ON THE BEKOLLE–BONAMI CONDITION

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Consider the system $\mathfrak{A}$ of the Carleson "squares"

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - |Q| \leq r \leq 1, |\theta - \theta_0| \leq |Q|/2\}$$

in the unit disc $\mathbb{D}$. Given a non-negative function $w$ on $\mathbb{D}$, and a subset $E$ of $\mathbb{D}$, we denote

$$\langle w \rangle_E = \frac{1}{m_2(E)} \int_E w(z) \, dm_2(z),$$

where $dm_2$ is Lebesgue area measure. The classes $\mathcal{B}_{p,q}$, $0 < p, q < \infty$, consist of $w$ such that

$$\sup_{Q \in \mathfrak{A}} \langle w^r \rangle_Q^{1/p} \langle w^{-q} \rangle_Q^{1/q} < \infty. \quad (1)$$

The classes $B_p = \mathcal{B}_{1,1/(p-1)}$, $1 < p < \infty$, were introduced by D. Bekolle and A. Bonami in [2]. They proved that for locally integrable non-negative weights $w$ on $\mathbb{D}$, and for $1 < p < \infty$, the Bergman projection operator $T : f \mapsto Tf$,

$$Tf(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} \, dm_2(\zeta),$$

acts continuously on $L^p(\mathbb{D}, w \, dm_2)$ if and only if $w \in B_p$. This result is similar to the Hunt–Muckenhoupt–Wheeden theorem (see [11], [7, Chapter 6]) that claims that the Hilbert transform is bounded on $L^p(\mathbb{R}, w \, dm)$ if and only if $w$ satisfies the condition $(A_p)$. The class $A_p$ consisting of functions $w$ satisfying $(A_p)$ is analogous to the class $B_p$, with squares $Q \subset \mathbb{D}$ in the definition (1) replaced by intervals of $\mathbb{R}$.

By the Hölder inequality, we have

$$A_{p_1} \subset A_p, \quad B_{p,q} \subset B_{p_1,q_1}, \quad p_1 \leq p, \; q_1 \leq q.$$

It is known (see [13]) that

$$A_p \subset \bigcup_{\varepsilon > 0} A_{p-\varepsilon}.$$

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On the other hand, \( B_{p,q} \not\subset \bigcup_{\varepsilon > 0} B_{p+\varepsilon, q+\varepsilon} \).

The aim of this article is to study what additional conditions on the weight \( w \in B_{p,q} \) imply that \( w \in \bigcup_{\varepsilon > 0} B_{p+\varepsilon, q+\varepsilon} \). In other words, we ask when the fact that \( T \) acts continuously on \( L^p(w \, dm_2) \) implies that \( T \) acts continuously on \( L^{p-\varepsilon}(w^{1+\varepsilon} \, dm_2) \), for small \( \varepsilon = \varepsilon(w) \).

Note that in applications of the Belkollé–Bonami theorem [1, 3, 10] the weight \( w \) is frequently equal to \( |\varphi|^\alpha \) for a univalent function \( \varphi \) and for real \( \alpha \).

Denote by \( \mathcal{A} \) the class of functions \( |f|^\alpha \), \( \alpha \in \mathbb{R} \), for \( f \) analytic in \( \mathbb{D} \), by \( \mathcal{M} \) the class of functions \( |f| \) for \( f \) meromorphic in \( \mathbb{D} \), by \( \mathcal{E}S \) the class of functions \( \exp u \) for \( u \) subharmonic in \( \mathbb{D} \), and by \( S \) the class of functions non-negative and subharmonic in \( \mathbb{D} \). Note that \( \mathcal{M} \cap B_{p,q} \subset \mathcal{A} \), \( p \geq 2 \).

