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Résumés
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Conférences plénières

Shift radix system: A bridge between number theory and symbolic dynamics

Shigeki Akiyama
Niigata

Shift radix system is a simple and deterministic dynamics on the lattice $\mathbb{Z}^n$, which is a generalization of linear recurrence. It is intimately connected to construction of good number systems as well as Markov partition of toral automorphism. I shall start with the definition, connection to other areas and discuss recent developments.

On the decimal expansion of an algebraic number

Yann Bugeaud
Strasbourg

It is commonly expected that the decimal expansion of an irrational algebraic number behaves, in many respects, like the one of almost all real numbers. For instance, it should include every finite block of digits from $\{0, \ldots, 9\}$. We are very far away from establishing such a strong assertion. However, there has been some recent progress. For instance, in a joint work with Adamczewski, we proved that, for C arbitrary and n large, at least Cn such blocks of length n do appear in the decimal expansion of every irrational algebraic number. We discuss this and related results.

Beta-expansions: Old and New

Karma Dajani
Utrecht

We give an overview of some of the old and new results describing the ergodic and arithmetic properties of algorithms generating expansions to non-integer base.

Répartitions des valeurs des fonctions arithmétiques réelles

Jean-Marc Deshouillers
Bordeaux

La fonction $\varphi(n)/n$ a une fonction de répartition G continue sur $[0,1]$ strictement croissante, purement singulière (Schoenberg, Erdös... dans les années 1930). Pour tout entier n, la fonction G n'est pas dérivable à droite au point $\varphi(n)/n$ (Erdös, 1974). Il se trouve que cette question a retrouvé un regain d'intérêt récent (travaux de Vincent Toulmonde [2005-2006], question de Hendrik Lenstra [2007]). Motivé par la question de Lenstra, Mehdi Hassani et moi avons repris ce type de question ; notre méthode d'attaque permet d'aborder des questions plus générales [e.g. répartition des valeurs de $\varphi(p-1)/(p-1)$, où p est premier et diverses variantes sur ce thème].

Digital expansions, prime numbers and uniform distribution modulo 1

Michael Drmota
Vienna

The purpose of this talk is to survey relations between prime numbers and digital expansions, in particular we focus on (uniform) distribution results on the sum-of-digits function $s_q(n)$ when n varies over primes $p \leq x$ or when the basis is a prime number. We discuss Newman's phenomenon of the Thue-Morse sequence, recent developments on Gelfond's problems (due to Mauduit and Rivat) and some extensions of these results.

On statistical testing of True Random Number Generators

Yuri Golubev
Marseille

The main goal in this talk in to provide a concise introduction to statistical tests for TRNG. We discuss two large classes of statistical tests: the so-called universal tests and generator oriented tests. The first one includes Maurer's family of tests and universal source coding tests. On the other hand, the second one is based to the information about the physical structure of TRNG. The talk is focused on jitter noise TRNGs on and testing with the help of dictionaries.

Distribution properties of digital functions over the Gaussian integers

Peter J. Grabner
Graz

We present three conceptually different methods to prove distribution results for block additive functions with respect to radix expansions of the Gaussian integers. Based on generating function approaches we obtain a central limit theorem and asymptotic expansions for the moments. Furthermore, these generating functions as well as ergodic skew products are used to prove uniform distribution in residue classes and modulo 1.

Quelques questions anciennes et nouvelles sur la répartition modulo 1. Old and new revisited

Jean-Pierre Kahane
Paris

Les “questions anciennes” se rattachent aux procédés de moyennisation, aux suites mal réparties, aux distributions non uniformes et aux usages de l’analyse de Fourier, dans l’esprit des articles des années 1963 à 1965 que j’ai faits en collaboration avec Salem ou Helson, ou seul. La plus simple, non posée dans ces articles, sera présentée ici, avec quelques bribes de solution : il s’agit de toutes les distributions possibles des suites multiples d’une suite donnée. Les “questions nouvelles” se rattachent au travail récent que j’ai fait avec Yitzhak Katznelson sur le comportement dans le groupe de Bohr (dual du cercle discret) de suites d’entiers aléatoires suivant que ce sont ou non des suites de Sidon. Les plus importantes sont très anciennes : par exemple, un Sidon peut-il être dense dans le groupe de Bohr ? Un sous-produit de notre étude concerne la distribution des suites f(n)/n ; je donnerai le résultat, qui est simple, et peut s’exprimer en parlant de “discrépance logarithmique”. La question est de savoir ce que veut dire cette notion et si elle a un intérêt. Suivant la marche du temps et l’intérêt des auditeurs il sera possible de se borner à l’un de ces deux sujets. L’exposé sera en français ; des transparents en anglais donneront l’essentiel.

