_{2n+2} - 2x_{2n+2}^{3/4}) \leq t(x_{2n+2} - 2x_{2n+2}^{3/4}) \leq \dots \\ \dots &\leq \tau(x_{2n+1} + x_{2n+1}^{3/4}) + \frac{x_{2n+2} - x_{2n+2}^{3/4} - x_{2n+1} + \varepsilon'x_{2n+1} + Y + 2A}{\underline{c}_{\beta-\eta,\eta/2}} \end{aligned}$$

from (3.54) and (3.55). Thus, $\tau(z_{2n+1}) \leq T'$. The inequality (3.53) and the monotonicity of $\underline{\varphi}_{\beta-\eta,\eta/2}$ then imply that, for all $x \in (-\infty, z_{2n+1} - \varepsilon'z_{2n+1} - Y] \subset (-\infty, x_{2n+2} - x_{2n+2}^{3/4}]$,

$$\begin{aligned} u(\tau(z_{2n+1}), x) &\geq \underline{\varphi}_{\beta-\eta,\eta/2}(z_{2n+1} - \varepsilon'z_{2n+1} - x_{2n+1} + \varepsilon'x_{2n+1} + 3A \dots \\ &\quad \dots - \underline{c}_{\beta-\eta,\eta/2}(\tau(z_{2n+1}) - \tau(x_{2n+1} + x_{2n+1}^{3/4}))). \end{aligned}$$

But it follows from (3.21), (3.28), (3.35) and (3.56) that

$$\begin{aligned} &z_{2n+1} - \varepsilon'z_{2n+1} - x_{2n+1} + \varepsilon'x_{2n+1} + 3A - \underline{c}_{\beta-\eta,\eta/2}(\tau(z_{2n+1}) - \tau(x_{2n+1} + x_{2n+1}^{3/4})) \\ &\leq \left(1 - \frac{\underline{c}_{\beta-\eta,\eta/2}}{\bar{c}_{\beta+\eta,\eta}} - \frac{\varepsilon}{2}\right) z_{2n+1} + 3A + \frac{3x_{2n+1} \underline{c}_{\beta-\eta,\eta/2}}{\bar{c}_{\beta+\eta,\eta}} + \frac{c_\beta}{c_\alpha}(x_{2n+1} + x_{2n+1}^{3/4} + M) \\ &\leq \left(\frac{\varepsilon}{8} - \frac{\varepsilon}{2}\right) z_{2n+1} + \left(6 + \frac{3c_\beta}{c_\alpha}\right) x_{2n+1} \leq -\frac{\varepsilon}{8} z_{2n+1} \leq -A, \end{aligned}$$

whence

$$u(\tau(z_{2n+1}), \cdot) \geq \underline{\varphi}_{\beta-\eta,\eta/2}(-A) \geq 1 - \frac{3\eta}{4} \geq 1 - \eta \quad \text{in } (-\infty, z_{2n+1} - \varepsilon'z_{2n+1} - Y]. \quad (3.57)$$

from the seventh assertion in (3.29).

Step 4: estimate of u on the left of x_{2n+2} at time $\tau(x_{2n+2} - 2x_{2n+2}^{3/4})$. With similar arguments as above, one has

$$\tau(x_{2n+1} + x_{2n+1}^{3/4}) \leq \tau(x_{2n+2} - 2x_{2n+2}^{3/4}) \leq T'$$

and

$$x_{2n+2} - \rho\varepsilon x_{2n+2} - Y \leq x_{2n+2} - x_{2n+2}^{3/4},$$

since $Y \geq 0$ in (3.51) and since $\rho\varepsilon x_{2n+2}^{1/4} \geq 1$ from (3.35). Thus, inequality (3.53) and the monotonicity of $\underline{\varphi}_{\beta-\eta,\eta/2}$ imply that, for all $x \in (-\infty, x_{2n+2} - \rho\varepsilon x_{2n+2} - Y]$,

$$u(\tau(x_{2n+2} - 2x_{2n+2}^{3/4}), x) \geq \underline{\varphi}_{\beta-\eta,\eta/2} \left(x_{2n+2} - \rho\varepsilon x_{2n+2} - x_{2n+1} + \varepsilon' x_{2n+1} + 3A \cdots \right. \\ \left. \cdots - \underline{c}_{\beta-\eta,\eta/2} (\tau(x_{2n+2} - 2x_{2n+2}^{3/4}) - \tau(x_{2n+1} + x_{2n+1}^{3/4})) \right).$$