**Theorem.** For \( 0 < p, q < \infty \) we have

(I) \( \mathcal{A} \cap B_{p,q} \subset \bigcup_{\varepsilon > 0} B_{p+\varepsilon, q+\varepsilon} \),

(II) \( \mathcal{E}S \cap B_{p,q} \subset \bigcup_{\varepsilon > 0} B_{p+\varepsilon, q+\varepsilon} \),

(III) \( S \cap B_{p,q} \not\subset \bigcup_{\varepsilon > 0} B_{p+\varepsilon, q+\varepsilon} \).

For \( 0 < p, q < 2 \) we have

(IV) \( \mathcal{M} \cap B_{p,q} \not\subset \bigcup_{\varepsilon > 0} B_{p+\varepsilon, q+\varepsilon} \).

**Proof.** (I) follows from (II).

(II) Let \( f = \exp u \), for \( u \) subharmonic in \( \mathbb{D} \), and let \( f \in B_{p,q} \), that is

\[
\sup_{Q \in \mathcal{A}} (f^p_Q)^{1/p} (f^{-q}_Q)^{1/q} < \infty.
\]

Since the function \( f^p \) is subharmonic, by the mean value inequality, for every \( Q \in \mathcal{A} \),

\[
\langle f^p \rangle_Q \geq c \left( \frac{\text{dist} (z, \partial Q)}{|Q|} \right)^2 \langle f \rangle_Q, \quad z \in Q.
\]

For every Carleson square \( Q \) denote by \( TQ \) the set \( Q \setminus (Q_1 \cup Q_2) \), where \( Q_1, Q_2 \in \mathcal{A} \), \( |Q_1| = |Q_2| = |Q|/2 \), \( (Q_1 \cup Q_2) \cap T = Q \cap T \). Put

\[
F(TQ) = \sup_{z \in TQ} f(z).
\]

Fix \( \gamma > 0 \). If a square \( Q \in \mathcal{A} \) contains two subsets \( E_1 \) and \( E_2 \) with \( m_2(E_1) \geq \gamma m_2(Q) \), \( m_2(E_2) \geq \gamma m_2(Q) \), \( \langle f^p \rangle_{E_1} \geq \lambda \sup_{E_2} f \), then (2) implies that \( \lambda \) is bounded uniformly in \( Q \in \mathcal{A} \). Therefore, the following claim is proved:
Claim. If two squares $Q_1, Q_2 \in \mathcal{A}$ are of comparable sidelength, and if the distance between them is bounded by a constant times the sidelength of $Q_1$, then $F(TQ_1)$ is comparable to $F(TQ_2)$.

Fix a dyadic system of Carleson squares $Q_{j,k}$, say

$$Q_{j,k} = \{ re^{i\theta} \in \mathbb{D} : 1 - 2\pi \cdot 2^{-k} \leq r \leq 1, \ |\theta - 2\pi \cdot 2^{-k}j| \leq \pi \cdot 2^{-k} \},$$

with trivial modification for $k = 1$. Put

$$g(z) = F(TQ_{j,k}), \quad z \in TQ_{j,k}.$$  \hfill (5)

Then $f \leq g$, and

$$\langle g^s \rangle_{TQ_{j,k}} \leq c(s) \langle f^s \rangle_{TQ_{j,k}}, \quad 0 < s < \infty,$$

$$\langle g^s \rangle_Q \leq c(s) \langle f^s \rangle_Q, \quad Q \in \mathcal{A}, \quad 0 < s < \infty. \hfill (6)$$

Next we verify that for some positive $\varepsilon, c$ independent of $j, k$,

$$\frac{1}{m_2(TQ_{j,k})} \int_{TQ_{j,k}} f(z)^{-q-e} dm_2(z) \leq c F(TQ_{j,k})^{-q-e}. \hfill (7)$$

For every $TQ_{j,k}$ we consider a rectangle $\Omega_{j,k}$ containing $TQ_{j,k}$ with $\operatorname{dist}(z, TQ_{j,k} \cup \mathbb{T}) \geq |Q_{j,k}|$, $z \in \partial \Omega_{j,k}$, and the conformal map $\omega_{j,k} : \mathbb{D} \rightarrow \Omega_{j,k}$ such that $\omega_{j,k}^{-1}(TQ_{j,k}) \subset r^2 \mathbb{D}$ for a constant $r < 1$. We write

$$u \circ \omega_{j,k} = u_{j,k} + \log F(TQ_{j,k}).$$

Then $u_{j,k}$ are subharmonic in $\mathbb{D}$, $u_{j,k}(z) \leq c$, with $c$ independent of $j, k$.

