ON THE CLOSURE OF POLYNOMIALS
IN WEIGHTED SPACES OF FUNCTIONS
ON THE REAL LINE

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Abstract. We describe the closure of polynomials in weighted
$L^p$ spaces of functions on the real line for even log-convex non-
quasianalytic weights: a function in the space can be approxi-
imated by polynomials if and only if it extends to an entire func-
tion of zero exponential type. This completes the series of in-
vestigations by Alkhizer, Mergelyan, Khachatryan, Koosis, and
Levinson-McKean.

1. INTRODUCTION

Let $W : \mathbb{R} \rightarrow [1, +\infty)$ be an even continuous function such that
\begin{equation}
\lim_{|x| \to +\infty} \frac{\log W(x)}{\log |x|} = +\infty.
\end{equation}

Consider the Banach spaces
\begin{align*}
L^p_W &= \left\{ f : \int_{-\infty}^{+\infty} \frac{|f(x)|^p}{W(x)^p} \, dx < +\infty \right\}, \quad 1 \leq p < +\infty, \\
L^\infty_W &= \left\{ f : \operatorname{esssup}_{x \in \mathbb{R}} \frac{|f(x)|}{W(x)} < +\infty \right\}, \\
C_0^W &= \left\{ f \in C(\mathbb{R}) : \lim_{|x| \to +\infty} \frac{f(x)}{W(x)} = 0 \right\}.
\end{align*}

Then $C_0^W$ is a closed subspace of $L^\infty_W$. Denote by $\mathcal{P}$ the set of all
polynomials. Our condition (1.1) guarantees that $\mathcal{P} \subset C_0^W$, $\mathcal{P} \subset L^p_W$,
$1 \leq p \leq +\infty$. Denote by $\mathcal{P}^\circ_W$ the closure of $\mathcal{P}$ in $L^p_W$, $1 \leq p \leq +\infty$.
Clearly, the closure of $\mathcal{P}$ in $C_0^W$ coincides with $\mathcal{P}^\circ_W$. Let $\mathcal{E}$ be the set of
entire functions of zero exponential type, $\mathcal{E}_W^p = \mathcal{E} \cap L^p_W$, $1 \leq p \leq +\infty$,
$\mathcal{E}_W^\infty = \mathcal{E} \cap C_0^W$.

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The classical Bernstein approximation problem is to find out whether the polynomials are dense in \( C^0_W \). Suppose that an even function \( W \) satisfies (1.1) and is log-convex, that is, the function \( x \mapsto \log W(\exp x) \) is convex on \( \mathbb{R} \). It is well-known (see \([5], [9, \text{Section VIB}]\)) that if
\[
\int_{-\infty}^{+\infty} \frac{\log W(x)}{x^2 + 1} \, dx = +\infty,
\]
then \( \mathcal{P}_W^\infty = C^0_W \), \( \mathcal{P}_W^p = L^p_W \), \( 1 \leq p < +\infty \). Otherwise, if
\[
\int_{-\infty}^{+\infty} \frac{\log W(x)}{x^2 + 1} \, dx < +\infty,
\]
then the polynomials are not dense in \( C^0_W \), \( L^p_W \), \( 1 \leq p < +\infty \), and a well-known theorem going back to M. Riesz ([14, Sections 16, 17], see also \([1, \text{Section 4}], [13, \text{Theorem 6}], [9, \text{Section VIB}], [10, \text{Theorem 1.5}]\)) claims that \( \mathcal{P}_W^p \subset \mathcal{E}_W^p \), \( 1 \leq p \leq +\infty \). An important problem is to describe \( \mathcal{P}_W^p \). For related results and discussions see \([6, 7, 4, 8, 11], [9, \text{Section VIH}]\).

The aim of this paper is to prove the equality
\[
\mathcal{P}_W^p = \mathcal{E}_W^p, \quad 1 \leq p \leq +\infty,
\]
(1.3)
for even log-convex \( W \) satisfying conditions (1.1) and (1.2). As a consequence, for each log-convex \( W \) satisfying just condition (1.1), we obtain that every entire function of zero exponential type belonging to one of the spaces \( L^p_W \), \( 1 \leq p < +\infty \), \( C^0_W \), can be approximated in this space by polynomials.

Earlier, equalities (1.3) were obtained for \( p = 2, +\infty \) under more restrictive assumptions on the weight function \( W \). I. O. Khachatryan ([6, 7], see also \([9, \text{Section VIH.2}]\)) proved that \( \mathcal{P}_W^\infty = \mathcal{E}_W^\infty \) for \( W \) satisfying (1.1) and (1.2), such that
\[
W(x) = \sum_{n \geq 0} a_n x^{2n}, \quad a_0 \geq 1, \quad a_k \geq 0, \quad k \geq 1.
\]
(1.4)
P. Koosis \([8, \text{Theorem IV}]\) proved that \( \mathcal{P}_W^\infty = \mathcal{E}_W^\infty \) for log-convex \( W \) satisfying (1.1) and (1.2), such that for every \( \eta > 1 \) there exists \( C_\eta \) with \( x^2 W(x) \leq C_\eta W(\eta x) \), \( x \geq 0 \). N. Levinson and H. P. McKean \([11, \text{Section 10a}]\) proved that \( \mathcal{P}_W^2 = \mathcal{E}_W^2 \) (i) for \( W \) of the form (1.4), satisfying (1.1) and (1.2), and (ii) for \( C^1 \)-smooth log-convex \( W \) satisfying (1.2), such that
\[
\lim_{x \to +\infty} \frac{xW'(x)}{\log x \cdot W(x)} = +\infty.
\]

We should also mention here (a special case of) the theorem of L. de Branges (see \([3, \text{Theorem 1}], [12, \text{Theorem 8}], [9, \text{p.215}]\) on
weighted approximation by the linear combinations of the exponentials. Suppose that \( W \) satisfies conditions (1.1) and (1.2). For \( A > 0 \), \( 1 \leq p \leq +\infty \), denote by \( L^p_W(A) \) the closure in \( L^p_W \) of the linear combinations of \( e^{i\lambda x} \), \( -A \leq \lambda \leq A \). Then \( E^p_W = \bigcap_{A>0} L^p_W(A) \).