On a theorem of Daboussi

Imre Kátai
Budapest

Daboussi [1] proved that for irrational α,

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{n\le x} f(n)e(n\alpha)\right| = 0, \tag{1}$$

where $e(y)=e^{2\pi iy}$, and $\mathcal{M}_1$ is the set of those multiplicative functions f taking on complex values and satisfying $|f(n)|\le 1$ $(n\in\mathbb{N})$.

This remarkable theorem has been extended in different directions, by Daboussi, Delange, Indlekofer, Montgomery and Vaughan, N.L. Bassily, De Koninck, Kátai, Huixue Lao.

Let $\mathcal{T}$ be the set of those functions $t:\mathbb{N}\to\mathbb{R}$ for which $x_n := t(n)+h(n)$ is uniformly distributed $\mod 1$ for every $h\in\mathcal{A}$, where $\mathcal{A}$ is the set of additive functions. Daboussi's theorem is equivalent to the assertion that $t(n)=\alpha n$ belongs to $\mathcal{T}$, if α is irrational.

One can give a very simple proof of (1), by using the Turán - Kubilius inequality, and this method gives a wide generalization of (1). For example, if $P(n) := \alpha_k n^k + \ldots + \alpha_1 n$, and at least one coefficient of $\alpha_1,\ldots,\alpha_k$ is irrational, then $P(n)\in\mathcal{T}$.

A natural conjecture is that $\alpha\sigma(n)$, $\alpha\varphi(n)$ belong to $\mathcal{T}$ if α is irrational. Presently we can prove it for those α which cannot be approximated with rational numbers very well.

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Good distribution of the values of sparse polynomials and joint distribution of monomials modulo a prime

Sergei Konyagin
Moscow

Let $x_1,\ldots,x_N$ be a finite sequence of elements from the interval $[0,1)$ and $E\subset[0,1)$. Denote

$$A(E) = |\{i\in\{1,\ldots,N\} : x_i\in E\}|.$$

The number

$$D_N = D_N(x_1,\ldots,x_N) = \sup_{0\le\alpha<\beta\le 1}\left|\frac{A([\alpha,\beta))}{N} - (\beta-\alpha)\right|$$

is called the discrepancy of the given sequence (see [**?**]).

Given a prime p, define the field of order p:

$$\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}.$$

Also, let $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$. For a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$, let the discrepancy of f be defined as

$$D(p, f) = D_p(f(0)/p, \dots, f(p-1)/p),$$

where we identify the residue $f(i) \in \mathbb{Z}_p$ with its representative from the set $\{0, \dots, p-1\}$. We will study the discrepancy of sparse polynomials, that is, polynomials with a bounded number of terms. The estimates for discrepancy of monomials follow from results of [**?**] or from sharper results of [**?**].

Assume that we have a sequence $\{p_j\}$ of primes and a sequence $\{f_j\}$ of polynomials where f_j maps the field $\mathbb{Z}_{p_j}$ into itself (this will be written as $f_j \in \mathbb{Z}_{p_j}[x]$). We say that a sequence $\{f_j\}$ is well-distributed if there is an $\varepsilon > 0$ such that for all j

$$D(p_j, f_j) \le p_j^{-\varepsilon}.$$

Our first purpose is to characterize well-distributed sequences of sparse polynomials. We strongly use and refine the main result of [**?**].

Next, in a joint paper with J. Bourgain, also related to [**?**], we study a joint distribution of a fixed number of monomials modulo a prime.

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Distribution functions of ratio sequences

Ladislav Mišík[1]
Ostrava

Let $X = \{x_1, x_2, \dots\} \subset \mathbb{N}$. The sequence

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

is called *the ratio block sequence* of the set X. It is formed by blocks $X_1, X_2, \dots, X_n, \dots$ where the n-th block is

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

By *distribution function* we mean any function $f\colon [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$ and f is nondecreasing in $[0,1]$.

Denote by $G(x_m/x_n)$ the set of all distribution functions $g(x)$ of the sequence (**??**).