Relations (2), (4), and (6) show that

$$\langle e^{-qu} \rangle_{Q_{j,k}} \leq c F(TQ_{j,k})^{-q}.$$ 

Moreover, by the Claim,

$$\langle e^{-qu} \rangle_{\Omega_{j,k}} \leq c F(TQ_{j,k})^{-q}.$$ 

Hence,

$$\int_{r \mathbb{D}} e^{-qu_{j,k}(z)} dm_2(z) \leq c.$$

If (7) is false, then for some $c$ and for every $n$ there exists a function $u_n$ subharmonic in $\mathbb{D}$, such that

$$u_n(z) \leq c, \quad z \in \mathbb{D}, \hfill (8)$$

$$\int_{r \mathbb{D}} e^{-qu_{n}(z)} dm_2(z) \leq c, \hfill (9)$$

and

$$\int_{r \mathbb{D}} e^{-(q + \frac{1}{n})u_n(z)} dm_2(z) \rightarrow \infty, \quad n \rightarrow \infty. \hfill (10)$$
Consider the measures $\mu_n = \Delta g_n$. By (8) and (9), there exist $\varepsilon, \delta > 0$ such that for every disc $D$ of radius $2\delta$ centered at a point of $rD$, and for every $n$,

$$\mu_n(D) \leq \frac{2}{q + 2\varepsilon}. $$

We cover $rD$ by a finite union of small discs $D_s$ with

$$\max_{n,s} \mu_n(2D_s) \leq \frac{2}{q + 2\varepsilon},$$

where $2D_s \subset rD$ are the discs concentric with $D_s$ with radii twice those of $D_s$.

Using the Riesz representation, for every $n,s$ we obtain

$$u_n(z) = u_{n,s}(z) + v_{n,s}(z) = u_{n,s}(z) + \int_{2D_s} \log|z - \zeta| \, d\mu_n(\zeta), \quad (11)$$

where $u_{n,s}$ is harmonic in $2D_s$. Since $v_{n,s}(z)$ is non-positive in $2D_s$, condition (9) implies that

$$\int_{2D_s} e^{-q u_{n,s}(z)} \, dm_2(z) \leq c.$$  

By the mean value property,

$$u_{n,s}(z) \geq c, \quad z \in D_s. \quad (12)$$

Denote

$$w_{n,s}(z) = \exp\left[-(q + \varepsilon) \int_{2D_s} \log|z - \zeta| \, d\mu_n(\zeta)\right].$$

By Cartan’s lemma (see [4, Chapitre III], [9, Lemma 6.17]),

$$m_2\{z \in D_s : w_{n,s}(z) > t\} \leq Ct^{-(q + 2\varepsilon)/(q + \varepsilon)}, \quad t > 1,$$

with an absolute constant $C$, and hence,

$$\int_{D_s} w_{n,s}(z) \, dm_2(z) \leq c(\varepsilon).$$

By (11) and (12),

$$\int_{D_s} e^{-(q + \varepsilon) u_n(z)} \, dm_2(z) \leq c(\varepsilon),$$

and

$$\int_{rD} e^{-(q + \varepsilon) u_n(z)} \, dm_2(z) < \infty,$$

that contradicts to (10). Thus, (7) is proved.