Combining this result with equality (1.3), for log-convex \( W \) satisfying (1.1), we obtain that \( P^p_W = \bigcap_{A>0} L^p_W(A) \); in other words, every function in \( L^p_W \) that can be approximated by finite linear combinations of \( e^{i\lambda x} \), \( -A \leq \lambda \leq A \), with arbitrary small \( A > 0 \), can also be approximated by polynomials.

Examples of (non-monotonic) even weights \( W \) such that \( P^\infty_W \subsetneq E^\infty_W \subsetneq C^0_W \), \( P^0_W \subsetneq E^0_W \subsetneq L^2_W \) are given in [7], [11, Section 11], [9, Section VI.3]. An open question remains: is it true that the equality (1.3) holds for even weight functions \( W \) just increasing on \( [0, +\infty) \) and satisfying (1.1) and (1.2)?

In the next section we give a proof of equality (1.3) for \( p = +\infty \) using a classical result of de Branges. Another proof of this fact (using a result by Khachatryan) and the proof of (1.3) for \( 1 \leq p < +\infty \) are given in Section 3. Section 4 contains the proofs of auxiliary statements.

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2. POLYNOMIAL APPROXIMATION IN \( C^0_W \)

**Theorem 2.1.** Suppose that \( W \) is an even log-convex function on the real line satisfying (1.1), (1.2). Then \( P^\infty_W = E^\infty_W \).

**Proof.** We can identify elements of \((C^0_W)^*\) with elements \( \mu \) of the space \( M(\mathbb{R}) \) of finite complex Borel measures \( \mathbb{R} \), with the duality being

\[
\langle \mu, f \rangle_{C^0_W} = \int_{-\infty}^{+\infty} \frac{f(x)}{W(x)} \, d\mu(x).
\]

Fix \( F \in E^\infty_W \). By the Hahn–Banach theorem, to prove that \( F \in P^\infty_W \), we need to verify that \( \langle \mu, F \rangle_{C^0_W} = 0 \) for every finite (complex) measure \( \mu \) vanishing on the polynomials. By polarization, it suffices to prove that \( \langle \mu, F \rangle_{C^0_W} = 0 \) only for real measures \( \mu \) of variation less than or equal to 1, such that \( \langle \mu, P \rangle_{C^0_W} = 0 \). Denote the set of such measures by \( \Sigma(W) \).

For every measure \( \mu \) denote by \( \tilde{\mu} \) the measure defined by \( \tilde{\mu}(E) = \mu(\{x \in \mathbb{R} : -x \in E\}) \), \( E \subset \mathbb{R} \), and put

\[
\mu_\pm = \frac{\mu \pm \tilde{\mu}}{2}.
\]
Since the weight $W$ is even,
\[ \mu \in \Sigma(W) \implies \tilde{\mu} \in \Sigma(W) \implies \mu_{\pm} \in \Sigma(W). \]
Denote
\[ \Sigma_{\pm}(W) = \{ \mu_{\pm} : \mu \in \Sigma(W) \} = \{ \mu \in \Sigma(W) : \mu = \mu_{\pm} \}. \]

For every measure $\mu$ we have $\mu = \mu_{+} + \mu_{-}$, hence $\Sigma(W) \subset \Sigma_{+}(W) + \Sigma_{-}(W)$, and we need only to verify that $\langle \mu, F \rangle_{C_{0}^{W}} = 0$ for $\mu \in \Sigma_{+}(W) \cup \Sigma_{-}(W)$.

The sets $\Sigma_{\pm}(W)$ are convex and weak* compact subsets of the space $M(\mathbb{R})$ dual to $C_{W}^{0}$, and by the Krein–Milman theorem, it suffices to verify the equality $\langle \mu, F \rangle_{C_{W}^{0}} = 0$ only for the extreme points $\mu$ of $\Sigma_{\pm}(W)$. Also, we note that for some entire functions $F_{1}, F_{2}$, we have $F(z) = F_{1}(x^{2}) + zF_{2}(x^{2})$, with both summands in the right-hand side belonging to $C_{W}^{0}$.

Consider the auxiliary weight
\[ U(x) = \begin{cases} x^{-1/2}W(x^{1/2}), & x \geq 0, \\ +\infty, & x < 0, \end{cases} \]
and define a map
\[ I : [M(\mathbb{R})]_{-} \to M((0, +\infty)) \]
by the relation
\[ \langle I\mu, f \rangle_{C_{0}^{0}} = \langle \mu, x \mapsto xf(x^{2}) \rangle_{C_{W}^{0}}, \quad f \in C_{W}^{0}, \]
where $C_{W}^{0}$ is defined by analogy with $C_{0}^{0}$.

Since the functions $x \mapsto h_{1}(x^{2}) + xh_{2}(x^{2})$, for continuous $h_{1}, h_{2}$ with finite support on $\mathbb{R}$, form a dense subset of $C_{W}^{0}$, we see that $I$ is an isometrical isomorphism. Furthermore, $\mu \in [M(\mathbb{R})]_{-}$ belongs to $\Sigma_{-}(W)$ if and only if $I\mu$ belongs to $\Sigma(U)$. Hence, the fact that $\mu$ is an extreme point of $\Sigma_{-}(W)$ implies that $I\mu$ is an extreme point of $\Sigma(U)$.