We can attach to each term X_n of the sequence of blocks (X_n) the following distribution function

$$F(X_n, x) = \frac{\#\{i : i \le n,\ \frac{x_i}{x_n} \le x\}}{n}.$$

Denote $G(X_n)$ the set of all distribution functions of the sequence of single blocks (X_n). It is the set of all functions $g(x)$ for which there exists an increasing sequence of indices (n_k) such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

almost everywhere (abbrev. a.e.) in $[0,1]$. Study of the set $G(X_n)$ began few years ago in paper [**?**].

In this talk we recall the most important known properties of $G(X_n)$ and we discuss some extensions of these properties. We present also some new results on sets $G(X_n)$ and $G(x_m/x_n)$ and we mention some open problems in the theory.

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Polynomials and ergodic theorems

Radhakrishnan Nair
Liverpool

Let (X, β, μ, T) be a dynamical system. By this we mean (X, β, μ) is a measure space and if $T^{-1}A = \{x : Tx \in A\}$ then $T^{-1}A \in \beta$ and $\mu(T^{-1}A) = \mu(A)$ for all $A \in \beta$. To a sequence of natural numbers $\mathbf{a} = (a_n)_{n=1}^{\infty} \subseteq \mathbf{N}$, a dynamical system (X, β, μ, T) , and β measurable function $f : X \to \mathbf{C}$ associate the averages

$$A_N f(x) = \frac{1}{N} \sum_{k=1}^{N} f(T^{a_k} x). \qquad (N = 1, 2, \cdots)$$

We will say a sequence $\mathbf{a}$ is L^p *good universal* if for each dynamical system (X, β, μ, T) and each $f \in L^p(X, \beta, \mu)$ the limit $\lim_{N\to\infty} A_N f(x)$ exists μ almost everywhere.

The pointwise ergodic theorem says if $a_n = n$ $(n = 1, 2, \cdots)$ then $\mathbf{a}$ is L^1 good universal. A. Bellow, H. Furstenberg and M. Herman asked if the sequence $a_n = n^2$ $(n = 1, 2, \cdots)$ is L^2 good universal. The answer to this question was shown to be yes by J. Bourgain [Bo1],[Bo3]. In fact he showed that for any natural number r if $a_n = n^r$ $(n = 1, 2, \cdots)$ that $\mathbf{a}$ is L^p good universal for every $p > 1$. Later he showed that if $a_n = p_n$ (the n^{th} rational prime) the sequences $\mathbf{a}$ is L^p good universal with $p > \frac{1}{2}(1 + \sqrt{3})$[Bo2]. This hypothesis was weakened to $p > 1$ by M. Wierdl [W].

Later the author showed that if ϕ is a polynomial with rational coefficients mapping the natural numbers to themselves then $\phi(p_n)$ is L^p good universal also for $p > 1$[Na1][Na2]. What happens for $p = 1$ is an open question. In addition the author gave a general condition on $\mathbf{a}$ involving the hypothesis that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i a_n \theta} = 0,$$

for non-integer θ, to ensure it is L^p good universal for $p > 1$[Na6]. This result has a number of applications [AN][Na3] [Na4][Na5]. Bourgain also offered a putative proof that if $T_1, \cdots, T_l$ are commuting maps of the measure space (X, β, μ) and $f \in L^p(X, \beta, \mu)$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T_1^k \cdots T_l^{k^l} x) \tag{1}$$

exits μ almost everywhere [Bo1]. Let $\rho(k) = \alpha_l k^l + \cdots + \alpha_1 k$. The argument in [Bo1] suggested as a proof of (1) relies fundamentally on estimates for the Weyl sum

$$K_N(\alpha) = \sum_{k=1}^{N} e^{2\pi i \rho(k)}$$

on a sumset of $\alpha = (\alpha_l, \cdots, \alpha_1) \in [0, 1)^l$ called the minor arcs of level N. Unfortunately this estimate does not seem to be correctly proved in [Bo1]. A valid proof of this estimate appears in [Na9] completing the proof of (1). These ideas can also be used to show that if $f \in L^p$ $p > 1$ then setting

$$Sf(x) = (\sum_{k \geq 1} |A_{N+1}f(x) - A_N f(x)|^2)^{\frac{1}{2}}$$

there exist a constant $C > 0$ such that if $||f||_2 = (\int_X |f|^2 d\mu)^{\frac{1}{2}}$