By (2), (5), (6), (7), we have $g \in B_{q,p}$, and for some $c_1, \varepsilon > 0$,

$$\langle f^{p+\varepsilon} \rangle_Q^{1/(p+\varepsilon)} \langle f^{-q-\varepsilon} \rangle_Q^{1/(q+\varepsilon)} \leq c\langle g^{p+\varepsilon} \rangle_Q^{1/(p+\varepsilon)} \langle g^{-q-\varepsilon} \rangle_Q^{1/(q+\varepsilon)}, \quad Q \in \mathcal{A}.$$
To complete the proof of (II) it remains to verify that for every positive \( g \in B_{p,q} \), which is constant on each \( TQ_{j,k} \), we have \( g \in B_{p+\varepsilon,q} \) for some \( \varepsilon = \varepsilon(g,p,q) \); after that, repeating the argument, we obtain \( g \in B_{p+\varepsilon,q+\varepsilon_1} \) for some \( \varepsilon_1 = \varepsilon_1(g,p,q,\varepsilon) \).

First, we choose \( s \) such that \( 0 < 1/s < \min(p,q) \), and define \( h = g^{1/s} \in B_{ps,q} \subset B_{ps,1} \). Then, using the Cauchy-Schwarz inequality, we get

\[
\langle h \rangle_Q \leq \langle h^{ps} \rangle_Q^{1/ps} \leq K \langle h \rangle_Q , \quad Q \in \mathfrak{A}.
\]  

(13)

Next we use a reverse Hölder inequality (cf. [8], [5]): for some \( \varepsilon, c > 0 \) depending only on \( K, p, s, \)

\[
\langle h^{ps+\varepsilon} \rangle_Q^{1/(ps+\varepsilon)} \leq c \langle h \rangle_Q, \quad Q \in \mathfrak{A}.
\]  

(14)

Inequalities (13), (14) imply that

\[
\langle g^{p+\varepsilon} \rangle_Q^{1/(p+\varepsilon)} \leq c \langle g^p \rangle_Q^{1/p}, \quad Q \in \mathfrak{A},
\]

\[
\sup_{Q \in \mathfrak{A}} \langle g^{p+\varepsilon} \rangle_Q^{1/(p+\varepsilon)} \langle g^{-q} \rangle_Q^{1/q} \leq \sup_{Q \in \mathfrak{A}} \langle g^p \rangle_Q^{1/p} \langle g^{-q} \rangle_Q^{1/q} < \infty,
\]

and hence, \( g \in B_{p+\varepsilon,q} \). Thus, (II) is proved modulo (14).

Finally, we verify that (13) implies (14) for \( h \) which are constant on each \( TQ_{j,k} \). Denote \( t = ps > 1 \). Fix a dyadic square \( Q = Q_{j,k} \), and, without loss of generality, assume that \( \langle h \rangle_Q = 1 \). Next, we fix a large \( N \), and modify \( h \) by making it equal to \( \langle h \rangle_{Q_{j,N}} \) on \( Q_{j,N} \subset Q \). Inequality (13) still holds, and we need to verify that for small \( \gamma > 0 \), \( \langle h^{t+\gamma} \rangle_Q \) is bounded uniformly in \( N \).

We use the standard Calderon-Zygmund decomposition. For every \( \lambda \geq 1 \) denote by \( H(\lambda) \) the set of all \( z \in Q \) such that \( h(z) \geq \lambda \), and consider the set \( \mathfrak{A}(\lambda) \) of maximal dyadic squares \( Q' \subset Q \) such that \( \langle h \rangle_{Q'} \geq \lambda \). Denote the union of these squares by \( \mathfrak{H}(\lambda) \). Then

\[
\langle h \rangle_{Q'} \leq 5\lambda, \quad Q' \in \mathfrak{A}(\lambda),
\]  

(15)

and

\[
H(4\lambda) \subset \mathfrak{H}(\lambda)
\]  

(16)