The extreme points of $\Sigma(U)$ are described by L. de Branges [2], [9, Section VIF], (One considers there continuous weight functions; in the situation under consideration, the description is the same.) For every such measure $I\mu$ there exists a transcendental entire function $E$ of zero exponential type having only simple real zeros $0 < x_{1}^{2} < x_{2}^{2} < \ldots$ such that
\[ \sum_{k \geq 1} \frac{U(x_{k}^{2})}{|E'(x_{k}^{2})|} < +\infty, \]
\[ \langle I\mu, f \rangle_{C_{0}^{0}} = \sum_{k \geq 1} \frac{f(x_{k}^{2})}{E'(x_{k}^{2})}, \quad f \in C_{W}^{0}. \]
Put $B(z) = E(z^2)$. Then
\begin{equation}
\frac{\sum_{k \in \mathbb{Z}} W(x_k)}{|B'(x_k)|} < +\infty,
\end{equation}
\begin{equation}
\langle \mu, F \rangle_{C_0^W} = \langle \mu, x \mapsto xF_2(x^2) \rangle_{C_0^W} = \sum_{k \in \mathbb{Z}} \frac{F(x_k)}{B'(x_k)},
\end{equation}
where $x_k = -x_{k-1}$, $k \leq 0$.

Next, consider the auxiliary weight
\begin{equation}
V(x) = \begin{cases} 
W(x^{1/2}), & x \geq 0, \\
+\infty, & x < 0,
\end{cases}
\end{equation}
and define an isometrical isomorphism
\begin{equation}
J : [M(\mathbb{R})]_+ \to M((0, +\infty))
\end{equation}
by the relation
\begin{equation}
\langle J\mu, f \rangle_{C_0^V} = \langle \mu, x \mapsto f(x^2) \rangle_{C_0^W}, 
\end{equation}
f $\in C_0^V$.

For every extreme measure $\mu$ of $\Sigma_+(W)$, $J\mu$ is an extreme point of $\Sigma(V)$, and there exists a transcendental entire function $E$ of zero exponential type having only simple real zeros $0 \leq x_1^2 < x_2^2 < \ldots$ such that
\begin{equation}
\sum_{k \geq 1} \frac{V(x_k^2)}{|E'(x_k^2)|} < +\infty,
\end{equation}
\begin{equation}
\langle J\mu, f \rangle_{C_0^V} = \sum_{k \geq 1} \frac{f(x_k^2)}{E'(x_k^2)}, 
\end{equation}
f $\in C_0^V$.

We consider two cases: $E(0) = 0$ and $E(0) \neq 0$. In the first case, put $B(z) = E(z^2)/z$. Then
\begin{equation}
\frac{\sum_{k \in \mathbb{Z}} W(x_k)}{|B'(x_k)|} < +\infty,
\end{equation}
\begin{equation}
\langle \mu, F \rangle_{C_0^W} = \langle \mu, x \mapsto F_1(x^2) \rangle_{C_0^W} = \sum_{k \in \mathbb{Z}} \frac{F(x_k)}{B'(x_k)},
\end{equation}
where $x_k = -x_{2-k}$, $k \leq 0$. 

In the second case, put \( B(z) = zE(z^2) \). Then

\[
\sum_{k \in \mathbb{Z}} \frac{x_k^2 W(x_k)}{|B'(x_k)|} < +\infty,
\]

\[
\langle \mu, F \rangle_{C_0^W} = \langle \mu, x \mapsto F_1(x^2) \rangle_{C_0^W} = \sum_{k \in \mathbb{Z}} \frac{x_k^2 F(x_k)}{B'(x_k)},
\]

where \( x_0 = 0, x_k = -x_{-k}, k < 0 \).

As a result of (2.1)–(2.3), to prove the theorem it remains to verify the following statement. For every even log-convex function \( W \) satisfying (1.1), (1.2), for every \( F \in \mathcal{E}_W^F \), and for every transcendental entire function \( B \) of zero exponential type having only simple real zeros \( x_k, k \in \mathbb{Z}, \) such that for an entire function \( E \),

\[
\text{either } B(z) = E(z^2) \text{ or } B(z) = zE(z^2),
\]

and such that

\[
\sum_{k \in \mathbb{Z}} \frac{W(x_k)}{|B'(x_k)|} < +\infty,
\]

we have

\[
\sum_{k \in \mathbb{Z}} \frac{F(x_k)}{B'(x_k)} = 0.
\]

An essential part of the proof consists in the use of

**Lemma 2.2.** For \( W \) and \( B \) as above, for some real number \( C = C(W, B) \) and for every \( y \geq 1 \),

\[
\int_{-\infty}^{+\infty} \frac{y \log W(x)}{y^2 + x^2} \, dx \leq C + \int_{-\infty}^{+\infty} \frac{y \log |B(x)/x|}{y^2 + x^2} \, dx.
\]

Using this lemma, to be proved below, and the facts that \( F \in \mathcal{E}_W^F \) is of zero exponential type, and that the function \( z \mapsto B(z)/z \) is outer in the upper half-plane, we get

\[
|F(iy)| \leq \exp \int_{-\infty}^{+\infty} \frac{|y| \log |F(x)|}{y^2 + x^2} \, dx = o(1) \exp \int_{-\infty}^{+\infty} \frac{|y| \log W(x)}{y^2 + x^2} \, dx
\]

\[
= o(1) \exp \int_{-\infty}^{+\infty} \frac{|y| \log |B(x)/x|}{y^2 + x^2} \, dx = o(1) \left| \frac{B(iy)}{y} \right|, \quad |y| \to +\infty.
\]