But, as in Step 3, it follows from (3.21), (3.28), (3.35) and (3.50) applied at $x_{2n+2} - 2x_{2n+2}^{3/4}$, that

$$x_{2n+2} - \rho\varepsilon x_{2n+2} - x_{2n+1} + \varepsilon' x_{2n+1} + 3A - \underline{c}_{\beta-\eta,\eta/2} (\tau(x_{2n+2} - 2x_{2n+2}^{3/4}) - \tau(x_{2n+1} + x_{2n+1}^{3/4})) \\ \leq \left(\frac{\rho\varepsilon}{4} + \frac{\rho\varepsilon}{2} - \rho\varepsilon \right) x_{2n+2} + \frac{2\underline{c}_{\beta-\eta,\eta/2} x_{2n+2}^{3/4}}{\bar{c}_{\beta+\eta,\eta}} \leq -\frac{\rho\varepsilon}{4} x_{2n+2} + 2x_{2n+2}^{3/4} \leq -A,$$

whence

$$u(\tau(x_{2n+2} - 2x_{2n+2}^{3/4}), \cdot) \geq \underline{\varphi}_{\beta-\eta,\eta/2}(-A) \geq 1 - \frac{3\eta}{4} \text{ in } (-\infty, x_{2n+2} - \rho\varepsilon x_{2n+2} - Y]. \quad (3.58)$$

Step 5: estimates of $\tau(x_{2n+2})$ and of u on the left of x_{2n+2} at time $\tau(x_{2n+2})$. Remember that $u(\tau(x_{2n+2} - 2x_{2n+2}^{3/4}), \cdot) \geq \alpha$ in \mathbb{R} , that $\underline{\varphi}_{\alpha-\eta,3\eta/4} \leq 1 - 3\eta/4$ in \mathbb{R} and that $\underline{\varphi}_{\alpha-\eta,3\eta/4} \leq \alpha$ in $[A, +\infty)$ from the eighth assertion in (3.29). Thus,

$$\forall x \in \mathbb{R}, \quad u(\tau(x_{2n+2} - 2x_{2n+2}^{3/4}), x) \geq \underline{\varphi}_{\alpha-\eta,3\eta/4} (x - (x_{2n+2} - \rho\varepsilon x_{2n+2} - Y) + A)$$

from (3.58), provided that (3.51) holds. Since $\underline{\varphi}_{\alpha-\eta,3\eta/4}(x - \underline{c}_{\alpha-\eta,3\eta/4}t)$ is a subsolution of the equation satisfied by u , the maximum principle yields

$$u(t, x) \geq \underline{\varphi}_{\alpha-\eta,3\eta/4} \left(x - x_{2n+2} + \rho\varepsilon x_{2n+2} + Y + A - \underline{c}_{\alpha-\eta,3\eta/4} (t - \tau(x_{2n+2} - 2x_{2n+2}^{3/4})) \right) \quad (3.59)$$

for all $(t, x) \in [\tau(x_{2n+2} - 2x_{2n+2}^{3/4}), +\infty) \times \mathbb{R}$. In particular, at

$$t = \tau(x_{2n+2}) \geq \tau(x_{2n+2} - 2x_{2n+2}^{3/4}) \quad \text{and} \quad x = x_{2n+2},$$

one gets that

$$\theta = u(\tau(x_{2n+2}), x_{2n+2}) \geq \underline{\varphi}_{\alpha-\eta,3\eta/4} \left(\rho\varepsilon x_{2n+2} + Y + A - \underline{c}_{\alpha-\eta,3\eta/4} (\tau(x_{2n+2}) - \tau(x_{2n+2} - 2x_{2n+2}^{3/4})) \right),$$

whence

$$\tau(x_{2n+2}) \leq \tau(x_{2n+2} - 2x_{2n+2}^{3/4}) + \frac{\rho\varepsilon x_{2n+2} + Y + A}{\underline{c}_{\alpha-\eta,3\eta/4}}$$

since $\underline{\varphi}_{\alpha-\eta,3\eta/4}$ is decreasing and $\underline{\varphi}_{\alpha-\eta,3\eta/4}(0) = \theta$. It follows then from (3.21), (3.50) and (3.55) applied at $x = x_{2n+2} - 2x_{2n+2}^{3/4}$, that