(here we use that \( h \) is constant on \( TQ_{j,k} \)). By (13), (15) and (16) we get

\[
\int_{H(4\lambda)} h(z)^t dm_2(z) \leq \int_{\mathfrak{H}(\lambda)} h(z)^t dm_2(z) = \sum_{Q' \in \mathfrak{A}(\lambda)} \int_{Q'} h(z)^t dm_2(z)
\]

\[
\leq \sum_{Q' \in \mathfrak{A}(\lambda)} (5\lambda)^{t-1} K^t \int_{Q'} h(z) dm_2(z), \quad \lambda \geq 1.
\]
Furthermore, for every $Q' \in \mathfrak{A}(\lambda),$
\[
\int_{Q' \cap H(\lambda/2)} h(z) \, dm_2(z) \leq \frac{\lambda}{2} m_2(Q'),
\]
and hence,
\[
\int_{Q'} h(z) \, dm_2(z) \leq 2 \int_{Q' \cap H(\lambda/2)} h(z) \, dm_2(z).
\]
Thus,
\[
\int_{H(\lambda)} h(z) \, dm_2(z) \leq 2 \cdot (5\lambda)^{t+1} K^t \int_{H(\lambda/2)} h(z) \, dm_2(z), \quad \lambda \geq 1. \tag{17}
\]
Therefore, for every $0 < \gamma < 1/2,$
\[
\int_{Q} h(z)^{t+\gamma} \, dm_2(z) \leq \sum_{n \geq 3} 2^n \gamma \int_{H(2^n) \cap H(2^{n+1})} h(z) \, dm_2(z)
\]
\[
\leq c m_2(Q) + \gamma \sum_{n \geq 3} 2^n \gamma \int_{H(2^n)} h(z) \, dm_2(z)
\]
\[
\leq c m_2(Q) + c(K, t) \gamma \sum_{n \geq 3} 2^n \gamma \int_{H(2^n-3)} h(z) \, dm_2(z)
\]
\[
\leq c m_2(Q) + \frac{c(K, t) \gamma}{2t-1+\gamma} \int_{Q} h(z)^{t+\gamma} \, dm_2(z).
\]
For sufficiently small $\gamma, 0 < \gamma \leq \gamma_0(K, t),$ we get
\[
\int_{Q} h(z)^{t+\gamma} \, dm_2(z) \leq c(K, t) m_2(Q),
\]
and (14) is proved for $\varepsilon \leq \gamma_0(K, ps)/s.$