Therefore, we can write the interpolation formula

\[
R(z) = \frac{z F(z)}{B(z)} - \sum_{k \in \mathbb{Z}} \frac{x_k F(x_k)}{B'(x_k)(z - x_k)} = 0
\]
(R is an entire function of zero exponential type tending to 0 along the imaginary axis, hence R = 0). Setting z = 0, we deduce

$$\sum_{k \in \mathbb{Z}} \frac{F(x_k)}{B'(x_k)} = 0.$$ 

Relation (2.5) is proved, and the proof of the theorem is completed. □

**Proof of Lemma 2.2. A.** By (2.4), \(|B(z)| = |B(-z)|, z \in \mathbb{C}\), and to prove the lemma, it suffices to verify that for some C and for every \(y \geq 1\),

$$\int_0^{+\infty} \frac{y \log W(x)}{y^2 + x^2} \, dx \leq C + \int_0^{+\infty} \frac{y \log |B(x)/x|}{y^2 + x^2} \, dx. \quad (2.6)$$

To verify (2.6), we write down all the zeros of B on (0, +∞) as 0 < x_1 = \exp t_1 < x_2 = \exp t_2 < \ldots < x_k = \exp t_k < \ldots, 1 \leq k < +\infty, and denote

$$K = \sum_{k \geq 1} \frac{W(x_k)}{|B'(x_k)|}.$$ 

Furthermore, put

$$p'_k = \frac{W(x_k)}{|B'(x_k)|},$$

$$p_k = \max(p'_k, p'_{k+1}).$$

Then

$$\sum_{k \geq 1} p_k \leq 2K. \quad (2.7)$$

For every \(k \geq 1\) denote

$$B_k(x) = \frac{xB(x)}{(1 - \frac{x^2}{x_k^2})(1 - \frac{x^2}{x_{k+1}^2})}. \quad (2.8)$$

Then

$$B'(x_k) = -\frac{2B_k(x_k)}{x_k^2} (1 - \frac{x_k^2}{x_{k+1}^2}) = 2B_k(x_k)\left(\frac{1}{x_{k+1}^2} - \frac{1}{x_k^2}\right),$$

$$B'(x_{k+1}) = -\frac{2B_k(x_{k+1})}{x_{k+1}^2} (1 - \frac{x_{k+1}^2}{x_k^2}) = 2B_k(x_{k+1})\left(\frac{1}{x_k^2} - \frac{1}{x_{k+1}^2}\right).$$
and we get
\[
\log \frac{W(\exp t_k)}{|B_k(\exp t_k)|} = \log \frac{W(\exp t_k)}{|B'(\exp t_k)|} + \log 2 + \log(e^{-2t_k} - e^{-2t_{k+1}}) \\
= \log(2p_k) + \log(e^{-2t_k} - e^{-2t_{k+1}}),
\]
\[
\log \frac{W(\exp t_{k+1})}{|B_k(\exp t_{k+1})|} = \log \frac{W(\exp t_{k+1})}{|B'(\exp t_{k+1})|} + \log 2 + \log(e^{-2t_k} - e^{-2t_{k+1}}) \\
= \log(2p_{k+1}) + \log(e^{-2t_k} - e^{-2t_{k+1}}).
\]

**B.** Recall the relation (2.4). By the Hadamard factorization theorem we have
\[
E(x^2) = \prod_{k \geq 1} \left(1 - \frac{x^2}{x_k^2}\right).
\]
Since the function \( t \mapsto \log |1 - e^{2t - 2x}| \) is concave on each of the intervals \((-\infty, s), (s, +\infty))\), using (2.4) and (2.8) we obtain that the function \( t \mapsto \log |B_k(\exp t)| \) is concave on \([t_k, t_{k+1}]\). Furthermore, the function \( t \mapsto \log |W(\exp t)| \) is convex on \((-\infty, +\infty)\). Therefore,
\[
\log \frac{W(\exp t)}{|B_k(\exp t)|} \leq \log(2p_k) + \log(e^{-2t_k} - e^{-2t_{k+1}}), \quad t_k \leq t \leq t_{k+1}.
\]

Put
\[
R(t) = \log \frac{W(\exp t) \exp(t - A)}{|B(\exp t)|},
\]
for some parameter \( A > 0 \) to be defined later on. Then
\[
R(t) \leq \log(e^{-2t_k} - e^{-2t_{k+1}}) + \log(2p_k) + 2t - A \\
- \log(e^{2(t - t_k)} - 1) - \log(1 - e^{2(t - t_{k+1})}), \quad t_k \leq t \leq t_{k+1}.
\]
Since
\[
t + \log(e^{-t_k} + e^{-t_{k+1}}) - \log(e^{-t_k} + e^{-t_{k+1}}) - \log(1 + e^{t_{k+1} - t}) = \log(1 + e^{t_{k+1} - t}) - \log(1 + e^{t_{k+1} - t}) \\
\leq 0, \quad t_k \leq t \leq t_{k+1},
\]
we get
\[
R(t) \leq \log(e^{-t_k} - e^{-t_{k+1}}) + \log(2p_k) + t - A \\
- \log(e^{t_k - t_{k+1}} - 1) - \log(1 - e^{t_k - t_{k+1}}), \quad t_k \leq t \leq t_{k+1}. \tag{2.9}
\]
To prove (2.6), we need to verify that
\[
\sum_{k \geq 1} \int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} R(\log x) \, dx \leq O(1), \quad y \to +\infty. \tag{2.10}
\]

**C.** Put \( \beta_k = t_{k+1} - t_k \). Suppose that \( \beta_k \leq \log 2 \).
Then
\[
\int_{\exp t_k}^{\exp t_{k+1}} \log(xe^{-t_k} - 1) \, dx = \int_{t_k}^{t_{k+1}} \log(e^{-t_k} - 1) e^t \, dt
\]
\[
= e^{t_k} \int_0^{\beta_k} \log(e^s - 1) e^s \, ds \geq -e^{t_k} \beta_k (\log \frac{1}{\beta_k} + C),
\]
for some absolute constant $C$. Analogously,
\[
\int_{\exp t_k}^{\exp t_{k+1}} \ log(1 - xe^{-t_{k+1}}) \, dx \geq -e^{t_k} \beta_k (\log \frac{1}{\beta_k} + C).
\]
Hence,
\[
\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} \left[ \log(xe^{-t_k} - 1) + \log(1 - xe^{-t_{k+1}}) \right] \, dx
\]
\[
\geq -\frac{ye^{t_k}}{y^2 + e^{2t_k}} \cdot 2\beta_k \left( \log \frac{1}{\beta_k} + C \right). \tag{2.11}
\]