$$||Sf||_2 \leq C||f||_2.$$

In the 1950's, motivated by H. Weyl's result that if ρ is a polynomial with one coefficient other than $\rho(0)$ irrational then $(\rho(k))_{k=1}^{\infty}$ is uniformly distributed modulo one, J.F. Koksma and R. Salem [KS] showed, subject to growth conditions on the Fourier coefficients of $f \in L^2([0,1))$, that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x + \rho(n)) = \int_0^1 f(x)dx, \tag{2}$$

almost everywhere. Some years ago R. C. Baker raised with the author, the question of whether the Fourier coefficient condition in Salem's theorem could be weakened. The same question was raised with me by M. Weber again in 2000. The result (2), subject to no Fourier series condition, follows now from (1).

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Inversive pseudorandom numbers: New generation methods and new tests

Harald Niederreiter
Singapore

Inversive generators form a family of nonlinear methods for the generation of uniform pseudorandom numbers. The name derives from the fact that inversive methods achieve nonlinearity by the use of multiplicative inverses in finite fields and residue class rings of the integers.

Because of certain deficiencies of the standard recursive inversive and digital inversive generators with regard to strong pseudorandomness properties, we introduce new inversive methods for generating pseudorandom numbers to overcome this difficulty. We also describe new and more powerful lattice tests for the assessment of structural properties of pseudorandom numbers. The new inversive pseudorandom numbers pass these strong lattice tests for a wide range of dimensions. In addition, binary sequences with strong pseudorandomness properties such as small correlation and well-distribution measures can be derived from the new generators.

This talk is based on joint papers with J. Rivat, A. Sárközy, and A. Winterhof.

Distribution functions of sequences

Oto Strauch [2]
Bratislava

Let x_n, $n = 1, 2, \dots$ be a sequence from the unit interval $[0, 1)$ and define:

- $A([0, x); N; x_n)$ is the number $n \leq N$ for which $x_n \in [0, x)$;
- $F_N(x) = \frac{A([0,x);N;x_n)}{N}$ is the step distribution function of $x_1, \dots, x_N$, while $F_N(1) = 1$;

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[2] This research was supported by the VEGA Grant No. 2/7138/27.

- $g(x)$ is a distribution function (abbreviating d.f.) of the sequence x_n, $n = 1, 2, \dots$ if an increasing sequence of positive integers $N_1, N_2, \dots$ exists such that $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ a.e. on $[0, 1]$;
- $G(x_n)$ is the set of all d.f.s of the given sequence x_n, $n = 1, 2, \dots$.

We shall identify the notion of the distribution of a sequence x_n with the set $G(x_n)$, i.e. the distribution of x_n is known if we know the set $G(x_n)$. The following methods can be used for computing $G(x_n)$:

- Directly by definition of $G(x_n)$ (see Th. **??**).
- Using $G(f(x_n)), G(h(x_n)), \dots$ for some mappings $f, h, \dots$ defined on $[0, 1]$ (see Th. **??**).
- Using $G((u_n, v_n))$, where $f(u_n, v_n) = x_n$ for some map f (see Th. **??**).
- Using the set $G(F)$ of all solutions $g(x)$ of $\int_0^1\int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$, where

$$\lim_N \frac{1}{N^2}\sum_{m,n=1}^{N} F(x_m, x_n) = 0$$

(see Th. **??**).

In this expository paper we present the following four theorems:

Theorem 1. *Let a real-valued function $f(x)$ be strictly increasing for $x \geq 1$ and let $f^{-1}(x)$ be its inverse function. Assume that* $\lim_{x\to\infty} f'(x) = 0$, $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$, $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)} = \psi(w)$ *for $w(k) \to w$, and $\psi(1) > 1$. Then every d.f. $g(x) \in G(\{f(n)\})$ has the form*

$$g_w(x) = \frac{\min(\psi(x), \psi(w)) - 1}{\psi(w)} + \frac{1}{\psi(w)}\frac{\psi(x) - 1}{\psi(1) - 1}, \text{ where } w \in [0, 1].$$

Theorem 2. *Let u_n and v_n be two sequences in $[0, 1)$ and $G((u_n, v_n))$ denote the set of all d.f.s of the two-dimensional sequence (u_n, v_n). If $x_n = u_n + v_n \bmod 1$, then every d.f. $g(t) \in G(x_n)$ has the form*

$$g(t) = \int_{0\leq x+y<t} 1.\mathrm{d}g(x, y) + \int_{1\leq x+y<1+t} 1.\mathrm{d}g(x, y), \text{where } g(x, y) \in G((x_n, y_n))$$

assuming that all the used Riemann-Stieltjes integrals exist.