$$\frac{x_{2n+2} - 3x_{2n+1}}{\bar{c}_{\beta+\eta,\eta}} \leq \frac{x_{2n+2} - 3x_{2n+1}}{\bar{c}_{\beta+\eta,\eta}} + \tau(x_{2n+1}) \leq \tau(x_{2n+2}) \leq \cdots \\ \cdots \leq \frac{x_{2n+1} + x_{2n+1}^{3/4} + M}{c_\alpha} + \frac{x_{2n+2} - 2x_{2n+2}^{3/4} + Y + 3A}{\underline{c}_{\beta-\eta,\eta/2}} + \frac{\rho\varepsilon x_{2n+2} + Y + A}{\underline{c}_{\alpha-\eta,3\eta/4}}. \quad (3.60)$$

Lastly, inequality (3.59) applied at $t = \tau(x_{2n+2}) \geq \tau(x_{2n+2} - 2x_{2n+2}^{3/4})$ implies that

$$\begin{aligned} \forall x \leq x_{2n+2} - \varepsilon' x_{2n+2} - Y, \quad u(\tau(x_{2n+2}), x) &\geq \varphi_{\alpha-\eta, 3\eta/4} \left(-\frac{\varepsilon}{2} x_{2n+2} + \rho \varepsilon x_{2n+2} + A \right) \\ &\geq \varphi_{\alpha-\eta, 3\eta/4}(-A) \geq 1 - \eta \end{aligned} \quad (3.61)$$

since $\varphi_{\alpha-\eta, 3\eta/4}$ is decreasing and because of the last assertion in (3.29) and because of (3.35).

Conclusion of the proof of Lemma 3.1

As already underlined, for all $N \geq N_0$, the estimates

$$\begin{cases} u(\tau(x_{2n}, \cdot)) \leq \alpha + \eta & \text{in } [x_{2n} + \varepsilon x_{2n}, x_{2n+1} - \varepsilon x_{2n+1}], \\ u(\tau(z_{2n}, \cdot)) \leq \alpha + \eta & \text{in } [z_{2n} + \varepsilon z_{2n}, x_{2n+1} - \varepsilon x_{2n+1}], \\ |u(\tau(x_{2n+1}, \cdot)) - \beta| \leq \eta & \text{in } [x_{2n+1} + \varepsilon x_{2n+1}, x_{2n+2} - \varepsilon x_{2n+2}], \\ |u(\tau(z_{2n+1}, \cdot)) - \beta| \leq \eta & \text{in } [z_{2n+1} + \varepsilon z_{2n+1}, x_{2n+2} - \varepsilon x_{2n+2}] \end{cases}$$

follow from (3.35), (3.38) and (3.39).

Now, since $u(\tau(x_{2N_0}), -\infty) = 1$, there exists a nonnegative real number $X_{2N_0} = X_{2N_0}(\varepsilon, \eta) \geq 0$ such that

$$u(\tau(x_{2N_0}, \cdot)) \geq 1 - \eta \quad \text{in } (-\infty, x_{2N_0} - \varepsilon' x_{2N_0} - X_{2N_0}].$$

In other words, X_{2N_0} plays the role of X in (3.42), with $n = N_0$. It follows then from (3.48) and (3.49) that

$$u(\tau(z_{2N_0}, \cdot)) \geq 1 - \frac{\eta}{4} \geq 1 - \eta \quad \text{in } (-\infty, z_{2N_0} - \varepsilon' z_{2N_0} - X_{2N_0}] \quad (3.62)$$

and that

$$u(\tau(x_{2N_0+1}, \cdot)) \geq 1 - \frac{\eta}{4} \geq 1 - \eta \quad \text{in } (-\infty, x_{2N_0+1} - \varepsilon' x_{2N_0+1} - X_{2N_0}]. \quad (3.63)$$

As a consequence, property (3.51) is fulfilled with $Y = X_{2N_0}$ and $n = N_0$. It follows then from (3.57) and (3.61) that

$$u(\tau(z_{2N_0+1}, \cdot)) \geq 1 - \frac{3\eta}{4} \geq 1 - \eta \quad \text{in } (-\infty, z_{2N_0+1} - \varepsilon' z_{2N_0+1} - X_{2N_0}] \quad (3.64)$$

and that

$$u(\tau(x_{2N_0+2}, \cdot)) \geq 1 - \eta \quad \text{in } (-\infty, x_{2N_0+2} - \varepsilon' x_{2N_0+2} - X_{2N_0}]. \quad (3.65)$$