Since
\[
\log(e^{-t_k} - e^{-t_{k+1}}) + t = t - t_{k+1} + \log(e^{\beta_k} - 1) \leq \log(2\beta_k), \quad t_k \leq t \leq t_{k+1},
\]
we obtain
\[
\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} \left[ \log(e^{-t_k} - e^{-t_{k+1}}) + \log x + \log(2p_k) - A \right] \, dx
\]
\[
\leq \int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} \left[ \log \beta_k + \log(4p_k) - A \right] \, dx. \tag{2.12}
\]
Furthermore,
\[
\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} \, dx \geq \frac{ye^{t_k}(e^{\beta_k} - 1)}{y^2 + e^{2t_k} + 2\beta_k} \geq e^{-2\beta_k} \beta_k \frac{ye^{t_k}}{y^2 + e^{2t_k}}. \tag{2.13}
\]
Finally, by (2.7),
\[
\log \beta_k + \log(4p_k) \leq \log(\log 2) + \log(8K).
\]
Therefore, by (2.9), (2.11)–(2.13), for $A \geq \log(\log 2) + \log(8K)$, we get
\[
\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} R(\log x) \, dx
\]
\[
\leq \frac{ye^{t_k}}{y^2 + e^{2t_k}} \left[ \beta_k e^{-2\beta_k} (\log \beta_k + \log(4p_k) - A) + 2\beta_k (\log \frac{1}{\beta_k} + C) \right]
\]
\[
\leq \frac{1}{2} \left[ \beta_k \log \frac{p_k}{\beta_k} + \beta_k (C_1 - A) \right],
\]
for some $C_1$ independent of $k$, $A$. 

D. If \( \beta_k > \log 2 \), then by (2.7), (2.9), we get
\[
R(t) + A \leq \log(1 - e^{-\beta_k}) + \log(4K) + t - t_k - \log(e^{t_k} - 1) - \log(1 - e^{-t_{k+1}})
\leq \log(4K) - \log(1 - e^{t_k}) - \log(1 - e^{-t_{k+1}})
\leq \log(4K) - 2\log(1 - 2^{-1/2}) = C_2, \quad t \in \left[ t_k + \frac{\log 2}{2}, t_{k+1} - \frac{\log 2}{2} \right],
\]
with \( C_2 \) independent of \( k, A \). Arguing as in part C, we obtain
\[
\int_{\exp t_k}^{\sqrt{2}\exp t_k} \frac{y}{y^2 + x^2} R(\log x) \, dx \leq (C_3 - C_4 A) \frac{ye^{t_k}}{y^2 + e^{2t_k}},
\]
\[
\int_{\exp t_{k+1}}^{\exp t_k} \frac{y}{y^2 + x^2} R(\log x) \, dx \leq (C_3 - C_4 A) \frac{ye^{t_{k+1}}}{y^2 + e^{2t_{k+1}}},
\]
for some \( C_3, C_4 > 0 \) independent of \( k, A \).

Hence, for sufficiently big \( A \) independent of \( k \), we have
\[
\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} R(\log x) \, dx \leq \frac{\beta_k}{2} \log \frac{p_k}{\beta_k}, \quad (2.14)
\]
if \( \beta_k \leq \log 2 \), and
\[
\int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} R(\log x) \, dx \leq 0, \quad (2.15)
\]
if \( \beta_k > \log 2 \). As a result of (2.14) and (2.15),
\[
\sum_{k \geq 1} \int_{\exp t_k}^{\exp t_{k+1}} \frac{y}{y^2 + x^2} R(\log x) \, dx \leq \sum_{k \geq 1} \frac{\beta_k}{2} \log^+ \frac{p_k}{\beta_k}
\leq \frac{1}{2e} \sum_{k \geq 1} p_k \leq \frac{K}{e},
\]
because \( x \log^+ \frac{1}{x} \leq \frac{1}{e} \), \( 0 < x < +\infty \). This proves (2.10) and the whole lemma.

\[\square\]

3. POLYNOMIAL APPROXIMATION IN \( L^p_W \), \( 1 \leq p < +\infty \)

**Theorem 3.1.** Suppose that \( W \) is an even log-convex function on the real line satisfying (1.1), (1.2). Then \( \mathcal{P}_W^p = \mathcal{E}_W^p \), \( 1 \leq p < +\infty \).

Notice that as a consequence of condition (1.2), \( \mathcal{E}_W^p \) is a closed subspace of \( L^p_W \), and the point evaluations \( F \mapsto F(z), z \in \mathbb{C} \), are bounded linear functionals on \( \mathcal{E}_W^p \).
For the proof of Theorem 3.1, we need the following three results, whose proofs are deferred to Section 4.

**Lemma 3.2.** Under conditions of Theorem 3.1 we have

$$\lim_{|z| \to +\infty} \frac{W(x+1)}{W(x)} = 1.$$  \hspace{1cm} (3.1)

**Lemma 3.3.** Let $F \in \mathcal{E}$ and either $1 \leq p < r < +\infty$ or $1 < p < r \leq +\infty$. Suppose that $\mathcal{P}_W^p = \mathcal{E}_W^r$ and $W$ satisfies (3.1). If the function $z \mapsto z F(z)$ belongs to $\mathcal{E}_W^r$, then $F \in \mathcal{P}_W^p$.