Let $\underline{g}_F(x) = \inf_{g\in G(F)} g(x)$, $\overline{g}_F(x) = \sup_{g\in G(F)} g(x)$ and $\Omega(F)$ be the set of all points (x, y) lying between Graphs of $\underline{g}_F$ and $\overline{g}_F$.

Theorem 3. *Assume that $\underline{g}_F, \overline{g}_F \in G(F)$ and that for every $(x, y) \in \Omega(F)$ there exists unique $g \in G(F)$ such that $(x, y) \in \mathrm{Graph}(g)$. Then for every sequence $x_n \in [0, 1)$ we have $G(x_n) = G(F)$ if and only if*
(i) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} F(x_m, x_n) = 0,$
(ii) $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n = \int_0^1 (\overline{g}_F(x) - \underline{g}_F(x))\mathrm{d}x.$

Let $x_n = \{\xi(3/2)^n\}$ and denote $f(x) = \{2x\}$ and $h(x) = \{3x\}$. Since $f(x_n)$ and $h(x_n)$ formed the same sequence x_n, then every $g(x) \in G(x_n)$ satisfies the functional equation $g_f(x) = g_h(x)$ for every $x \in [0, 1]$, where $g_f(x) = g(f_1^{-1}(x)) + g(f_2^{-1}(x)) - g(1/2)$, $g_h(x) = g(h_1^{-1}(x)) + g(h_2^{-1}(x)) + g(h_3^{-1}(x)) - g(1/3) - g(2/3)$, with inverse functions $f_1^{-1}(x) = x/2, f_2^{-1}(x) = (x+1)/2$, $h_1^{-1}(x) = x/3, h_2^{-1}(x) = (x+1)/3, h_3^{-1}(x) = (x+2)/3$.

Theorem 4. *Let d.f.* g_1 *be absolutely continuous solution of* $g_f(x) = g_h(x)$ *for* $x \in [0,1]$. *Then the absolutely continuous d.f.* $g(x)$ *satisfies* $g_f(x) = g_h(x) = g_1(x)$ *if and only if* $g(x)$ *has the form*

$$g(x) = \begin{cases} \Psi(x), & \textit{for } x \in [0,1/6], \\ \Psi(1/6) + \Phi(x-1/6), & \textit{for } x \in [1/6,2/6], \\ \Psi(1/6) + \Phi(1/6) + g_1(1/3) - \Psi(x-2/6) & \\ +\Phi(x-2/6) - g_1(2x-1/3) + g_1(3x-1), & \textit{for } x \in [2/6,3/6], \\ 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) & \\ -\Psi(x-3/6) + g_1(2x-1), & \textit{for } x \in [3/6,4/6], \\ -\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_1(1/2) & \\ -\Phi(x-4/6) + g_1(2x-1), & \textit{for } x \in [4/6,5/6], \\ -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x-5/6) & \\ -\Phi(x-5/6) - g_1(2x-5/3) + g_1(3x-2), & \textit{for } x \in [5/6,1], \end{cases}$$

where $\Psi(x) = \int_0^x \psi(t)\mathrm{d}t$, $\Phi(x) = \int_0^x \phi(t)\mathrm{d}t$, *for* $x \in [0,1/6]$, *and* $\psi(t)$, $\phi(t)$ *are Lebesgue integrable functions on* $[0,1/6]$ *satisfying* $0 \le \psi(t) \le 2g_1'(2t)$, $0 \le \phi(t) \le 2g_1'(2t+1/3)$, $2g_1'(2t) - 3g_2'(3t+1/2) \le \psi(t) - \phi(t) \le -2g_1'(2t+1/3) + 3g_2'(3t)$, *for almost all* $t \in [0,1/6]$.

Note that Theorem **??** generalizes Koksma theorem in [**?**, p. 58, Th. 7.7]; Theorems **??** and **??** are in [**?**]; Theorem **??** is from [**?**] and Theorem **??** is in [**?**].

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