(III) Just consider the function $f,$ $f(z) = |z|^{2/q}(\log(A/|z|))^{2/q}$ for $A > 1$ to be determined later on. Then $f \in B_{p,q} \setminus \bigcup_{\varepsilon > 0} B_{p,q+t,\varepsilon},$ $0 < p < \infty.$ (In fact, $f \in C(D), 1/f \in L^q(D) \cap C(D \setminus \{0\}), 1/f \notin L^{q+\varepsilon}(D).$) It remains to verify that for sufficiently large $A,$ the function $f$ is subharmonic in $D,$ or, what is equivalent, the function $f_1, f_1(z) = (\log r)^{s-r-ss}, r = |z|,$ with $0 < s < \infty,$ is subharmonic for sufficiently large $r,$ which is equivalent, in its turn, to the fact that the function $f_2, f_2(r) = f_1(\exp r) = r^s e^{-rs}$ is convex for large $r$:
\[
f_2'(r) = sr^{s-1} e^{-rs} - sr^s e^{-rs},
\]
\[
f_2''(r) = s(s-1)r^{s-2} e^{-rs} - 2s^2 r^{s-1} e^{-rs} + s^2 r^s e^{-rs} \geq 0, \quad r > r(s).
(IV) We start with the following elementary calculation. Take small positive $x, y$ such that $0 < 2x < y$. Given $0 < p, q < 2$, we fix a natural number $N$ such that $1 \leq Np < 2$, and estimate the integrals
\[
I(p + \varepsilon) = \int_{\mathbb{D}} \left( \frac{z^{3N} - y^{3N}}{(z^3 - x^3)^N} \right)^{p+\varepsilon} dm_2(z), \quad 0 < \varepsilon < \frac{2}{N} - p,
\]
\[
J = \int_{\mathbb{D}} \left( \frac{z^{3N} - y^{3N}}{(z^3 - x^3)^N} \right)^q dm_2(z).
\]
We have
\[
I(p + \varepsilon) = \int_{|z| \leq x/2} + \int_{x/2 < |z| \leq 3x/2} + \int_{3x/2 < |z| \leq 2y} + \int_{|z| > 2y} = I_1 + I_2 + I_3 + I_4,
\]
\[
I_1 \asymp \left( \frac{y}{x} \right)^{3N(p+\varepsilon)} x^2,
\]
\[
I_2 \leq c \left( \frac{y}{x} \right)^{3N(p+\varepsilon)} x^2,
\]
\[
I_3 \leq c \left( \frac{y}{x} \right)^{3N(p+\varepsilon)} x^2,
\]
\[
I_4 \asymp 1.
\]
Thus, if
\[
x \leq y^{3N(p+\varepsilon)/(3N(p+\varepsilon)-2)},
\]
then
\[
I(p + \varepsilon) \asymp \left( \frac{y}{x} \right)^{3N(p+\varepsilon)} x^2.
\]
Analogously,
\[
J = \int_{|z| \leq y/2} + \int_{y/2 < |z| \leq 2y} + \int_{|z| > 2y} = J_1 + J_2 + J_3,
\]
\[
J_1 \leq cy^2,
\]
\[
J_2 \leq cy^2,
\]
\[
J_3 \asymp 1.
\]
Hence,
\[
J \asymp 1.
\]
Choose sequences \(\{x_k\}, \{y_k\}\) such that
\[
0 < x_k = y_k^{3Np/(3Np-2)}, \quad (y_k/x_k)^{1/k} \to \infty, \quad k \to \infty,
\]
and define
\[ \Phi_k(z) = \frac{z^{3N} - y_k^{3N}}{(z^3 - x_k^3)^N}. \]

Then
\[
\int_{\mathbb{D}} |\Phi_k(z)|^p dm_2(z) \sim \int_{\mathbb{D}} |\Phi_k(z)|^{-q} dm_2(z) \approx 1,
\]
\[
\int_{\mathbb{D}} |\Phi_k(z)|^{p+\frac{1}{q}} dm_2(z) \to \infty, \quad k \to \infty.
\]

For \( w \in \mathbb{D} \) denote by \( \varphi_w \) the Möbius function \( \varphi_w(z) = (z - w)/(1 - zw) \). For \( w_k \in [0, 1] \) sufficiently rapidly tending to 1 we put
\[ \Phi = \prod_k \Phi_k \circ \varphi_{w_k}. \]

Then
\[
\langle |\Phi|^p \rangle_Q \approx \langle |\Phi|^{-q} \rangle_Q \times 1, \quad Q \in \mathfrak{A},
\]
\[
\langle |\Phi|^{p+\frac{1}{q}} \rangle_Q \to \infty, \quad k \to \infty,
\]
for squares \( Q_k \) such that \( w_k \in TQ_k \), \( \text{dist}(w_k, \partial TQ_k) \geq |Q_k|^p \). Thus, \( |\Phi| \in \mathcal{M} \cap \mathcal{B}_{pq} \), but \( |\Phi| \notin \bigcup_{\varepsilon > 0} \mathcal{B}_{p+\varepsilon,q} \). Hence, \( \mathcal{M} \cap \mathcal{B}_{pq} \notin \bigcup_{\varepsilon > 0} \mathcal{B}_{p+\varepsilon,q} \).