**Lemma 3.4.** Let $W$ be monotone increasing on $\mathbb{R}_+$ and monotone decreasing on $\mathbb{R}_-$, and satisfy (1.1). Suppose that $F \in \mathcal{P}_W^p$, $1 \leq p \leq +\infty$. If the function $z \mapsto z F(z)$ belongs to $\mathcal{E}_W^r$, then it belongs to $\mathcal{P}_W^p$ as well.

The last lemma together with a result of Khachatrian [7, Corollary 2] formulated below gives another proof of Theorem 2.1.

**Proposition 3.5** (Khachatrian). In the conditions of Theorem 3.1, suppose that $F \in \mathcal{E}$. If the function $z \mapsto z^2 F(z)$ belongs to $\mathcal{E}_W^r$, then $F \in \mathcal{P}_W^p$.

Since [7] is rather inaccessible, in Section 4 we show how to derive this proposition from some results given in [9].

The second proof of Theorem 2.1. We start with $G \in \mathcal{E}_W^\infty$, and define

$$G_1(z) = \frac{G(z) - G(0)}{z}, \quad G_2(z) = \frac{G_1(z) - G_1(0)}{z}.$$  \hspace{1cm} (3.2)

By Proposition 3.5, $G_2 \in \mathcal{P}_W^\infty$. Furthermore, applying Lemma 3.4 twice, we obtain $G_1 \in \mathcal{P}_W^\infty, G \in \mathcal{P}_W^\infty$. Thus, $\mathcal{E}_W^\infty = \mathcal{P}_W^\infty$.

Furthermore, Lemma 3.4 together with Theorem 2.1 gives a proof of Theorem 3.1.

**Proof of Theorem 3.1.** We start with the case $1 < p < +\infty$. Take $G \in \mathcal{E}_W^p$, and define $G_1$ as in (3.2). Applying Lemma 3.3 with $r = +\infty$ and Theorem 2.1, we obtain that $G_1 \in \mathcal{P}_W^p$. By Lemma 3.4, $G \in \mathcal{P}_W^p$. Thus, $\mathcal{E}_W^p = \mathcal{P}_W^p, 1 < p < +\infty$.

If $p = 1, G \in \mathcal{E}_W^\infty$, then we apply Lemma 3.3 with $r = 2$ and the (already proved) equality $\mathcal{P}_W^p = \mathcal{E}_W^p$ to get $G_1 \in \mathcal{P}_W^1$. Again by Lemma 3.4, $G \in \mathcal{P}_W^1$. Thus, $\mathcal{E}_W^1 = \mathcal{P}_W^1$. \hspace{1cm} \Box
4. The proofs of auxiliary statements

Proof of Lemma 3.2. Without loss of generality, assume that \( W \) is \( C^1 \)-smooth. Put \( \varphi = \log W \). Since \( W \) is log-convex, the function \( x \mapsto x \varphi'(x) \) increases, and elementary inequalities

\[
\varphi'(x) \geq \frac{t \varphi'(t)}{x} \geq \frac{\varphi'(t)}{3}, \quad t \leq x \leq 3t,
\]

\[
\varphi(x) \geq \int_t^x \varphi(y) \, dy \geq \frac{(x-t) \varphi'(t)}{3} \geq \frac{t \varphi'(t)}{3}, \quad 2t \leq x \leq 3t,
\]

\[
\int_{3t}^{2t} \frac{\varphi(x)}{x^2} \, dx \geq \frac{t \cdot t \varphi'(t)}{9t^2 \cdot 3} \geq \frac{\varphi'(t)}{27}, \quad t \geq 0,
\]

together with (1.2) show that

\[
\lim_{x \to +\infty} \varphi'(x) = 0.
\]

Therefore,

\[
\lim_{|x| \to +\infty} \frac{W(x+1)}{W(x)} = 1.
\]

\( \square \)

Proof of Lemma 3.3. Since

\[
\int_{-\infty}^{+\infty} \frac{|F(x)|^p}{(W(x))^p} \, dx \leq \int_{|t| \leq 1} \frac{|F(x)|^p}{(W(x))^p} \, dx + \int_{|t| > 1} \frac{|x|^p \cdot |F(x)|^p}{(W(x))^p} \, dx < +\infty,
\]

we obtain \( F \in \mathcal{E}_W^p \). For \( \varepsilon > 0 \) put

\[
F_\varepsilon(z) = \frac{1}{\varepsilon} \int_0^\varepsilon F(z+t) \, dt.
\]

By (3.1) it follows easily that the shift operator is continuous in \( L^s_W \), for \( 1 \leq s < +\infty \), and in \( C_0^0_W \), and therefore \( F_\varepsilon \to F \) in \( L^p_W \) as \( \varepsilon \to 0 \). Furthermore, the functions \( z \mapsto (z+i)F_\varepsilon(z), \varepsilon > 0 \), belong to both \( C^0_W \) and \( L^p_W \), because again by (3.1),

\[
|x+i| \cdot |F_\varepsilon(x)| \leq \frac{CW(x)}{\varepsilon} \int_0^\varepsilon \frac{|x+t+i| \cdot |F(x+t)|}{W(x+t)} \, dt
\]

\[
\leq \frac{CW(x)}{\varepsilon^{1/p}} \left( \int_0^\varepsilon \frac{|x+t+i|^p \cdot |F(x+t)|^p}{(W(x+t))^p} \, dt \right)^{1/p}
\]

\[
= \frac{CW(x)}{\varepsilon^{1/p}} \left( \int_0^{x+\varepsilon} \frac{|x+i|^p \cdot |F(t)|^p}{(W(t))^p} \, dt \right)^{1/p} = o(W(x)), \quad |x| \to +\infty,
\]
and
\[
\int_{-\infty}^{\infty} \frac{|x+i|P \cdot |F_\varepsilon(x)|^p}{(W(x))^{p/2}} dx = \frac{1}{\varepsilon^p} \int_{-\infty}^{\infty} \left( \int_0^\varepsilon |F(x+t)|^p dt \right)^p \frac{|x+i|^p}{(W(x))^p} dx
\]
\[
\leq \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \left( \int_0^\varepsilon \frac{|x-t+i|^p}{(W(x-t))^p} dt \right) |F(x)|^p dx
\]
\[
= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \left( \int_0^\varepsilon \frac{|x+t+i|^p}{(W(x+t))^p} dt \right) |F(x)|^p dx
\]
\[
\leq C \int_{-\infty}^{\infty} \frac{|x+i|^p |F(x)|^p}{(W(x))^p} dx < +\infty.
\]