By an immediate induction, one gets that the above four estimates (3.62)-(3.65) hold for all $n \geq N_0$. Hence, since $\varepsilon' = \varepsilon/2 > 0$ and $\lim_{m \rightarrow +\infty} x_m = \lim_{m \rightarrow +\infty} z_m = +\infty$, there exists an integer $N_1 = N_1(\varepsilon, \eta) \geq N_0$ such that

$$\forall n \geq N_1, \quad \begin{cases} u(\tau(x_{2n}, \cdot)) \geq 1 - \eta & \text{in } (-\infty, x_{2n} - \varepsilon x_{2n}], \\ u(\tau(z_{2n}, \cdot)) \geq 1 - \eta & \text{in } (-\infty, z_{2n} - \varepsilon z_{2n}], \\ u(\tau(x_{2n+1}, \cdot)) \geq 1 - \eta & \text{in } (-\infty, x_{2n+1} - \varepsilon x_{2n+1}], \\ u(\tau(z_{2n+1}, \cdot)) \geq 1 - \eta & \text{in } (-\infty, z_{2n+1} - \varepsilon z_{2n+1}]. \end{cases}$$

Furthermore, by an immediate induction, it also follows that the estimates (3.46), (3.56) and (3.60) hold for all $n \geq N_0$ with $X = Y = X_{2N_0}$. Therefore, since $\lim_{m \rightarrow +\infty} x_m = \lim_{m \rightarrow +\infty} x_{m+1}/x_m = \lim_{m \rightarrow +\infty} z_m/x_m = +\infty$, one gets that

$$\left\{ \begin{array}{l} c_\alpha \leq \liminf_{n \rightarrow +\infty} \frac{z_{2n}}{\tau(z_{2n})} \leq \limsup_{n \rightarrow +\infty} \frac{z_{2n}}{\tau(z_{2n})} \leq \bar{c}_{\alpha+\eta,\eta}, \\ c_\alpha \leq \liminf_{n \rightarrow +\infty} \frac{x_{2n+1}}{\tau(x_{2n+1})} \leq \limsup_{n \rightarrow +\infty} \frac{x_{2n+1}}{\tau(x_{2n+1})} \leq \bar{c}_{\alpha+\eta,\eta}, \\ \underline{c}_{\beta-\eta,\eta/2} \leq \liminf_{n \rightarrow +\infty} \frac{z_{2n+1}}{\tau(z_{2n+1})} \leq \limsup_{n \rightarrow +\infty} \frac{z_{2n+1}}{\tau(z_{2n+1})} \leq \bar{c}_{\beta+\eta,\eta}, \\ \left(\underline{c}_{\beta-\eta,\eta/2}^{-1} + \rho \varepsilon \underline{c}_{\alpha-\eta,3\eta/4}^{-1} \right)^{-1} \leq \liminf_{n \rightarrow +\infty} \frac{x_{2n+2}}{\tau(x_{2n+2})} \leq \limsup_{n \rightarrow +\infty} \frac{x_{2n+2}}{\tau(x_{2n+2})} \leq \bar{c}_{\beta+\eta,\eta}. \end{array} \right.$$

Because of (3.28), there exists an integer $N = N(\varepsilon, \eta) \geq N_1$ such that

$$\forall n \geq N, \quad \left| \frac{x_{2n}}{\tau(x_{2n})} - c_\beta \right| + \left| \frac{z_{2n}}{\tau(z_{2n})} - c_\alpha \right| + \left| \frac{x_{2n+1}}{\tau(x_{2n+1})} - c_\alpha \right| + \left| \frac{z_{2n+1}}{\tau(z_{2n+1})} - c_\beta \right| \leq \varepsilon.$$

That completes the proof of Lemma 3.1. \square

Remark 3.3 The behaviour of the solution u in the region where it is less than θ is close in some sense to that of the solution v of the heat equation, as expected. The function v oscillates at large time between α and β , infinitely many times: such a nontrivial dynamics is well-known for the heat equation, see [14]. However, the difficulty in the above proof came from the nonlinear reaction term $f(u)$ and from the estimates of the position and average speed of the solution u as time runs.

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