**Remark 1.** We can refine somewhat assertions (III)–(IV) of Theorem: for \( 0 < p, q < \infty \) we have
(i) \( \mathcal{S} \cap \mathcal{B}_{pq} \subset \bigcup_{\varepsilon > 0} \mathcal{B}_{p+\varepsilon,q} \),
(ii) \( \mathcal{S} \cap \mathcal{B}_{pq} \notin \bigcup_{\varepsilon > 0} \mathcal{B}_{pq+\varepsilon} \),
for \( 0 < p, q < 2 \) we have
(iii) \( \mathcal{M} \cap \mathcal{B}_{pq} \notin \bigcup_{\varepsilon > 0} \mathcal{B}_{p+\varepsilon,q} \cup \mathcal{B}_{p+\varepsilon,q} \).

The proof of (i) is similar to that of the part (II) of Theorem. Instead of (3) we use that for \( 0 < p < \infty \), and for functions \( f \), non-negative and subharmonic on the unit disc \( \mathbb{D} \), we have \( f^p(0) \leq c(p) \langle f^p \rangle_{\mathbb{D}} \).

**Remark 2.** The class \( B_1 \) consists of non-negative functions \( w \) on \( \mathbb{D} \) such that for some \( K = K(w) \),
\[
\langle w \rangle_Q \leq K \cdot w(z), \quad z \in Q, \quad Q \in \mathfrak{A}.
\]

Denote by \( \mathcal{H} \) the class of functions \( \exp(f) \) for \( f \) harmonic in \( \mathbb{D} \). J. Rubio de Francia [14] (see also [6]) extended the factorization theorem of P. Jones [12] and obtained that for every \( w \in \mathcal{B}_{pq} \), \( 0 < p, q < \infty \), there exist \( w_1, w_2 \in B_1 \) such that
\[
w = w_1^{1/p} w_2^{-1/q}.
\]
(It is clear that $w_1^{1/p} w_2^{-1/q} \in \mathcal{B}_{pq}$ for any $w_1, w_2 \in B_1$). It appears to be unknown whether such a factorization is possible for $w \in \mathcal{E} \cap \mathcal{B}_{pq}$ with $w_1, w_2 \in \mathcal{E} \cap B_1$ (or if an analogous statement holds with $\mathcal{E}$ replaced by $A$). If true, this would provide a short proof of the inclusion $\mathcal{E} \cap \mathcal{B}_{pq} \subset \bigcup_{\varepsilon > 0} \mathcal{A}_{p+\varepsilon q+\varepsilon}$. Indeed, we would only need to verify that for any $w \in \mathcal{S} \cap B_1$ there exists $\varepsilon > 0$ such that $w^{1+\varepsilon} \in B_1$. Fix a dyadic square $Q \in \mathcal{A}$ and assume that $\langle w \rangle_Q = 1$. For every $n \geq 1$ we consider the maximal dyadic subsquares $Q^n_j$ of $Q$ such that $TQ^n_j$ intersects with the set $\{ z : w(z) > 2^n \}$. Since $w \in \mathcal{S} \cap B_1$, for some $c, \delta > 0$,

$$2^m m_2(Q) \asymp \int_{Q^n_j} w(z) \, dm_2(z) \leq c \cdot 2^n \int_{\{ z : w(z) > 2^n \}} dm_2(z). \quad (18)$$

Then for small $\varepsilon > 0$,

$$\int_Q w(z)^{1+\varepsilon} \, dm_2(z) \leq c m_2(Q) + \varepsilon \sum_{n \geq 0} 2^n \int_{\{ z : w(z) > 2^n \}} w(z) \, dm_2(z)$$

$$\leq c m_2(Q) + \varepsilon \sum_{n \geq 0} 2^n \sum_j \int_{Q^n_j} w(z) \, dm_2(z)$$

$$\leq c m_2(Q) + \epsilon \sum_{n \geq 0} 2^n \int_{\{ z : w(z) > 2^n \} \cap Q^n_j} dm_2(z)$$

$$\leq c m_2(Q) + \frac{c \varepsilon}{1 + \varepsilon} \int_Q w(z)^{1+\varepsilon} \, dm_2(z),$$

and we are done.

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REFERENCES


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