Since \( r > p \), the functions \( z \mapsto (z+i)F_\varepsilon(z) \), \( \varepsilon > 0 \), belong also to \( L^r_W \). Fix \( \varepsilon > 0 \). By conditions of the lemma, \( P_\varepsilon^r = \mathcal{E}_W^r \), and we can find a sequence of polynomials \( Q_n \) approximating the function \( z \mapsto (z+i)F_\varepsilon(z) \) in \( \mathcal{E}_W^r \). Since the point evaluation \( F \mapsto F(-i) \) is bounded on \( \mathcal{E}_W^r \), we get \( Q_n(-i) \to 0 \), \( n \to \infty \). Now, without loss of generality we may assume that \( Q_n(-i) = 0 \), and hence, \( Q_n(z) = (z+i)P_n(z) \) for some polynomials \( P_n \). Thus,
\[
\int_{-\infty}^{\infty} \frac{|(x+i)F_\varepsilon(x) - (x+i)P_n(x)|^r}{(W(x))^p} dx \to 0, \quad n \to +\infty,
\]
if \( r < +\infty \), and
\[
\sup_{x \in \mathbb{R}} \frac{|(x+i)F_\varepsilon(x) - (x+i)P_n(x)|}{W(x)} \to 0, \quad n \to +\infty.
\]
if \( r = +\infty \). By Hölder’s inequality,
\[
\int_{-\infty}^{\infty} \frac{|F_\varepsilon(x) - P_n(x)|^p}{(W(x))^p} dx
\]
\[
\leq \left( \int_{-\infty}^{\infty} \frac{|(x+i)F_\varepsilon(x) - (x+i)P_n(x)|^r}{(W(x))^p} dx \right)^{p/r} 
\]
\[
\times \left( \int_{-\infty}^{\infty} \frac{dx}{|x+i|^{pr/(r-p)}} \right)^{(r-p)/r} \to 0, \quad n \to +\infty,
\]
if \( r < +\infty \), and
\[
\int_{-\infty}^{\infty} \frac{|F_\varepsilon(x) - P_n(x)|^p}{(W(x))^p} dx
\]
\[
\leq \left( \sup_{x \in \mathbb{R}} \frac{|(x+i)F_\varepsilon(x) - (x+i)P_n(x)|}{W(x)} \right)^p \int_{-\infty}^{\infty} \frac{dx}{|x+i|^p} \to 0, \quad n \to +\infty,
\]
if \( r = +\infty \). Here we use that either \( r < +\infty \) and \( pr > r - p \), or \( r = +\infty \) and \( p > 1 \).
Thus, \( F_\varepsilon \in \mathcal{P}_W^p, \varepsilon > 0 \), and as a consequence, \( F \in \mathcal{P}_W^p \).

Proof of Lemma 3.4. Denote \( G(z) = zF(z) \). For every \( \lambda \in \mathbb{C} \), the function \( z \mapsto \frac{G(z) - G(\lambda)}{z - \lambda} \) belongs to \( \mathcal{P}_W^p \). Suppose that \( G \in \mathcal{E}_W^q \setminus \mathcal{P}_W^p \).

Define
\[
\mathfrak{A} = \text{clos} \text{ Lin} \left[ \{ G(\eta z), 0 < \eta \leq 1 \} \cup \mathcal{P} \right].
\]

Then,
\[
\text{for every } \lambda \in \mathbb{C}, \ H \in \mathfrak{A}, \ \begin{cases} \text{the function } z \mapsto \frac{H(z) - H(\lambda)}{z - \lambda} \text{ belongs to } \mathcal{P}_W^p \end{cases}
\]

(4.1)

We can identify bounded linear functionals on \( L^p_W \), \( 1 \leq p < +\infty \),

with elements of \( L^q_{1/W} \), \( 1/p + 1/q = 1 \),

\[
L^q_{1/W} = \left\{ f : \int_{-\infty}^{+\infty} |f(x)|^q (W(x))^q \, dx < +\infty \right\}, \quad 1 < q < +\infty,
\]

\[
L^\infty_{1/W} = \left\{ f : \text{ess sup}_{x \in \mathbb{R}} |f(x)|W(x) < +\infty \right\},
\]

the duality being
\[
\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)g(x) \, dx, \quad f \in L^q_{1/W}, \ g \in L^p_W.
\]

From now on we restrict ourselves to the case \( 1 \leq p < +\infty \). In the case \( p = \infty \) we identify bounded linear functionals on \( C^0_W \) with complex Borel measures \( \mu \) such that
\[
\int_{-\infty}^{+\infty} W(x) \, d\mu(x) < +\infty,
\]

and the rest of the proof is similar to that for the case \( 1 \leq p < +\infty \).

Using the Hahn–Banach theorem, choose \( g \in L^q_{1/W} \) such that
\[
\langle g, G \rangle \neq 0, \quad \langle g, \mathcal{P}_W^p \rangle = 0.
\]

By (4.1), for every \( \lambda \in \mathbb{C} \) we get
\[
\int_{-\infty}^{+\infty} g(x) \frac{G(x) - G(\lambda)}{x - \lambda} \, dx = 0,
\]

\[
G(\lambda) \int_{-\infty}^{+\infty} \frac{g(x)}{x - \lambda} \, dx = \int_{-\infty}^{+\infty} \frac{g(x)G(x)}{x - \lambda} \, dx, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

(4.2)
By the Lebesgue dominated convergence theorem,
\[
\lim_{|t| \to +\infty} \int_{-\infty}^{+\infty} g(x)G(x) \frac{it}{x-\xi} \, dx = - \int_{-\infty}^{+\infty} g(x)G(x) \, dx \neq 0, \quad (4.3)
\]
and hence,
\[
\int_{-\infty}^{+\infty} \frac{g(x)}{x-\xi} \, dx \neq 0 \quad (4.4)
\]
for \( t \) real and \(|t|\) large enough.

Next, take \( H \in \mathfrak{A} \) such that \( \langle g, H \rangle = 0 \). By (4.1), for every \( \lambda \in \mathbb{C} \) we obtain as before:
\[
H(\lambda) \int_{-\infty}^{+\infty} \frac{g(x)}{x-\lambda} \, dx = \int_{-\infty}^{+\infty} \frac{g(x)H(x)}{x-\lambda} \, dx, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (4.5)
\]
\[
\lim_{|t| \to +\infty} \int_{-\infty}^{+\infty} g(x)H(x) \frac{it}{x-\xi} \, dx = - \int_{-\infty}^{+\infty} g(x)H(x) \, dx = 0. \quad (4.6)
\]

Then we divide (4.2) by (4.5), taking account of observation (4.4), and use (4.3), (4.6), to get
\[
|H(it)| = o(1)|G(it)|, \quad |t| \to +\infty. \quad (4.7)
\]

One immediate consequence of this relation is that for every \( n \geq 0 \), we have
\[
|t|^n = o(1)|G(it)|, \quad |t| \to +\infty. \quad (4.8)
\]

Furthermore, if \( \dim(\mathfrak{A}/\mathcal{P}_W^p) \geq 2 \), then we may take
\[
G_1 \in \mathfrak{A} \setminus (\mathbb{C}G + \mathcal{P}_W^p)
\]
and \( f, f_1 \in L^1_{\mathfrak{A}/W} \) such that
\[
\langle f, G \rangle \neq 0, \quad \langle f, \mathbb{C}G_1 + \mathcal{P}_W^p \rangle = 0, \quad \langle f_1, G_1 \rangle \neq 0, \quad \langle f_1, \mathbb{C}G + \mathcal{P}_W^p \rangle = 0.
\]

Applying the above argument we deduce
\[
|G_1(it)| = o(1)|G(it)|, \quad |t| \to +\infty,
\]
\[
|G(it)| = o(1)|G_1(it)|, \quad |t| \to +\infty,
\]
which is impossible.

Thus, \( \dim(\mathfrak{A}/\mathcal{P}_W^p) = 1 \), and we have \( \mathfrak{A} = \mathbb{C}G + \mathcal{P}_W^p \). Given \( 0 < \eta < 1 \), put \( G_\eta(z) = G(\eta z) \). Since \( W(x) \) increases with \(|x|\), by dominated convergence we obtain \( G_\eta \to G \) in \( \mathcal{E}_W^p \) as \( \eta \to 1 \). Since \( G \notin \mathcal{P}_W^p \), for some \( \eta < 1 \) we should have \( G_\eta \notin \mathcal{P}_W^p \), and
\[
G_\eta = H(\eta) + A(\eta)G
\]
for some $H(\eta) \in \mathcal{P}_W^0$, $A(\eta) \in \mathbb{C} \setminus \{0\}$. By (4.7) we obtain
\[ |G(i\eta)| = |G_\eta(i\eta)| = |A(\eta) + o(1)| \cdot |G(i\lambda)|, \quad |\lambda| \to +\infty.\]
Hence, for every $M > \log A(\eta)/\log(1/\eta)$ there exists $C_M$ such that
\[ |G(i\eta)| \leq C_M(|\eta| + 1)^M, \quad \eta \in \mathbb{R}, \]
which contradicts (4.8). Thus, $G \in \mathcal{P}_W^0$, and the proof is completed. \qed

**Proof of Proposition 3.5.** We make extensive use of the material in [9, Section VI H.2]. First, by an argument in the proof of the theorem on page 226 of [9], for some positive $C$, for some function $S$ of the form (1.4), and for any $\eta$, $0 < \eta < 1$, we have
\[ CW(x) \leq S(x) \leq 1 + \frac{x^2}{1-\eta^2} W\left(\frac{x}{\eta}\right), \quad x \in \mathbb{R} \quad (4.9)\]
Denote $G(z) = z^2 F(z)$, $W_\eta(x) = W(x/\eta)$, $F_\eta(x) = F(\eta x)$. Since $G \in \mathcal{E}_W^0$, by the first inequality in (4.9), we get $G \in \mathcal{E}_W^0$. By the result of Khachatryan mentioned in Section 1 (for the proof see pages 223–226 of [9]), we obtain $G \in \mathcal{P}_S^0$. Hence, as in the proof of Lemma 3.3, we can find a sequence of polynomials $P_n$, such that $z \mapsto z^2 P_n(z)$ approximate $G$ in $\mathcal{E}_W^0$. Applying the second inequality in (4.9), we conclude that $P_n$ tend to $F$ in $\mathcal{E}_W^0$, and hence, $F \in \mathcal{P}_W^0$, $F_\eta \in \mathcal{P}_W^0$. Finally, since $\eta < 1$ is arbitrary, and $F_\eta \to F$ as $\eta \to 1$, we obtain that $F \in \mathcal{P}_W^0$. \qed

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